MATH105: Numbers and Sets

Basic information and an introduction is given in the file of Lecture 1. The brief syllabus is given there as:

Syllabus

Basic propositional logic

Natural numbers

Sets and maps

Equivalence relations and quotients

Rational and real numbers

Countability ~~and~~

Complex numbers.

Basic Propositional logic is simply a shorthand used in writing mathematics. The first basic symbols are

$\vee$ or

$\wedge$ and

$\Longrightarrow$ implies

$\Longleftarrow$ is implied by

Examples

If x is a real number, $x > 0 \vee x < 0 \vee x = 0$

" " $x > 0$ or $x < 0$ or $x = 0$

If x is a real number, $x \geqslant 0 \vee x \leqslant 0$

" " $x \geqslant 0$ or $x \leqslant 0$

$(1 < 2) \wedge (2 < 3)$ — $1 < 2$ and $2 < 3$

If x, y, z are real numbers $((x < y) \wedge (y < z)) \Longrightarrow x < z$

" " ~~the~~ and if $x < y$ and $y < z$, ~~we have~~ then $x < z$

(2)

If x is a real number, $x^2 = 1 \iff (x = 1 \vee x = -1)$

" " $x^2 = 1$ if and only if $x = 1$ or $x = -1$

If x is a real number, $x = 1 \Rightarrow x^2 = 1$

If x is a real number, if $x = 1$ then $x^2 = 1$

All of the statements above are true.

Which of the following are true?

If x is real, $x^2 = 4 \Rightarrow x = 2$ — False

$(2 < 3) \vee (3 < 2)$ — True

If x is real, $x^2 = x \iff x^2 - x = 0$ — True

$x = 1 \Rightarrow x^2 = x$ — True

$x^2 = x \Rightarrow x = 1$ — False

If x is real,

If x is real, $(x^2 = x) \Rightarrow (x = 0 \vee x = 1)$ — True

$(x = 0 \vee x = 1) \Rightarrow (x^2 = x)$ — True

$x > 1 \Rightarrow x > 0$ — True

$x > 1 \iff -x > -1$ — False

$(x = 0 \wedge x = 1) \Rightarrow x = 2$ — False!

We often determine whether or not statements are true by following chains of implications between statements which are true (or not).

<u>For example</u> Suppose we have statements A, B, C, D, E and the following implications hold

$A \Rightarrow B \qquad A \Rightarrow C \qquad D \Rightarrow C \qquad B \Rightarrow E$

If A holds, which of the following hold?

B (Yes) C (Yes) D (can't tell) E (yes)

$B \vee C$ (Yes) ~~E~~ $B \wedge C$ (Yes) $C \vee D$ (Yes, - because C holds)

$B \wedge E$ (Yes)

If the implications hold and B holds, which of the following hold?

A (can't tell) B (Yes) C (can't tell) D (can't tell)

E (Yes) $A \vee E$ (Yes, because E holds) $C \wedge D$ (can't tell)

Important Suppose $A \Rightarrow B$

Then if A holds - that is - if A is true - then B also holds/is true

But if B ~~too~~ is true we cannot tell whether or not B is true without more information.

Example Suppose A is "$x=0$"
B is "$x^2 = x$"

Then $A \Rightarrow B$ $x=0 \Rightarrow x^2 = x$

But it is not true that $x^2 = x \Rightarrow x = 0$

But suppose A is "$x=0 \vee x=1$" and B is "$x^2 = x$"

Then $(x=0 \vee x=1) \Rightarrow x^2 = x$

and this time we do also have $x^2 = x \Rightarrow x=0 \vee x=1$

We have both $A \Rightarrow B$ and $B \Rightarrow A$

That is, $A \Leftrightarrow B$

Negation

If A is a statement then $\neg A$ is the statement "not A"

Examples If A is "It will rain today" then $\neg A$ is the statement "It will not rain today"

In the following examples, x and y and z are real numbers

$\neg(x<y)$ iff $x \geq y$ – which can also be written as $y \leq x$

$\neg(x \leq y)$ is $y<x$ – also written as $x>y$

What about $\neg(x<y<z)$?

