All questions are similar to homework problems.

MATH102 Solutions September 2006
Section A

1. The Taylor series of $f(x) = x^{-1} = (2 + (x-2))^{-1} = 2^{-1}(1 + (x-2)/2)^{-1}$ is

$$2^{-1} - 2^{-2}(x-2) + 2^{-3}(x-2)^2 - \cdots = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1}(x-2)^n.$$

This can also be worked out by computing all derivatives of $f$ at $x = 2$.
[3 marks]
a) When $x = 3$ the series is convergent and equal to $f(3) = \frac{1}{3}$.
[1 mark]
b) When $x = 4$ the series is not convergent to $f(4) = \frac{1}{4}$. In fact, the series is not convergent.
[1 mark]
No explanation is required in a) or b).
$5 = 3 + 1 + 1$ marks

2(i) Separating the variables, we have

$$\int y dy = -\int e^x dx,$$

$$\frac{y^2}{2} = e^x + C,$$

or

$$y = \pm\sqrt{2C + 2e^x}.$$

Putting $y(0) = 1$ gives $2C = -1$ and $y = +\sqrt{2e^x - 1}$.
2(ii) Using the integrating factor method, and the integrating factor is

$$\exp\left(\int dx\right) = e^x.$$

So the equation becomes

$$\frac{d}{dx}(ye^x) = e^{2x}.$$

Integrating gives

$$ye^x = \frac{1}{2}e^{2x} + C.$$

So the general solution is

$$y = \frac{1}{2}e^x + Ce^{-x}.$$

Putting $y(0) = 2$ gives $C = \frac{3}{2}$ and $y = \frac{1}{2}e^x + \frac{3}{2}e^{-x}$.
3 marks for (i) 4 marks for (ii).

[7 marks]

3. Try $y = e^r x$. Then

$$r^2 + 4r + 3 = 0 \Rightarrow (r+1)(r+3) = 0 \Rightarrow r = -1 \text{ or } r = -3.$$

So the general solution is

$$y = Ae^{-x} + Be^{-3x}.$$

[2 marks]
So $y' = -Ae^{-x} - 3Be^{-3x}$ and the initial conditions $y(0) = 1$, $y'(0) = 2$ give

$$A + B = 1, \quad -A - 3B = 2 \to -2B = 3,\ A = 1 - B \Rightarrow B = -\frac{3}{2},\ A = \frac{5}{2}.$$

So

$$y = \frac{5}{2}e^{-x} - \frac{3}{2}e^{-3x}.$$

[3 marks]
$2 + 3 = 5$ marks
4. We have

$$\lim_{(x,y)\to(0,0),y=0} \frac{x^4}{x^4 + y^4} = \lim_{x\to 0} \frac{x^4}{x^4} = 1,$$

$$\lim_{(x,y)\to(0,0),x=0} \frac{x^4}{x^4 + y^4} = \lim_{y\to 0} \frac{0}{y^4} = 0.$$

So the limits along two different lines as $(x, y) \to (0, 0)$ are different, and the overall limit does not exist.
[4 marks]

5.

$$\frac{\partial f}{\partial x} = -\frac{2x}{(x^2+y)^2}, \quad \frac{\partial f}{\partial y} = -\frac{1}{(x^2+y)^2},$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{2}{(x^2+y)^2} + \frac{8x^2}{(x^2+y)^3},$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{4x}{(x^2+y)^3},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{4x}{(x^2+y)^3}$$

so that these last two are equal, and

$$\frac{\partial^2 f}{\partial y^2} = \frac{2}{(x^2+y)^3}.$$

[4 marks]

6. We have

$$\frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 1,\ \frac{\partial v}{\partial x} = 1,\ \frac{\partial v}{\partial y} = -1.$$

By the Chain Rule,

$$\begin{aligned}\frac{\partial g}{\partial x}(x,y) &= \frac{\partial f}{\partial u}(u,v)\frac{\partial u}{\partial x}(x,y) + \frac{\partial f}{\partial v}(u,v)\frac{\partial v}{\partial x}(x,y)\\ &= \frac{\partial f}{\partial u}(u,v) + \frac{\partial f}{\partial v}(u,v).\end{aligned}$$

Similarly,

$$\frac{\partial g}{\partial y}(x,y) = \frac{\partial f}{\partial u}(u,v) - \frac{\partial f}{\partial v}(u,v).$$

