MATH105 Problem Sheet 10

Due Wednesday 10th December

1. Find the numbers whose continued fraction expansions are

a)

$$\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\ddots}}}}$$

b)

$$\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\ddots}}}}$$

2. Consider the polynomial $f(x) = x^3 + x + 3$

a) Show that f is strictly increasing.

b) Show that $x^3 + x + 3 = 0$ has no rational solutions *Hint*: first show that it has no integer solutions. Then write $x = \frac{p}{q}$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$, $q \neq 1$ and p and q coprime. Suppose that x is a solution, and obtain a contradiction by showing that p and q have a common prime factor.

c) Write down a Dedekind cut representing the unique real root of f

3. Let $f(x) = x^3 - 12x + 2$.

a) Show that f has exactly 3 real zeros, one in $(-4, -3)$, one in $(0, 1)$ and one in $(3, 4)$. You may use calculus if you like.

b) Write down a Dedekind cut representing each of the zeros.

4. Let $a_n \in \mathbb{Z}_+$ for all $n \geq 0$, that is, a_n is any sequence of strictly positive integers. Let

$$p_{-1} = 1, \quad q_{-1} = 0 = p_0, \quad q_0 = 1,$$

and for $n \geq 1$,

$$p_n = p_{n-2} + a_n p_{n-1}, \quad q_n = q_{n-2} + a_n q_{n-1}.$$

Prove by induction that

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n$$

for all $n \geq 0$.

Hint: this one has only one base case. So start by proving it when $n = 0$. Also you only need to use the inductive definition of p_{n+1} and q_{n+1} *not* the inductive definitions of p_n and q_n.

I will collect solutions at the lecture on Wednesday 10th December. Any solutions which are not handed in then, or by 5 p.m. that day in the folder outside room 516 will not be marked.

Possible tutorial problems

Some or all of these may be used in the problem class, if desired. They can also used as practice problems, either for practising for the homework problems, or for later practice during revision, or both.

5. Find the numbers whose continued fraction expansions are

a)

$$\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\ddots}}}}$$

b)

$$\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\ddots}}}}$$

6. Consider the polynomial $f(x) = x^3 + 2x + 5$

a) Show that f is strictly increasing.

b) Show that $x^3 + 2x + 5 = 0$ has no rational solutions *Hint*: first show that it has no integer solutions. Then write $x = \frac{p}{q}$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$, $q \neq 1$ and p and q coprime. Suppose that x is a solution, and obtain a contradiction by showing that p and q have a common prime factor.

c) Write down a Dedekind cut representing the unique real root of f

7. Let $f(x) = x^3 - 12x + 1$.

a) Show that f has exactly 3 real zeros, one in $(-4, -3)$, one in $(0, 1)$ and one in $(3, 4)$. You may use calculus if you like.

b) Write down a Dedekind cut representing each of the zeros.