$x<y<z$ is the same as $\underset{A}{(x<y)} \wedge \underset{B}{(y<z)}$

This is not true if and only if either $x \geq y$ or $y \geq z$ or both

So $\neg(x<y<z)$ is $\underset{\neg A}{(x \geq y)} \vee \underset{\neg B}{(y \geq z)}$

In general $\neg(A \wedge B)$ is $(\neg A) \vee (\neg B)$.

What about $\neg(x>3 \vee x<-1)$?

It is not true that $(x>3 \text{ or } x<-1)$ if and only if $(-1 \leq x \leq 3)$.

So $\neg(\underset{A}{x>3} \vee \underset{B}{x<-1})$ is $-1 \leq x \leq 3$, which can also be written as $\underset{\neg B}{(-1 \leq x)} \wedge \underset{\neg A}{(x \leq 3)}$ and also as $\underset{\neg B}{(x \geq -1)} \wedge \underset{\neg A}{(3 \geq x)}$ and as $\underset{\neg A}{(x \leq 3)} \wedge \underset{\neg B}{(-1 \leq x)}$ ...

In general $\neg(A \wedge B)$ is $(\neg A) \vee (\neg B)$

Examples $\neg(x=0 \vee x=1)$ is $x \neq 0 \wedge x \neq 1$

$$\neg(x=0 \vee x=1) \Longleftrightarrow (x \neq 0 \wedge x \neq 1)$$

$$\neg(-1 < x < 1) \Longleftrightarrow \neg(-1 < x \wedge x < 1) \Longleftrightarrow (x \leq -1 \vee x \geq 1)$$

Theorems

A Theorem is a statement which is true

e.g. $2 < 3$ is a theorem. $2 < 1$ is a false statement, so not a theorem.

e.g. $x^2 > 4 \Longleftrightarrow (x > 2 \vee x < -2)$ is a theorem.

If we want to prove a theorem C, we might start with a theorem A - a statement that we know to be true - and try to deduce C from it.

If A is true and $A \Rightarrow C$ then C is true.

If A is true and $A \Rightarrow B$ and $B \Rightarrow C$ then C is true.

We could have a longer chain of implications. e.g. If $A \Rightarrow B_1$ and $B_1 \Rightarrow B_2$ and $B_2 \Rightarrow C$ all hold and A is true, then C is true.

It is a good idea to use $\Longleftrightarrow$ whenever possible. That way, we might end up proving a stronger theorem than we want.

Examples

Theorem If x is real number, $x^2 - 3x + 2 = 0 \Rightarrow x = 1 \vee x = 2$

Proof $x^2 - 3x + 2 = 0 \Longleftrightarrow (x-1)(x-2) = 0$

$\Longleftrightarrow x - 1 = 0 \vee x - 2 = 0$ (if a product of 2 real numbers is zero, at least one of the numbers in the product must be zero)

$\Longleftrightarrow x = 1 \vee x = 2.$

$\Longleftrightarrow x = 1 \vee x = 2.$ □

So $x^2 - 3x + 2 = 0 \Longleftrightarrow x = 1 \vee x = 2$, which is This is stronger than the theorem originally stated, which is fine. We would still obtain the theorem originally stated, if we used $\Rightarrow$ instead of $\Longleftrightarrow$, all the way through.