Similarly,

$$\begin{aligned}\frac{\partial^2 g}{\partial x^2}(x,y) &= \left(\frac{\partial^2 f}{\partial u^2}(u,v) + \frac{\partial^2 f}{\partial u \partial v}(u,v)\right) + \left(\frac{\partial^2 f}{\partial v \partial u}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v)\right)\\ &= \frac{\partial^2 f}{\partial u^2}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v) + 2\frac{\partial^2 f}{\partial v \partial u}(u,v).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^2 g}{\partial y^2}(x,y) &= \left(\frac{\partial^2 f}{\partial u^2}(u,v) - \frac{\partial^2 f}{\partial u \partial v}(u,v)\right) - \left(\frac{\partial^2 f}{\partial v \partial u}(u,v) - \frac{\partial^2 f}{\partial v^2}(u,v)\right)\\ &= \frac{\partial^2 f}{\partial u^2}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v) - 2\frac{\partial^2 f}{\partial u \partial v}(u,v).\end{aligned}$$

Adding, we obtain

$$\frac{\partial^2 g}{\partial x^2}(x,y) + \frac{\partial^2 g}{\partial y^2}(x,y) = 2\left(\frac{\partial^2 f}{\partial u^2}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v)\right).$$

[6 marks]

7. For

$$f(x,y,z) = x^2yz$$

we have

$$\nabla f(x,y,z) = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}.$$

So

$$\nabla f(2,1,1) = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}.$$

3 marks

The tangent plane at $(2,1,1)$ is

$$\nabla f(2,1,1).((x-2)\mathbf{i} + (y-1)\mathbf{j} + (z-1)\mathbf{k} = 0,$$

or

$$4(x-2)+4(y-1)+4(z-1)=0,$$

or

$$x+y+z-4=0.$$

[2 marks]
[$3+2=5$ marks.]

8. For

$$f(x,y)=x^2y-2yx+2y^2-3y,$$

we have

$$\frac{\partial f}{\partial x}=2xy-2y, \quad \frac{\partial f}{\partial y}=x^2-2x+4y-3.$$

[2 marks]

So at a stationary point,

$$y(x-1)=0=x^2-2x+4y-3 \Leftrightarrow y=0=(x+1)(x-3) \text{ or } (x-1=0=4(y-1)$$

$$\Leftrightarrow (x,y)=(-1,0) \text{ or } (3,0) \text{ or } (1,1).$$

[2 marks]

$$A=\frac{\partial^2 f}{\partial x^2}=2y, \ B=\frac{\partial^2 f}{\partial y \partial x}=2x-2, C=\frac{\partial^2 f}{\partial y^2}=4.$$

For $(x,y)=(-1,0)$ or (3,0) we have $A=0$ and $B\neq 0$. So $AC-B^2<0$ and these points are saddle points.

For $(x,y)=(1,1)$, we have $A=2$, $B=0$, $C=4$. So $A>0$, $AC-B^2>0$ and $(1,-1)$ is a minimum
[4 marks]
[$2+2+4=8$ marks]

9. For

$$f(x,y)=\frac{1}{x^2-y^2},$$

we have

$$\frac{\partial f}{\partial x}=-\frac{2x}{(x^2-y^2)^2}, \quad \frac{\partial f}{\partial y}=\frac{2y}{(x^2-y^2}.$$

So

$$f(2,1)=\frac{1}{3}, \quad \frac{\partial f}{\partial x}(2,1)=-\frac{4}{9}, \quad \frac{\partial f}{\partial y}(2,1)=\frac{2}{9}.$$

So the linear approximation is

$$\frac{1}{3}-\frac{4}{9}(x-2)+\frac{2}{9}(y-1).$$

[It would be acceptable to realise that

$$f(x,y) = (3 + 4(x-2) + 4(x-2)^2 - 2(y-1) - (y-1)^2)^{-1}$$
$$= \frac{1}{3}(1 + \frac{4}{3}(x-1) - \frac{2}{3}(y-1) + \frac{4}{3}(x-2)^2 - \frac{1}{3}(y-1)^2)^{-1}$$

and to expand out.]
[4 marks]

10. In polar coordinates $(r,\theta)$, $D$ is the set where $r \leq 1$ and $0 \leq \theta \leq 2\pi$ (by choice of argument). Also, $x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$. So

$$\int\int_D (x^2+y^2)^{3/2} dxdy = \int_0^{2\pi} \int_0^1 r^4 drd\theta = \int_0^{2\pi} \left[\frac{1}{5}r^5\right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi} \frac{1}{5} d\theta = \frac{2\pi}{5}.$$

