All questions are similar to homework problems.

MATH102 Solutions September 2007

Section A

1. The Taylor series of

$$f(x) = x^{-1} = (3 + (x-3))^{-1} = 3^{-1}(1 + (x-3)/3)^{-1}$$

is

$$\frac{1}{3} - \frac{x-3}{9} + \frac{(x-3)^2}{27} - \frac{(x-3)^3}{81} \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{(x-3)^n}{3^{n+1}}.$$

This can also be worked out by computing all derivatives of f at $x = 3$.
[3 marks]
a) When $x = 0$ the series is not convergent.
[1 mark]
b) When $x = 4$ the series is convergent (to $f(4) = \frac{1}{4}$).
[1 mark]
No explanation is required in a) or b).
$5 = 3 + 1 + 1$ marks

2(i) Separating the variables, we have

$$\int e^y dy = \int x^2 dx,$$

$$e^y = \frac{x^3}{3} + C.$$

Putting $x = 1$ and $y = 0$ gives

$$1 = \frac{1}{3} + C$$

or $C = \frac{2}{3}$. So we obtain

$$y = \ln\left(\frac{x^3+2}{3}\right).$$

It is acceptable to leave the answer in the form $e^y = (x^3 + 3)/3$.
2(ii) Using the integrating factor method, putting the equation in standard form, we have

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{3}{x}.$$

The integrating factor is

$$\exp\left(\int (2/x)dx\right) = e^{2\ln x} = x^2.$$

So the equation becomes

$$\frac{d}{dx}(yx^2) = 3x.$$

Integrating gives

$$x^2 y = \int 3x dx = \frac{3}{2}x^2 + C.$$

So the general solution is

$$y = \frac{3}{2} + Cx^{-2}.$$

Putting $y(1) = 0$ gives $C = -\frac{3}{2}$ and

$$y = \frac{3}{2} - \frac{3}{2}x^{-2}.$$

3 marks for (i) 5 marks for (ii).
[8 marks]

3. Try $y = e^r x$. Then

$$r^2 - 4r + 3 = 0 \Rightarrow (r+3)(r+1) = 0 \Rightarrow r = -3 \text{ or } r = -1.$$

So the general solution is

$$y = Ae^{-3x} + Be^{-x}.$$

[2 marks]
So $y' = -3Ae^{-3x} + -Be^{-x}$ and the initial conditions $y(0) = 2$, $y'(0) = 1$ give

$$A + B = 2, \quad -3A - B = 1 \Rightarrow -2A = 3, \ B = 2 - A \Rightarrow A = -\frac{3}{2}, \ B = \frac{7}{2}.$$

So

$$y = -\frac{-3}{2}e^{-3x} + \frac{7}{2}e^{-x}.$$

[3 marks]
[$2 + 3 = 5$ marks]
4. We have

$$\lim_{(x,y)\to(0,0),y=0} \frac{xy + 2x^2}{x^2 + xy + y^2} = \lim_{x\to 0} \frac{2x^2}{x^2} = 2,$$

$$\lim_{(x,y)\to(0,0),x=0} \frac{xy + 2x^2}{x^2 + xy + y^2} = \lim_{x\to 0} \frac{0}{y^2} = 0.$$

So the limits along two different lines as $(x,y) \to (0,0)$ are different, and the overall limit does not exist.
[4 marks]

5.

$$\frac{\partial f}{\partial x} = 12x^2y - 4y^3,$$
$$\frac{\partial f}{\partial y} = 4x^3 - 12xy^2,$$
$$\frac{\partial^2 f}{\partial x^2} = 24xy,$$
$$\frac{\partial^2 f}{\partial y \partial x} = 12x^2 - 12y^2,$$
$$\frac{\partial^2 f}{\partial x \partial y} = 12x^2 - 12y^2$$

so that these last two are equal, and

$$\frac{\partial^2 f}{\partial y^2} = -24xy.$$

So we also have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

as required.[5 marks]

6. We have

$$\frac{\partial f}{\partial x} = -yz\sin(xyz),$$
$$\frac{\partial f}{\partial y} = 2yz - xz\sin(xyz),$$
$$\frac{\partial f}{\partial z} = y^2 - xy\sin(xyz)$$

[3 marks]

