MATH104 Exam May 2008, Solutions

All questions except q.1 are standard examples similar to classwork or homework.

1. sigma, psi, ϕ, γ. (1 mark each.)

2.

(a) $f(n) = -1 - 6n$.

(b) $f(n) = 6 + 3n$

(c) $2 \in S$ (take $n = 0$); $-6 \in S$ (take $n = -2$); $4 \notin S$ since this would require $n \in \mathbb{N}$ with $n^3 = 2$; $10 \in S$ (take $n = 2$).

(2 marks for (a), 4 marks for (b) and 6 marks for (c).)

3.

(a) $x \neq -4$ and $x \leq 2$. The statement (a) has a free variable x. (3 marks)

(b) There exist real numbers a, b, c such that $ab = ac$ and $b \neq c$. The original statement is false and this negation is true: take $a = 0, b = 1, c = 2$; then $ab = ac = 0$ and $b \neq c$. (4 marks)

(c) There exist $x > 1$ and $y > 2$ with $f(x) \leq 0$ or $f(y) \leq 4$ (3 marks)

(d) $\exists N \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $f(x) \leq N$. (3 marks)

4.

(a) f is *not injective* if there exist x, y in the domain of f such that $f(x) = f(y)$ but $x \neq y$. (2 marks)

(b) Let $x, y \in \mathbb{R}$ and suppose $f(x) = f(y)$. Then $2 + 4x = 2 + 4y$ and rearranging gives $4x = 4y$ so $x = y$. (3 marks)

(c) Let $x = 0, y = 2$. Then $f(x) = f(y) = 0$ but $x \neq y$. Hence f is not injective. (3 marks)

(d) Let $x, y \in \mathbb{R}, x > 1, y > 1$, Suppose that $f(x) = f(y)$. Then $x^2 - 2x = y^2 - 2y$, which gives $x^2 - y^2 - 2(x - y) = 0$, that is $(x - y)(x + y) - 2(x - y) = 0$, that is $(x - y)(x + y - 2) = 0$. But $x > 1$ and $y > 1$ so that the second factor is > 0. Hence the first factor is 0, giving $x = y$. (5 marks)

5.

S closed under addition means that, for all $x \in S$ and $y \in S$, it follows that $x + y \in S$. (1 mark)

For the given S, let $x, y \in S$ so that there exist integers k, ℓ with $3x = k$, $3y = \ell$. Thus $3(x + y) = k + \ell$ and since $k + \ell \in \mathbb{Z}$ we deduce that $x + y \in S$. Thus S is closed under addition. (3 marks)

(a) R is not an equivalence relation. Property (i) fails, since for example $1\,R\,1$ is false. (2 marks)

(b) R is an equivalence relation. For let x, y, and z be any integers. Then

i) $x - x = 0$ and $6|0$ since $0 = 0 \times 6$, so $x\,R\,x$.

ii) If $x\,R\,y$ then $x - y = 6k$, for some $k \in \mathbb{Z}$ so $y - x = 6(-k)$, and $-k \in \mathbb{Z}$ so $y\,R\,x$.

iii) If $x\,R\,y$ and $y\,R\,z$ then $x - y = 6k$ and $y - z = 6\ell$ for some integers k, ℓ so $x - z = (x - y) + (y - z) = 6(k + \ell)$, i.e. $x\,R\,z$.

(4 marks)

(c) R is not an equivalence relation. In this case only property (iii) fails. For example, $0\,R\,\frac{1}{2}$ and $\frac{1}{2}\,R\,1$ but $0\,R\,1$ is false since the difference is 1. (4 marks)

6.

$m|n$ means that there exists an integer k with $n = km$. (1 mark)

(a) $m|0$ since $0 = km$ hold for $k = 0$. On the other hand, if $0|n$ then $n = k \times 0 = 0$ no matter what k is. (2 marks)

(b) Let $a, b \in \mathbb{Z}$ and suppose $2|a$ and $4|b$. Then $a = 2k$, $b = 4\ell$ for integers k, ℓ. Hence $6a + b = 12k + 4\ell = 4(3k + \ell)$ so that $4|(6a + b)$ as required. (3 marks)

The converse is:
Let $a, b \in \mathbb{Z}$. If $4|(6a + b)$ then $2|a$ and $4|b$.
This is false, for example $a = 1, b = 2$ satisfy $4|(6a + b)$ but 2 is not a factor of a. (3 marks)

(c) The contrapositive of the proposition states that:
Let $a, b \in \mathbb{Z}$. If 4 does not divide $6a+b$ then either 2 does not divide a or 4 does not divide b.
Thus given 4 does not divide $6a + b$ and a is even, we know the first of these conclusions is wrong so the second must hold: 4 does not divide b. (3 marks)

(d) Suppose for a contradiction that $7a - 21b = 29$ where $a, b \in \mathbb{Z}$. Then $7(a - 3b) = 29$ from which it follows that $7|29$. However this is false, so we deduce that no such a, b exist. (3 marks)

7.

Context X is a T_2 space. (1 mark)

Hypothesis X is first countable and X is countably compact. (1 mark)

Conclusion X is T_3. (1 mark)

Contrapositive *Let X be a T_1 space. If X is not T_3 then X is not first countable or X is not countably compact.* (2 marks)

(a) Nothing: hypothesis of theorem not fully satisfied. (2 marks)

(b) X is not first countable. From contrapositive, since one half of conclusion is false. (2 marks)

(c) Nothing further can be deduced since the hypothesis and conclusion of the contrapositive are both true. (2 marks)

(d) Nothing: this is not the hypothesis of theorem or contrapositive. (2 marks)

However if the converse of the theorem is true then we can deduce that X is not T_3 since the converse says that if X is T_3 then X is both first countable and countably compact. (2 marks)

8.

a) False. ($x = 0$ gives $x^2 = 0$.) (2 marks)

b) True. (Take $n = -2$.) (3 marks)

c) False. (Take $x = -1$. Then there is no $y \in \mathbb{R}$ with $y^2 = -1$.) (2 marks)

d) True. (Take $m = -1$.) (3 marks)

e) True. (Take n to be any integer $> \frac{1}{\varepsilon^2}$.) (4 marks.)