[6 marks]

Section B

11.
(i)a) $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f^{(3)}(x) = \sin x$, $f^{(4)}(x) = \cos x$. So $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f^{(3)}(0) = 0$, and

$$P_3(x,0) = 1 - \frac{x^2}{2}, \quad R_3(x,0) = \frac{\cos c}{4!}x^4$$

for some $c$ between 0 and $x$.
[3 marks.]
We have $|cosc| \leq 1$ for all $c$. So

$$|R_3(x,0)| \leq \frac{x^4}{24}.$$

[3 marks.]
(i)b) For the same function $f(x) = \cos x$, we have $f(\pi) = -1$, $f'(\pi) = 0$, $f''(\pi) = 1$, $f^{(3)}(\pi) = 0$. So

$$P_3(x,\pi) = -1 + \frac{(x-\pi)^2}{2}, \quad R_3(x,\pi) = \frac{\cos c}{4!}(x-\pi)^4$$

for some $c$ between $\pi$ and $x$.
[3 marks.]
(i)c) We have $f(2\pi) = f(0) = 1$, $f'(2\pi) = 0$, $f''(2\pi) = -1$, $f^{(3)}(2\pi) = 0$, and

$$P_3(x,2\pi) = 1 - \frac{(x-2\pi)^2}{2}, \quad R_3(x,2\pi) = \frac{\cos c}{4!}(x-2\pi)^4$$

for some $c$ between $2\pi$ and $x$.
[3 marks]
(ii)a)

$$y^2 = 0.001 + 2(\cos x - 1).$$

For $P_3(x,0)$ and $R_3(x,0)$ as in (i)a), we have

$$y^2 = 0.001 - 2 + 2(P_3(x,0) + R_3(x,0)) = 0.001 - x^2 + 2R_3(x,0).$$

So

$$|0.001 - x^2 - y^2| \le \frac{x^4}{12}.$$

[3 marks.]
[3 + 3 + 3 + 3 + 3 = 15 marks.]

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 + 2r - 15 = (r+5)(r-3) = 0,$$

that is, $r = 3$ or $-5$. So the complementary solution is $Ae^{3x} + Be^{-5x}$.
[3 marks]
(i) We try $y_p = Cx + D$. Then $y_p' = C$ and $y_p'' = 0$. So $y_p'' + 2y_p' - 15y_p = 2C - 15Cx - 15D$. So $C = -\frac{1}{5}$ and $-15D - \frac{2}{5} = 2$ and $D = -\frac{12}{75} = -\frac{4}{25}$. So the general solution is

$$y = Ae^{3x} + Be^{-5x} - \frac{1}{5}x - \frac{4}{25}.$$

[3 marks]
This gives

$$y' = 3Ae^{3x} - 5Be^{-5x} - \frac{1}{5}.$$

So putting $x = 0$, the boundary conditions give

$$A + B - \frac{4}{25} = 1,\ 3A - 5B - \frac{1}{5} = 2\ \Rightarrow\ 8A - \frac{4}{5} - \frac{1}{5} = 7,\ B = \frac{29}{25} - A \Rightarrow A = 1,\ B = \frac{4}{25}.$$

So the solution is

$$y = e^{3x} + \frac{4}{25}e^{-5x} - \frac{1}{5}x - \frac{4}{25}.$$

[3 marks]
(ii) We try $y_p = C\sin x + D\cos x$. Then $y_p'(x) = C\cos x - D\sin x$ and $y_p'' = -C\sin x - D\cos x$. So

$$y_p'' + 2y_p' - 15y_p = (-16C - 2D)\sin x + (2C - 16D)\cos x = 5\sin x.$$

So $C = 8D$ and $-130D = 13$ So the general solution is

$$y = Ae^{-x} + Be^{-3x} - \frac{4}{5}\sin x - \frac{1}{10}\cos x.$$

[3 marks] This gives

$$y'(x) = 3Ae^{3x} - 5Be^{-5x} - \frac{4}{5}\cos x + \frac{1}{10}\sin x$$

So putting $x = 0$, the boundary conditions give

$$A+B-\frac{1}{10} = 1,\ 3A-5B-\frac{4}{5} = 2 \Rightarrow 8A-\frac{13}{10} = 7,\ B = \frac{11}{10}-A \Rightarrow A = \frac{83}{80},\ B = \frac{1}{16}.$$