By the Chain Rule,

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

So

$$\begin{aligned}\frac{dF}{dt}(0) &= \frac{\partial f}{\partial x}(2,-1,0) + \frac{\partial f}{\partial y}(2,-1,0) - \frac{\partial f}{\partial z}(2,-1,0)\\ &= 0 + 0 - 1 = -1.\end{aligned}$$

[2 marks]
[$3 + 2 = 5$ marks]

7. For

$$f(x,y,z) = \frac{x+y+z}{x^2+y^2+z^2}$$

we have

$$\nabla f(x,y,z) = \left(\frac{1}{x^2+y^2+z^2} - \frac{2x(x+y+z)}{(x^2+y^2+z^2)^2}\right)\mathbf{i}$$

$$+\left(\frac{1}{x^2+y^2+z^2} - \frac{2y(x+y+z)}{(x^2+y^2+z^2)^2}\right)\mathbf{j} + \left(\frac{1}{x^2+y^2+z^2} - \frac{2z(x+y+z)}{(x^2+y^2+z^2)^2}\right)\mathbf{k}.$$

So

$$\nabla f(1,1,1) = \left(\frac{1}{3} - \frac{6}{9}\right)(\mathbf{i}+\mathbf{j}+\mathbf{k}) = -\frac{1}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

3 marks

The derivative of f in the direction $\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ is

$$\frac{\nabla f(1,1,1).(\mathbf{i}-2\mathbf{j}-2\mathbf{k})}{\sqrt{1+(-2)^2+(-2)^2}} = -\frac{1}{3} \times \frac{-3}{3} = \frac{1}{3}.$$

[2 marks]
[$3+2=5$ marks.]

8. For

$$f(x,y) = x^2 + xy^2 - 4xy - 5x,$$

we have

$$\frac{\partial f}{\partial x} = 2x + y^2 - 4y - 5 \quad \frac{\partial f}{\partial y} = 2xy - 4x.$$

[2 marks]

So at a stationary point,

$$2x + y^2 - 4y - 5 = 2x(y-2) = 0$$

$$\Leftrightarrow (x,y) = (9/2, 2) \text{ or } (0,-1) \text{ or } (0,5).$$

[2 marks]

$$A = \frac{\partial^2 f}{\partial x^2} = 2,\ B = \frac{\partial^2 f}{\partial y \partial x} = 2y - 4, C = \frac{\partial^2 f}{\partial y^2} = 2x.$$

For $(x,y) = (9/2,2)$ $A=2$, $B=0$ and $C=9$. So $AC-B^2>0$ $A>0$ and $(9/2,2)$ is a local minimum.

For $(x,y)=(0,-1)$, we have $A=2$, $B=-6$, $C=0$. So $AC-B^2<0$, and $(0,-1)$ is a saddle.

For $(x,y)=(0,5)$, we have $A=2$, $B=6$, $C=0$. So $AC-B^2<0$, and $(0,5)$ is again a saddle.

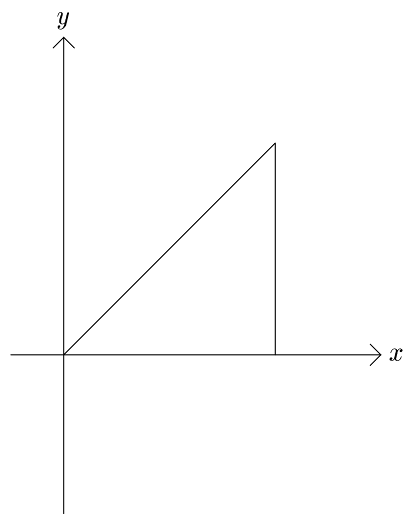
[4 marks]
[$2+2+4=8$ marks]

9. For

$$f(x,y) = \frac{1}{2x^2-y^2},$$

we have
$$\frac{\partial f}{\partial x} = \frac{-4x}{(2x^2-y^2)^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{(2x^2-y^2)^2}.$$
So
$$f(1,1) = 1, \quad \frac{\partial f}{\partial x}(1,1) = -4, \quad \frac{\partial f}{\partial y}(1,1) = 2.$$
So the linear approximation is
$$1 - 4(x-1) + 2(y-1).$$
[It would be acceptable to realise that
$$\begin{aligned} f(x,y) &= (1 + 4(x-1) + 2(x-1)^2 - 2(y-1) - (y-1)^2)^{-1} \\ &= (1 + 4(x-1) - 2(y-1) + 2(x-1)^2 - (y-1)^2)^{-1} \end{aligned}$$
and to expand out.]
[4 marks]

