MATH102 Solutions September 2005
Section A

1. The Taylor series of $f(x) = e^{2x}$ is

$$1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

This can be written down using the Taylor series of e^y at 0 with $y = 2x$, and can also be worked out directly by computing all derivatives of f at $x = 0$.
[3 marks]
a) When $x = 1$ the series is convergent and equal to $f(1)$.
[1 mark]
b) When $x = 100$ the series is also convergent and equal to $f(100)$. In fact the Taylor series is convergent nd equal to $f(x) = e^{2x}$ for all x
[1 mark]

No explanation is required in a) or b).

$5 = 3 + 1 + 1$ marks

2 Separating the variables, we have

$$\int y dy = -\int x dx,$$

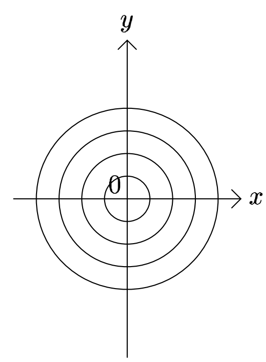
$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C,$$

or

$$y^2 + x^2 = 2C.$$

[2 marks]

The solution curves are circles centred on the origin of radius $\sqrt{2C}$, for any $C > 0$, as shown.



[3 marks]

For a solution with $y(1) = 1$ we have $1 + 1 = 2C$, that is $2C = 2$. So the solution satisfies $x2 + y^2 = 2$. For y as a function of x, we have $y = \sqrt{2 - x^2}$, taking positive square root, since $y(1) = 1 > 0$.

[1 mark]
[$2+3+1=6$ marks]

3. Try $y = e^r x$. Then

$$r^2 - 4 = 0 \Rightarrow r = \pm 2.$$

So the general solution is

$$y = Ae^{2x} + Be^{-2x}.$$

[2 marks]
So $y' = 2Ae^{2x} - 2Be^{-2x}$ and the initial conditions $y(0) = 1$, $y'(0) = -1$ give

$$A + B = 1, \quad 2A - 2B = -1 \to 4A = 1, \ A = 1 - B \Rightarrow A = \frac{1}{4}, B = \frac{3}{4}.$$

So

$$y = \frac{1}{4}e^{2x} + \frac{3}{4}e^{-2x}.$$

[3 marks]
$2+3=5$ marks

4. We take limits along the lines $y = 0$ and $x = 0$. We have:

$$\lim_{y=0, x\to 0} \frac{x^2 + y^2}{x^2 + 2y^2} = \lim_{x \to 0} \frac{x^2}{x^2} = 1,$$

$$\lim_{x=0, y\to 0} \frac{x^2 + y^2}{x^2 + 2y^2} = \lim_{x \to 0} \frac{y^2}{2y^2} = \frac{1}{2}.$$

So we get different limits along different lines of approach, and the overall limit does not exist.
[4 marks]

5.

$$\frac{\partial f}{\partial x} = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial f}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}.$$

[2 marks]

$$\frac{\partial^2 f}{\partial x^2} = \frac{-2x}{(x^2 + y^2)^2} + \frac{4x(y^2 - x^2)}{(x^2 + y^2)^3}$$

$$= \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2x}{(x^2 + y^2)^2} + \frac{8xy^2}{(x^2 + y^2)^3}$$

$$= \frac{-2x^3 + 6xy^2}{(x^2+y^2)^3}.$$

So

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

as required.
[4 marks]
[$2 + 4 = 6$ marks.]

6. By the Chain Rule,

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial F}{\partial u}(u,v) + \frac{\partial F}{\partial v}(u,v),$$
$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial F}{\partial u}(u,v) - \frac{\partial F}{\partial v}(u,v),$$

So

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 F}{\partial u^2}(u,v) + \frac{\partial^2 F}{\partial v \partial u}(u,v) + \frac{\partial^2 F}{\partial u \partial v}(u,v) + \frac{\partial^2 F}{\partial v^2}(u,v),$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 F}{\partial u^2}(u,v) - \frac{\partial^2 F}{\partial v \partial u}(u,v) - \frac{\partial^2 F}{\partial u \partial v}(u,v) + \frac{\partial^2 F}{\partial v^2}(u,v),$$

and so

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 2\left(\frac{\partial^2 F}{\partial u^2}(u,v) + \frac{\partial^2 F}{\partial v^2}(u,v)\right),$$

as required.
[6 marks]