So

$$y = \frac{83}{80}e^{3x} + \frac{1}{16}e^{-5x} - \frac{4}{5}\sin x - \frac{1}{10}\cos x.$$

[3 marks]
[$5 \times 3 = 15$ marks]

13a)

We have

$$\nabla f = 4x\mathbf{i} + 2y\mathbf{j}$$
$$\nabla g = (4-2x)\mathbf{i} - \mathbf{j}.$$

[2 marks]

The only stationary point of $f$ is where $4x = 2y = 0$, that is, $(x, y) = (0, 0)$. This is on the boundary of the region, which we call $R$. So any maximum or minimum of $f$ must be on the boundary
[3 marks]

To find maxima and minima on the parabola part of the boundary $y = 4x - x^2$, apart from the endpoints, we can either use Lagrange multipliers or straight substutution. Both methods give similar calculations. Using Lagrange multipliers, we consider the equation

$$\nabla f = \lambda \nabla g,$$

that is,

$$4x = \lambda(4-2x),\ 2y = -\lambda.$$

So

$$4x = -2y(4-2x) \Rightarrow x = -y(2-x) = x(x-4)(2-x).$$

So $x = 0$ or

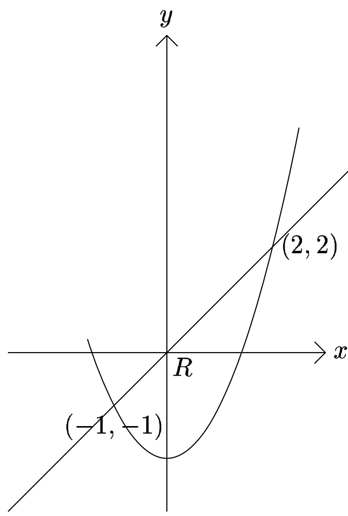
$$1 = -8 + 6x - x^2 \Rightarrow (x-3)^2 = 0.$$

So $x = 3$ and $y = 12 - 9 = 3$. Then $f(3, 3) = 27$.
[6 marks.]

On the straight part of the boundary, $f(x, y) = f(x, 0) = 2x^2$, which is largest at the endpoint $(4, 0)$ and smallest at the endpont $(0, 0)$. We have $f(0, 0) = 0$ and $f(4, 0) = 32$. So the maximum and minimum of $f$ over the whole region are 32 at $(4, 0)$ and 0 at $(0, 0)$.

[4 marks]
$2+3+6+4=15$ marks.

14a). The line $y=x$ meets the parabola $y=x^2-2$ when $x^2-x-2=0$, that is, $(x+1)(x-2)=0$. When $x=-1$ then $y=-1$ and when $x=2$ $y=2$ The parabola is to the left of the line. The region $R$ is as shown.




[3 marks]
The area $A$ is

$$\int_{-1}^{2}\int_{x^2-2}^{x} dydx = \int_{-1}^{2}(x+2-x^2)dx$$

$$= \left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_1^2 = 2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}=\frac{9}{2}$$

[4 marks]
14b) Then

$$\overline{x} = \frac{1}{A}\int_{-1}^{2}\int_{x^2-2}^{x} xdydx$$

$$= \frac{2}{9}\int_{-1}^{2}(x^2+2x-x^3)dx = \frac{2}{9}\left[\frac{x^3}{3}+x^2-\frac{x^4}{4}\right]_{-1}^{2}$$

$$= \frac{2}{9}\left(\frac{8}{3}-4+4+\frac{1}{3}-1+\frac{1}{4}\right) = \frac{2}{9}.\frac{9}{4} = \frac{1}{2}.$$

$$\overline{y} = \frac{1}{A}\int_{-1}^{2}\int_{x^2-2}^{x} ydydx$$

$$= \frac{2}{9}\int_{-1}^{2}\left[\frac{y^2}{2}\right]_{x^2-2}^{x} dx = \frac{1}{9}\int_{-1}^{2}(x^2+4x^2-4-x^4)dx$$

$$= \frac{1}{9}\left[\frac{5x^3}{3} - 4x - \frac{x^5}{5}\right]_{-1}^{2} = \frac{1}{9}\left(\frac{40}{3} - 8 - \frac{32}{5} + \frac{1}{3} - 4 + \frac{1}{5}\right)$$

$$= \frac{1}{9}.\frac{-18}{5} = -\frac{2}{5}$$

So

$$(\overline{xy}) = \left(\frac{1}{2}, -\frac{2}{5}\right).$$

[3 + 4 + 3 + 5 = 15 marks.]