10. The domain of integration is the triangle as shown



This integral can be written as $\int_0^1 \int_0^x dydx$ or $\int_0^1 \int_y^1 dxdy$. So we have
$$\begin{aligned} \int_0^1 \int_y^1 \cos(y/x)dxdy &= \int_0^1 \int_0^x \cos(y/x)dxdy \\ &= \int_0^1 [x\sin(y/x)]_{y=0}^{y=x} dx = \int_0^1 x(\sin 1)dx \\ &= \left[(\sin 1)\frac{x^2}{2}\right]_0^1 = \frac{\sin 1}{2}. \end{aligned}$$
[6 marks]

Section B

11. (i)

$$f'(y) = \frac{1}{2}(1-y)^{-3/2}, \quad f''(y) = \frac{3}{4}(1-y)^{-5/2}, \quad f^{(3)}(y) = \frac{15}{8}(1-y)^{-7/2},$$

$$f^{(4)}(y) = \frac{105}{16}(1-y)^{-9/2}, \quad f^{(5)}(y) = \frac{945}{32}(1-y)^{-11/2}.$$

[4 marks] So

$$f'(0) = \frac{1}{2}, \quad \frac{f''(0)}{2!} = \frac{3}{8}, \quad \frac{f^{(3)}(0)}{3!} = \frac{15}{48} = \frac{5}{16}, \quad \frac{f^{(4)}(0)}{24} = \frac{35}{128}.$$

So the fourth Taylor polynomial of f at 0 is

$$1 + \frac{1}{2}y + \frac{3}{8}y^2 + \frac{5}{16}y^3 + \frac{35}{128}y^4.$$

[4 marks

Putting $y = x^2$, the ninth Taylor polynomial of g at 0 has only even terms, and so is the same as the eighth Taylor polynomial, and is obtained by putting $y = x^2$ in the above, so that it is

$$1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8.$$

[2 marks]

Integrating, the tenth Taylor polynomial of $h(x) = \sin^{-1}(x)$ at 0 is

$$x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9.$$

[2 marks]
(ii) We have Now

$$R_4(y,0) = \frac{f^{(5)}(c)}{5!}y^{n+1} = \frac{315}{1280}(1-c)^{-11/2}y^{n+1}$$

for some c between 0 and y. Since $c \leq \frac{1}{4}$, $1 - c \geq \frac{3}{4}$ and $(1-c)^{-11/2} \leq (4/3)^{11/2}$. So

$$|R_n(y,0) \leq \frac{945}{5! \times 32}\frac{4^{11/2}}{3^{11/2}}y^5 = \frac{63}{256}\frac{4^{11/2}}{3^{11/2}}y^5$$

[3 marks]
[$4 + 4 + 2 + 2 + 3 = 15$ marks]

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 + 4r - 5 = (r+5)(r-1) = 0,$$

that is, $r = -5$ or 1. So the complementary solution is $Ae^{-5x} + Be^{x}$.
[3 marks]
(i) We try $y_p = Ce^{-x}$. Then $y_p' = -Ce^{-x}$ and $y_p'' = Ce^{-x}$. So $y_p'' + 4y_p' - 5y_p = -8C$. So $C = -\frac{1}{2}$ So the general solution is

$$y = Ae^{-5x} + Be^{x} - \frac{1}{2}e^{x}.$$

[2 marks]
This gives

$$y' = -5Ae^{5x} + Be^{x} + frac12e^{x}.$$

So putting $x = 0$, the boundary conditions give

$$A+B-\frac{1}{2} = 1, \quad -5A+B+\frac{1}{2} = -1 \quad \Rightarrow \quad 6A = 3, \quad B = \frac{3}{2}-A \Rightarrow A = \frac{1}{2}, \quad B = 1.$$