If x is real number,

Theorem $x^2-3x+2 \le 0 \Leftrightarrow 1 \le x \le 2$

Proof $x^2-3x+2 \le 0 \Leftrightarrow (x-1)(x-2) \le 0$

$\Leftrightarrow ((x-1 \geq 0 \wedge x-2 \le 0) \vee (x-1 \le 0 \wedge x-2 \geq 0))$

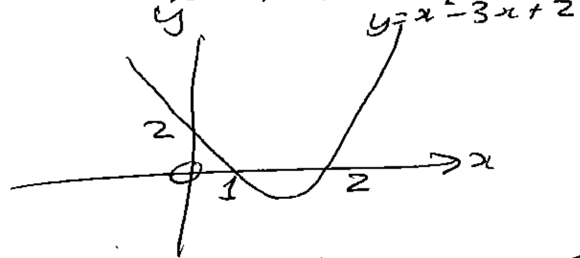
(the product of 2 real numbers is $\le 0 \Leftrightarrow$ one of them is ≥ 0 and the other is ≤ 0)

$\Leftrightarrow ((x \geq 1 \wedge x \le 2) \vee (x \le 1 \wedge x \geq 2))$

$\Leftrightarrow 1 \le x \le 2$ because no real number x satisfies $x \le 1 \wedge x \geq 2$

$\square$

A graph confirms this theorem



Theorem If x is a real number, $x^2-3x+2 > 12 \Leftrightarrow (x<-2 \vee x>5)$

Proof $x^2-3x+2 > 12 \Leftrightarrow x^2-3x-10 > 0$

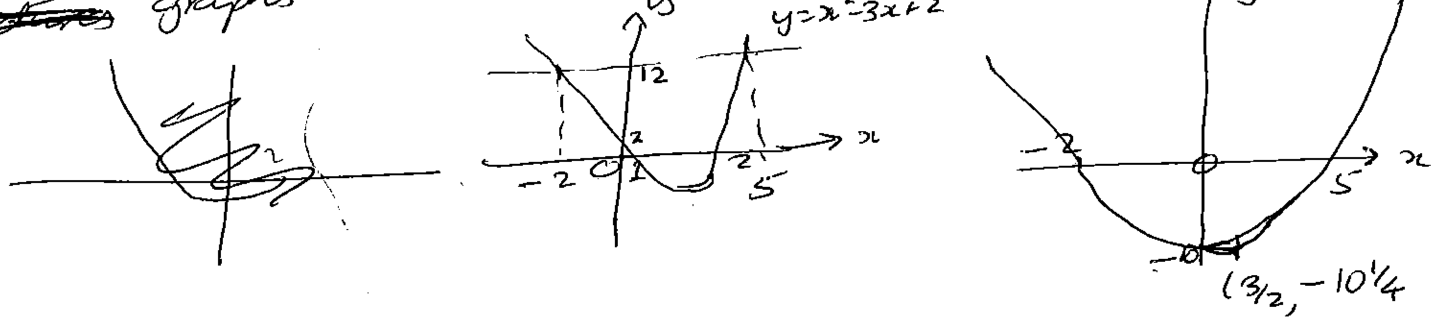
$\Leftrightarrow (x+2)(x-5) > 0 \Leftrightarrow (x+2>0 \wedge x-5>0) \vee (x+2<0 \wedge x-5<0)$

(A product of 2 real numbers is $>0 \Leftrightarrow$ both numbers are >0 or both <0)

$\Leftrightarrow (x>-2 \wedge x>5) \vee (x<-2 \wedge x<5)$

$\Leftrightarrow x>5 \vee x<-2$ $\square$

A graph confirms this theorem. Actually there are two natural ~~pictures~~ graphs one can draw: of $y=x^2-3x+2$ or $y=x^2-3x-10$



Theorem If x is a real number, $\left|\frac{1}{x+1}\right| < 1 \iff (x>0 \vee x<-2)$

Proof $\left|\frac{1}{x+1}\right| < 1 \iff \frac{1}{(x+1)^2} < 1$ (Modulus is always ≥ 0. The square of a number ≥ 0 ~~which is~~ is $<1 \iff$ the number itself is <1. Also $y^2 = (-y)^2$ for all real numbers y.)

