UNIVERSITY OF LIVERPOOL

MATH 104

EXAMINER: Prof. S.M. Rees, EXTENSION 44063.

TIME ALLOWED: One and a half hours.

Full marks will be obtained by complete answers to all questions.

1. Give the names of the following (lower case) Greek letters: δ, ζ. Write the lower case Greek letters *epsilon* and *omega*. [4 marks]

2.

For the following sets S, in (a) and (b), give a function $f(n)$ such that

$$S = \{f(n) \mid n \in \mathbb{N}\}.$$

(Recall that $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.)

(a) $S = \{0, -3, -6, -9, \ldots\}$.

(b) $S = \{0, 2, 6, 12, 20, 30, 42, 56 \ldots\}$.

(c) Write down all the elements in the set $\{(-1)^n \mid n \in \mathbb{N}\}$.

(d) Give a statement $P(x)$ such that the set $\{x^2+1 \mid x \in \mathbb{R}\}$ can be written in the form $\{x \in \mathbb{R} \mid P(x)\}$.

[12 marks]

3. Negate each of the following statements:

(a) $x > 10$ and $x < 11$.

(b) For $x \in \mathbb{R}$, $x^2 - 1 = 0 \Rightarrow x = 1$.

(c) There exists $x \in \mathbb{R}$ such that $x \geq 2$ and $f(x) \geq 1$.

(d) If $f(x)$ is an odd function then $f(x)$ is not an even function.

Determine which of these original statements (if any) have free variables. For those which do not have free variables (if any) state whether they are true or false, giving a brief reason. [15 marks]

4. Definition Let f be a (real-valued) function Then f is *injective* if for all x, y in the domain of f

$$f(x) = f(y) \implies x = y.$$

Definition Let f be a (real-valued) function with domain in $\mathbb{R}$. Then f is *strictly increasing* if for all x and y in the domain of f.

$$x < y \implies f(x) < f(y).$$

(a) Write down what it means for f *not* to be strictly increasing.

b) Determine whether the function $f(x) = |x|$, with domain $\mathbb{R}$, is injective.

c) Prove that the function $f(x) = \frac{x}{1-x}$, $x \neq 1$ is injective, but not strictly increasing.

d) Prove that if a real-valued function f with domain in $\mathbb{R}$ is strictly increasing, then f is injective.

[15 marks]

5. a) Write down carefully the meaning of the statement that a set S of numbers is *closed under addition.*

Determine whether or not the set $S = \{n^2 \mid n \in \mathbb{Z}\}$ is closed under addition.

b) Write down carefully what it means to say that $m|n$ (m divides n) when m and n are integers.

Let $S = \{n \in \mathbb{Z}|5 \text{ divides } n\}$. Show that:

S is closed under addition;

$n \in S \Rightarrow n^2 \in S$;

if $n \in \mathbb{Z}$ and $n+1 \in S$ or $n-1 \in S$, then $n^2 - 1 \in S$;

if $n \in \mathbb{Z}$ and $n+2 \in S$ or $n-2 \in S$, then $n^2 + 1 \in S$.

[16 marks]

6. Let R be a relation on a set X. Then R is defined to be an *equivalence relation* if for all $x, y, z \in X$ the following three conditions hold:

(i) $x\,R\,x$.

(ii) If $x\,R\,y$ then $y\,R\,x$.

(iii) If $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

Determine whether or not the following relations R on the given sets X are equivalence relations. You should justify your answers carefully, working directly from the definitions.

a) $X = \mathbb{R}$ $x\,R\,y$ means $x + y = 0$.

b) $X = \mathbb{R}$, $x\,R\,y$ means $|x| = |y|$.

c) $X = \mathbb{R}$, $x\,R\,y$ means $x - y \in \mathbb{Z}$.

[10 marks]

7. Consider the following theorem. You are not expected to understand what it means.

Theorem *Let X be a Banach space and let $T : X \to X$ be a linear map. Suppose that T is bounded. Then the spectrum of T is a closed non-empty bounded subset of $\mathbb{C}$.*

Identify the context, hypothesis, and conclusion of the theorem. State its converse. Assume that the *original* theorem (*not the converse*) is true. State, with a brief reason, what, if anything, the theorem tells you about:

(a) A Banach space X and a linear map $T : X \to X$ which is not bounded;

(b) A Banach space X and a linear map T whose spectrum is empty;

(c) A Banach space X and a linear map T whose spectrum is closed, non-empty and bounded.

(d) A Banach space X and a bounded linear map $T : X \to X$

Suppose now that the *converse* of the theorem is true. Does this alter your answer to (b)? Give a reason. [15 marks]

8. Determine whether each of the following statements is true or false, giving a brief reason in each case. Recall that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
We define $\mathbb{R}^+ = \{x \in \mathbb{R} \,|\, x > 0\}$.

(a) $\forall x \in \mathbb{R},\ x^2 - 1 \geq 0$.

(b) $\exists x \in \mathbb{R}^+,\ x < \frac{1}{x}$.

(c) $\exists a, b \in \mathbb{R}$ such that $(a+b)^2 \leq 5ab$ and $a \neq b$.

(d) $\forall x \in \mathbb{R}^+, \exists n \in \mathbb{N},\ \frac{1}{x} < n$.

(e $\exists n \in \mathbb{N}, \forall x \in \mathbb{R}^+,\ \frac{1}{x} < n$. [14 marks]