MATH105 Problem Sheet 7: Sets and maps

Due Wednesday 26th November

1. Find the inverse function of the following functions *in the cases where the inverse function exists.*

a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

b) $f : (-\infty, 0] \to [0, \infty)$ defined by $f(x) = x^2$.

c) $f : (0, \infty) \to (0, \frac{1}{2})$ defined by $f(x) = \dfrac{1}{x+2}$.

d) $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = 2e^x$.

2. In a class of 25 primary school children, the only pets that the children have are cats, dogs and hamsters. Three of the children have no pets at all, 15 have a cat, 15 have a dog, and 10 have a hamster. 8 have both a cat and a dog, 6 have both a cat and a hamster and 5 have both a dog and a hamster.

a) How many have all three?

b) How many have just a dog?

c) How many just have a cat?

d) How many just have a hamster?

3. Let $f : X \to Y$ be any map. Then we define, for any $B \subset Y$,

$$f^{-1}(B) = \{x : f(x) \in B\} \subset X$$

In order to make this definition, it is *not* necessary for f to have an inverse. But if f does have an inverse f^{-1}, then

$$f^{-1}(B) = \{x : f(x) \in B\} = \{f^{-1}(y) : y \in B\}$$

for any $B \subset Y$. Now show that for any B, $C \subset Y$:

a) $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$;

b) $f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$.

4. Let x_n be defined inductively for $n \in \mathbb{N}$ by

$$x_0 = 1, \quad x_{n+1} = \frac{2x_n + 3}{x_n + 2}$$

Prove by induction on n that, for all $n \in \mathbb{N}$,

(i) $x_n \geq 1$

(ii) $x_n^2 < 3$

Hint: Write $x_{n+1}^2 - 3$ in terms of x_n.

I will collect solutions at the lecture on Wednesday 26th November. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.

Possible tutorial problems

Some or all of these may be used in the problem class, if desired. They can also used as practice problems, either for practising for the homework problems, or for later practice during revision, or both.

5. Find the inverse function of the following functions *in the cases where the inverse function exists.*

a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 + x + x^2$.

b) $f : [-1/2, \infty) \to [3/4, \infty)$ defined by $f(x) = 1 + x^2$.

c) $f : (-1, \infty) \to (0, \infty)$ defined by $f(x) = \dfrac{1}{x+1}$.

6. In a group of 20 allotments, at least one of the following crops is grown on each of the plots: potatoes are grown on 15, courgettes on 12, and tomatoes on 10. Both potatoes and courgettes are grown on 8, potatoes and tomatoes on 7, and tomatoes and courgettes on 6.

a) On how many plots are all three of these crops grown?

b) On how many plots, are just potatoes grown out of these three crops?

c) On how many plots, are just courgettes grown out of these three crops?

d) On how many plots, are just tomatoes grown out of these three crops?

7. Let $f : X \to Y$ be any map. Then we define, for any $B \subset Y$,

$$f^{-1}(B) = \{x : f(x) \in B\} \subset X$$

In order to make this definition, it is *not* necessary for f to have an inverse. But if f does have an inverse f^{-1}, then

$$f^{-1}(B) = \{x : f(x) \in B\} = \{f^{-1}(y) : y \in B\}$$

for any $B \subset Y$. For any $A \subset X$, we also define

$$f(A) = \{f(x) : x \in A.$$

a) Show that for any $B \subset Y$,

$$f(f^{-1}(B)) \subset B.$$

b) Show that for any $A \subset X$,

$$A \subset f^{-1}(f(A)).$$

Give examples to show that equality might not hold in each case.

8. Let x_n be defined inductively for $n \in \mathbb{N}$ by

$$x_0 = 1, \quad x_{n+1} = \frac{3x_n + 2}{x_n + 3}$$

Prove by induction on n that, for all $n \in \mathbb{N}$,

(i) $x_n \geq 1$

(ii) $x_n^2 < 2$

Hint: Write $x_{n+1}^2 - 3$ in terms of x_n.

Extra question

This one comes from Eccles' book and is quite hard.

9. At an international conference of 100 people, 75 speak English, 60 speak Spanish, 45 speak Swahili (and everyone present speaks at least one of these languages). Write A, B and C for the numbers speaking English, Spanish and Swahili respectively.

a) Write each of the sets A, B, C and $A \cup B \cup C$ as a union of the sets

$$A \setminus (B \cup C),\ B \setminus (A \cup C),\ C \setminus (A \cup B),\ A \cap B \setminus C,\ B \cap C \setminus A,\ A \cap C \setminus B,\ A \cap B \cap C.$$

b) Show that the number of people speaking all three languages, that is, $|A \cap B \cap C|$, is ≤ 40. Hence, or otherwise, determine the maximum possible number of people who can speak only one language. In this case, how many speak only English, how many speak only Spanish and how many speak only Swahili?

c) What is the maximum number of people who speak only English. In this case, what can be said about the number who speak only Spanish and the number who speak only Swahili?

Hint: Write A and $B \cup C$ and $A \cup B \cup C$ as unions of the sets

$$A \setminus (B \cup C),\ (B \cup C) \setminus A,\ \ A \cap (B \cup C).$$