$\iff 1 < (x+1)^2$ (Inequalities are preserved when multiplying through by numbers >0 and $(x+1)^2 > 0$ because we know $(x+1)^2 \neq 0 \vee \frac{1}{(x+1)^2} < 1$)

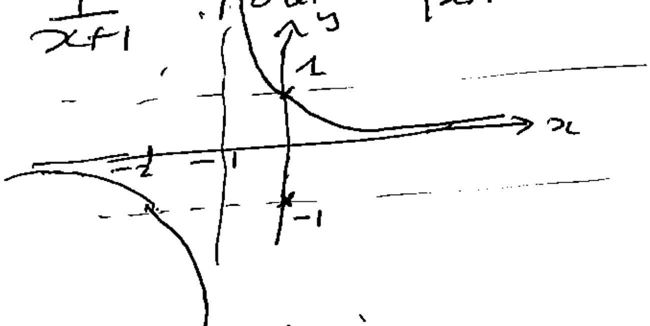
$\iff 1 < x^2+2x+1 \iff 0 < x^2+2x \iff 0 < x(x+2)$

$\iff (0<x \wedge 0<x+2) \vee (x<0 \wedge x+2<0)$

$\iff (0<x \wedge -2<x) \vee (x<0 \wedge x<-2)$

$\iff 0<x \vee x<-2$ $\square$

Again this is confirmed by drawing a graph. The easiest graph to draw is probably $y = \frac{1}{x+1}$. Note that $\left|\frac{1}{x+1}\right| < 1 \iff$
$-1 < \frac{1}{x+1} < 1.$



Theorem If x and y are real numbers then
$x^2+xy+y^2 \leq 0 \iff x=y=0$

Proof $x^2+xy+y^2 \leq 0 \iff (x+\frac{1}{2}y)^2 + \frac{3}{4}y^2 \leq 0$ (A square of any real number is ≥ 0 and $=0 \iff$ the number $=0$)

$\iff (x+\frac{1}{2}y)^2 + \frac{3}{4}y^2 = 0$

$\iff x+\frac{1}{2}y = 0 \wedge y = 0$

$\iff x=0 \wedge y=0$ $\square$

Negating implications

If A and B are statements, then $A \Rightarrow B$ is equivalent to

$$\neg B \Rightarrow \neg A.$$

<u>Example</u> Suppose x is a real number. Let A be the statement $x < 2$ and let B be the statement $x < 3$

$A \Rightarrow B$, that is $x < 2 \Rightarrow x < 3$ is a true statement

$\neg A$ is $x \geqslant 2$ $\quad \neg B$ is $x \geqslant 3$

$x \geqslant 3 \Rightarrow x \geqslant 2$ $\quad$ So we see in this example that $\neg B \Rightarrow \neg A$

$\neg B \Rightarrow \neg A$

But $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$ holds whatever the statements A and B are.

<u>Example</u> Again, suppose x is a real number.

Let A be the statement $x^2 < 4$ $\quad$ Let B be $x < 2$

$x^2 < 4 \Rightarrow x < 2$ is a true statement

$A \Rightarrow B$

$\neg A$ is $x^2 \geqslant 4$ $\quad \neg B$ is $x \geqslant 2$

$\neg B \Rightarrow \neg A$ $\quad x \geqslant 2 \Rightarrow x^2 \geqslant 4$ is true.

Note that $\neg A$ does not imply $\neg B$

$x^2 \geqslant 4$ does <u>not</u> imply $x \geqslant 2$

e.g. $x = -3$ satisfies $(-3)^2 \geqslant 4$ and $\neg(-3 \geqslant 2)$

$-3 < 2$

<u>Example</u> Let x and y be real numbers.

Let A be $x > 0 \wedge y > 0$ $\quad$ Let B be $xy > 0$

$A \Rightarrow B$. $\quad (x > 0 \wedge y > 0) \Rightarrow xy > 0$

$\neg B$ is $xy \leq 0$ $\quad \neg A$ is $(x \leq 0) \vee (y \leq 0)$

$\neg B \Rightarrow \neg A$ $\quad xy \leq 0 \Rightarrow (x \leq 0 \vee y \leq 0)$

But it is <u>not</u> true that $\neg A \Rightarrow \neg B$.