So the solution is

$$y = \frac{1}{2}e^{-5x} + e^{x} - \frac{1}{2}e^{-x}.$$

[3 marks]
(ii) We try $y_p = Cx^2 + Dx + E$. Then $y_p'(x) = 2Cx + D$ and $y_p'' = 2C$. So

$$y_p'' + 4y_p' - 5y_p = (2C + 4D - 5E) + x(8C - 5D) - 5Cx^2 = -5x^2 + 3x + 1.$$

Comparing coefficients, we obtain

$$-5C = -5, \quad 8C - 5D = 3, \quad 2C + 4D - 5E = 1.$$

So

$$C = 1, \quad D = 1, \quad E = 1$$

So the general solution is

$$Ae^{-5x} + Be^{x} + x^2 + x + 1.$$

[4 marks] This gives

$$y'(x) = -5Ae^{-5x} + Be^{x} + 2x + 1.$$

So putting $x = 0$, the boundary conditions giveB=

$$A+B+1 = 1, \quad -5A+B+1 = -1 \Rightarrow A = -B, \quad 6A = -2 \Rightarrow A = -\frac{1}{3}, \quad B = \frac{1}{3}.$$

So

$$y = -\frac{1}{3}e^{-5x} + \frac{1}{3}e^{x} + x^2 + x + 1.$$

[3 marks]
[$3 + 2 + 3 + 4 + 3 = 15$ marks]

13a)

We have

$$\nabla f = y\mathbf{i} + (x-1)\mathbf{j}$$
$$\nabla g = 2x\mathbf{i} + 6y\mathbf{j}.$$

[2 marks]

At a stationary point of f we have

$$y = x - 1 = 0 \Rightarrow (x, y) = (1, 0).$$

This is in the set where $g(x, y) < 3$. (The point is easily seen to be a saddle and so cannot be a maximum of minimum of f on the set where $g \leq 6$, but we shall not use this.)

[2 marks]

At a stationary point of f on $g = 3$, we have $\nabla f = \lambda g$, that is,

$$y = 2x\lambda, \quad x - 1 = 6y\lambda \Rightarrow 3y^2 - x(x-1) = 0.$$

On $g = 3$, we have $3y^2 = 3 - x^2$, so

$$-2x^2 + x + 3 = (3 - 2x)(1 + x) = 0.$$

So $x = -1$ or $x = \frac{3}{2}$. So the stationary points of f restricted to $g = 3$ are

$$(-1, \pm\sqrt{2/3}), \quad \left(\frac{3}{2}, \pm\frac{1}{2}\right).$$

[6 marks]

Now we check the values of f at all these points. We have

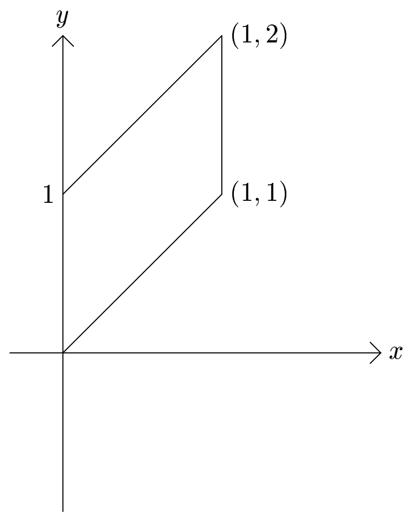
$$f(1,0) = 0, \quad f(-1, \sqrt{2/3}) = -2\sqrt{2/3}, \quad f(-1, -\sqrt{2/3}) = 2\sqrt{2/3},$$
$$f\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{1}{4}, \quad f\left(\frac{3}{2}, -\frac{1}{2}\right) = -\frac{1}{4}.$$

So the minimum value is $-2\sqrt{2/3}$ achieved at $(-1, \sqrt{2/3})$ and the maximum value is $2\sqrt{2/3}$, achieved at $(-1, -\sqrt{2/3})$.

[5 marks]

[$2 + 2 + 6 + 5 = 15$ marks.]

14a).The region R is as shown.



The weight W is

$$\int_0^1 \int_x^{x+1} xdydx = \int_0^1 xdx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

[5 marks]

14b) Then

$$\overline{x} = \frac{1}{W} \int_0^1 \int_x^{x+1} x^2 dydx$$

$$= 2\int_0^1 x^2 dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}.$$

[5 marks]

$$\overline{y} = \frac{1}{W} \int_0^1 \int_x^{x+1} xydydx$$

$$= 2\int_0^1 x\left[\frac{y^2}{2}\right]_x^{x+1} dx = \int_0^1 (2x^2 + x)dx$$

$$= \left[\frac{2x^3}{3} + \frac{x}{2}\right]_0^1 = \frac{7}{6}.$$

So

$$(\overline{x}, \overline{y}) = \left(\frac{2}{3}, \frac{7}{6}\right).$$

[5 marks]

[$3 \times 5 = 15$ marks.]