MATH104 Week 2: Thinking about the natural numbers

Not to be handed in

The symbol $\emptyset$ means the set with no elements in it. Here is a statement which is true:

a) Let $A \subseteq \mathbb{N}$ with $A \neq \emptyset$. Then A has a smallest element, that is, there is $a \in A$ such that $a \leq n$ for all $n \in A$.

Here is a statement which is *not* true:

b) Let $A \subseteq \mathbb{N}$ with $A \neq \emptyset$. Then A has a largest element, that is, there is $a \in A$ such that $n \leq a$ for all $n \in A$.

Why is it not true?
Here is a "proof" of b) which uses a). Identify at least one mistake in the proof:

- Let $B = \mathbb{N} \setminus A$, that is, $B = \{n \in \mathbb{N} | n \notin A\}$. Then, by a), B has a smallest element b. Then $b - 1 \in A$. So $b - 1$ is the largest element in A, and A has a largest element.

Here is a statement that is true, which is slightly different from b):

c) Let $A \subseteq \mathbb{N}$ with $A \neq \emptyset$, and suppose that there is $N \in \mathbb{N}$ (not in A) such that $n < N$ for all $n \in A$. Then A has a largest element, that is, there is $a \in A$ such that $b \leq a$ for all $b \in A$.

Here is part of a proof of c):

- Let $B = \{n \in \mathbb{N} | n > m \text{ for all } m \in A\}$ ...

Can you see how to finish this proof off? And why does this work for proving c) but not for b)? Why could we not define B in this way in order to prove b)?

The definition given in lectures of an odd integer is (or will be) that an integer is odd if it is not even. The definition of an even integer is that it is divisible by 2. For convenience, let us restrict to the natural numbers, and let $n \in \mathbb{N}$ be odd. By c), the set $A = \{m \in \mathbb{N} | m < n,\ m \text{ even }\}$ has a largest element a.

(i) Why is A nonempty? (We need that if we are to use c).)

(ii) Why is $a = n - 1$? (*Hint*: Why is $a + 2$ even?)

(iii) Is it true that for $m \in \mathbb{N}$, m is odd $\Leftrightarrow$ $m - 1$ is even?