MATH105 Problem Sheet 11

1. Determine which of the following sequences are increasing, which are decreasing and which are neither

a) $x_n = \dfrac{2}{n^2+1}$, $n \geq 1$

b) $x_n = 1 - \dfrac{2}{n^2+1}$, $n \geq 1$

c) $x_n = n^2 - 3n + 2$, $n \geq 1$

d) $x_n = \dfrac{1}{1-3^n}$, $n \geq 1$

e) $x_n = \sum_{k=1}^{n} \dfrac{1}{k!}$, $n \geq 1$

f) $x_n = \sum_{k=1}^{n} \dfrac{(-1)^{k+1}}{k^2}$, $n \geq 1$.

g) x_n defined inductively by $x_1 = 1$ and $x_{n+1} = \frac{1}{1+x_n}$ for all $n \geq 1$.

h) x_n defined inductively by $x_1 = 1$ and $x_{n+1} = \dfrac{x_n}{1+x_n}$ for all $n \geq 1$.

2. Determine which of the sequences in question 1 are bounded above and which are bounded below.
Hint for 1e): Show that

$$x_n \leq \sum_{k=1}^{n} \frac{1}{2^{k-1}} = \sum_{k=0}^{n-1} \frac{1}{2^k}.$$

3. Why do the sequences in 1a), 1b), 1d), 1e) and 1h) have a limit in $\mathbb{R}$? Why does the sequence in 1c) not have a limit in $\mathbb{R}$?

4.

a) For the sequences in 1f) and 1g) show that for all $n \geq 1$

$$x_{2n} < x_{2n+2} < x_{2n+1} < x_{2n-1}$$

b) Explain why the sequences x_{2n} and x_{2n-1} have limits in both these cases

c) In the case of 1f) show that $\lim_{n\to\infty} x_{2n-1} - x_{2n} = 0$ and explain why the sequence x_n also has a limit in this case.

d) The sequence in 1g) also has a limit but this is a bit harder to prove. What do you think the limit is in this case?

5. Determine which of the following sets are countable and which are uncountable. You may assume that the image of a countable set is countable, and that the domain of a map with uncountable image is uncountable

a) $(0, \infty)$

b) $[0, 1]$

c) $\mathbb{Q} \cap [0, 1]$

d) The set of even integers

6. Assuming that $A \times B$ is countable for any countable sets A and B, prove by induction that if A is countable then so is A^n for all $n \in \mathbb{Z}_+$, where $A^n = \{(a_1, \cdots a_n) : a_i \in A,\ 1 \leq i \leq n\}$.

7. In contrast show that if A is countable with at least two elements then the set of sequences of elements in A is uncountable, that is $\{(a_i) : a_i \in A \forall i \in \mathbb{Z}_+\}$ is uncountable.
Hint: Mimic the proof that $\mathbb{R}$ is uncountable.

8. Prove that the following numbers are algebraic

- $3 - \sqrt{2}$
- $\sqrt{3} - 1$
- $\sqrt{3} - \sqrt{2}$

  Hint: This last one is a root of an equation of degree four. To find the equation, it might help to compute $(\sqrt{3} - \sqrt{2})^2$.