MATH105 Problem Sheet 5: Integers, divisors and primes

Due Wednesday 5th November

1. Prove by induction on n that $2^n + 3^n < 4^n$ for all integers $n \geq 2$.

2. Find the g.c.d and l.c.m. of 364 and 434, first using the Euclidean algorithm and then by prime factorisation. Also, writing d for the g.c.d., used the Euclidean algorithm to find integers a and b such that $d = 434a + 364b$.

Most of the rest of the questions use the list of the first 1000 primes which is included with this problem sheet, and which is also available on the module webpage. Write p_n for the n'th prime in the list, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on. A *prime gap* is a set of the form $(p_n, p_{n+1}) \cap \mathbb{Z}$ for any $n \geq 2$. Such a set is always nonempty, because apart from $p_1 = 2$, all the primes are odd, and so there must be at least one even integer between two consecutive primes > 2. For example, $\{8, 9, 10\} = (7, 11) \cap \mathbb{Z} = (p_4, p_5) \cap \mathbb{Z}$ is a prime gap.

3. Write down all the integers in the following prime gaps:

a) $(p_5, p_6) \cap \mathbb{Z}$;

b) $(p_6, p_7) \cap \mathbb{Z}$.

The rest of the questions concern the prime gap $(1327, 1361) \cap \mathbb{Z} = (p_{217}, p_{218}) \cap \mathbb{Z}$, which is larger than any other prime gap $(p_n, p_{n+1}) \cap \mathbb{Z}$ for $n < 1000$.

4. Explain why every integer in $(1327, 1361) \cap \mathbb{Z}$ is divisible by some positive prime ≤ 31 (assuming that $(1327, 1361) \cap \mathbb{Z}$ is a prime gap).

5. Let $n \in \mathbb{Z}_+$ with $n < 1327$ and explain why the g.c.d. of n and 1327 is 1. Using the Euclidean algorithm, find integers a and b such that $a \times 1327 + bn = 1$ where

a) $n = 17$;

b) $n = 31$.

In order to help with the last question, you are advised to check your arithmetic.

6. Write $1327 = qn + r$ for each of the following n. You can use the first step of your calculation in the previous question in the first two parts. In order to do the next question without too much stress, you are advised to check your arithmetic.

a) $n = 17$;

b) $n = 31$;

c) $n = 13$;

d) $n = 19$;

e) $n = 23$;

f) $n = 29$.

7. Find the prime factorisation of each of the integers in the prime gap $(1327, 1361) \cap \mathbb{Z}$ which is divisible by one or more of: 31, 29, 23, 19, 17, 13. You can use your previous calculations, and can also use the list of primes provided, as proof that factors found are prime.

I will include prime factorisations of all the numbers in the gap in the solutions but don't waste your time!

I will collect solutions at the lecture on Wednesday 5th November. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.

Possible tutorial problems

Some or all of these may be used in the problem class, if desired. They can also used as practice problems, either for practising for the homework problems, or for later practice during revision, or both.

8. Prove by induction on n that $3^n + 4^n < 5^n$ for all integers $n \geq 3$.
Hint: write $3^{n+1} = 3 \times 3^n$, which is less than 4×3^n and $4^{n+1} = 4 \times 4^n$.

9. Find the g.c.d and l.c.m. of 378 and 450, first using the Euclidean algorithm and then by prime factorisation. Also, writing d for the g.c.d., used the Euclidean algorithm to find integers a and b such that $d = 378a + 450b$.

The *length* of a prime gap $(p_n, p_{n+1}) \cap \mathbb{Z}$ is $p_{n+1} - p_n$, that is, one more than the number of integers in the prime gap. The length of the prime gap $(1327, 1361) \cap \mathbb{Z}$ considered above is therefore 34.

10. Show that $n! + k$ is divisible by k for any integer k with $2 \leq k \leq n$. Deduce that the numbers $n! + k$, for $2 \leq k \leq n$ are contained in a prime gap of length $\geq n$

This shows that arbitrarily long prime gaps exist. The proof is given in
http://mathworld.wolfram.com/PrimeGaps.html .