1. Give the names of the following (lower case) Greek letters: σ, ψ. Write the lower case Greek letters *phi* and *gamma*. [4 marks]

2.

For the following sets S, in (a) and (b), give a function $f(n)$ such that

$$S = \{f(n) \mid n \in \mathbb{N}\}.$$

(Recall that $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.)

(a) $S = \{-1, -7, -13, -19, \ldots\}$.

(b) $S = \{m \in \mathbb{N} \mid m \geq 6 \text{ and } m \text{ is a multiple of } 3\}$.

(c) For the set $S = \{n^3 + 2 \mid n \in \mathbb{Z}\}$, state, with a brief reason, whether each of the following numbers is in S: (i) 2, (ii) -6, (iii) 4, (iv) 10.

[12 marks]

3. Negate each of the following statements:

(a) $x = -4$ or $x > 2$.

(b) For all $a, b, c \in \mathbb{R}$, $ab = ac \Rightarrow b = c$.

(c) If $x > 1$ and $y > 2$ then $f(x) > 0$ and $f(y) > 4$.

(d) $\forall N \in \mathbb{N}, \exists x \in \mathbb{R}, f(x) > N$.

In cases (a) and (b) also indicate, with a reason, whether the given statement is true or false, or has free variables. [13 marks]

4.

Definition: Let f be a (real-valued) function. Then f is *injective* if for all x, y in the domain of f,

$$f(x) = f(y) \implies x = y.$$

(a) Write down what it means for f *not* to be injective.

(b) Prove that the function $f(x) = 2 + 4x, \ x \in \mathbb{R}$ is injective.

(c) Prove that the function $f(x) = x^2 - 2x, \ x \geq 0$ is not injective.

(d) Prove that the function $f(x) = x^2 - 2x, \ x > 1$ is injective.

[13 marks]

5. (a) Write down carefully the meaning of the statement that the set S (of real numbers) is *closed under addition.* Prove from your definition that the set

$$S = \{x \in \mathbb{R} \mid 3x \in \mathbb{Z}\}$$

is closed under addition.

(b) Let R be a relation on a set X. Then R is defined to be an *equivalence relation* if for all $x, y, z \in X$ the following three conditions hold:

$x\,R\,x$.

If $x\,R\,y$ then $y\,R\,x$.

If $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

Determine whether or not the following relations R on the given sets X are equivalence relations. You should justify your answers carefully, working directly from the definitions.

(i) $X = \mathbb{R}$, $x\,R\,y$ means $x < y$.

(ii) $X = \mathbb{Z}$, $x\,R\,y$ means $6|(x-y)$.

(iii) $X = \mathbb{R}$, $x\,R\,y$ means $-1 < x - y < 1$.

[14 marks]

6.

Write down carefully what it means to say that $m|n$ (m divides n) when m and n are integers.

(a) Show from your definition that $m|0$ for every integer m but that $0|n$ only if $n = 0$.

(b) Prove the following proposition:
Let a and b be integers. If $2|a$ and $4|b$ then $4|(6a+b)$.
Also state the converse of this proposition and either prove the converse or give a counterexample.

(c) What, if anything, can you deduce from the proposition in (b) given that a and b are integers such that 4 does not divide $6a+b$ and a is even? Give a clear reason for your answer.

(d) Prove that there do not exist integers a, b with $7a - 21b = 29$.

[15 marks]

7. Consider the following theorem. You are not expected to understand what it means.

Theorem *Let X be a T_2-space. If X is first countable and X is countably compact, then X is T_3.*

Identify the context, hypothesis, and conclusion of the theorem. State its contrapositive.

State, with a brief reason, what, if anything, the theorem tells you about a T_2-space X which is:

(a) first countable;

(b) countably compact but not T_3;

(c) neither first countable nor T_3;

(d) not first countable.

Suppose now that the *converse* of the theorem is true. Does this alter your answer to (d)? Give a reason. [15 marks]

8. Determine whether each of the following statements is true or false, giving a brief reason in each case. Recall that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
In part (e), $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.

(a) $\forall x \in \mathbb{R},\ x^2 > 0$.

(b) $\exists n \in \mathbb{Z},\ n \leq -1$ and $(n-1)^2 = 9$.

(c) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y^2 = x$.

(d) $\exists m \in \mathbb{Z},\ \forall n \in \mathbb{N},\ m < n$.

(e) $\forall \varepsilon \in \mathbb{R}^+,\ \exists n \in \mathbb{N},\ \frac{1}{\sqrt{n}} < \varepsilon$. [14 marks]