7. For

$$f(x,y,z) = xy^2 + yz^2 - 2xyz,$$

we have

$$\nabla f(x,y,z) = (y^2 - 2yz)\mathbf{i} + (2xy + z^2 - 2xz)\mathbf{j} + (2yz - 2xy)\mathbf{k}$$

[2 marks] So

$$\nabla f(1,2,1) = 3\mathbf{j}.$$

The tangent plane at (1,2,1) is

$$3(y-2) = 3y - 6 = 0.$$

[3 marks]
[$3 + 2 = 5$ marks.]

8. For

$$f(x,y) = 2x^2 - 2x^2y + y^2,$$

we have

$$\frac{\partial f}{\partial x} = 4x - 4xy, \quad \frac{\partial f}{\partial y} = -2x^2 + 2y.$$

[2 marks]

So at a stationary point,

$$x(1-y) = 0 = 2x^2+2y \Rightarrow x = y-0 \text{ or } (y = 1 \text{ and } x^2 = 1) \Rightarrow (x, y) = (0, 0) \text{ or } (1, 1) \text{ or } (-1, 1).$$

[2 marks]

$$A = \frac{\partial^2 f}{\partial x^2} = 4 - 4y, \ B = \frac{\partial^2 f}{\partial y \partial x} = -4x, C = \frac{\partial^2 f}{\partial y^2} = 2.$$

For $(x, y) = (0, 0)$ we have $A = 4 > 0$, $B = 0$, $C = 2$ and $AC - B^2 = 8 > 0$. So $(0, 0)$ is a minimum.

For $(x, y) = (\pm 1, 1))$ $A = 0$, $B = \pm 4$, $C = 2$. So $AC - B^2 = -16 < 0$, and both these points are saddles.

[4 marks]

[$2 + 2 + 4 = 8$ marks]

9. For

$$f(x, y) = \frac{1}{x - y^2},$$

we have

$$\frac{\partial f}{\partial x} = \frac{-1}{(x - y^2)^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{(x - y^2)^2}.$$

So

$$f(2, 1) = 1, \quad \frac{\partial f}{\partial x}(2, 1) = -1, \quad \frac{\partial f}{\partial y}(2, 1) = 2.$$

So the linear approximation is

$$1 - (x - 2) + 2(y - 1).$$

[It would be acceptable to realise that

$$f(x, y) = (1 + (x - 2) - 2(y - 1) - (y - 1)^2)^{-1}$$
$$= (1 + (x - 2) - 2(y - 1) - (y - 1)^2)^{-1}$$

and to expand out.]

[4 marks]

10.

$$\int\int_R x^2 dy dx = \int_{-1}^{1} \int_{x^2-1}^{0} x^2 dy dx$$
$$= \int_{-1}^{1} \left[x^2 y\right]_{x^2-1}^{0} dx = \int_{-1}^{1} (x^2 - x^4) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = 2\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4}{15}.$$

[6 marks]

Section B

11. (i) We have $f^{(2k)}(x) = (-1)^k \sin x$ and $f^{(2k+1)}(x) = (-1)^k \cos x$. So $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$. So:
a) $P_4(x) = x - \frac{1}{6}x^3$ and $R_4(x) = \frac{1}{5!}x^5 \cos c$ for some c between 0 and x.
b) For any $k \geq 1$

$$P_{2k}(x) = \sum_{r=0}^{k-1} (-1)^r \frac{x^{2r+1}}{(2r+1)!}, \quad R_{2k}(x) = (-1)^k \frac{x^{2k+1}}{(2k+1)!} \cos c$$

for some c between 0 and x.
[6 marks]

Since $|\cos c| \leq 1$ for all c, and $\cos c \geq 0$ for $c \in [0, \frac{\pi}{2}]$ we have

$$0 \leq R_4(x) \leq \frac{x^5}{(5)!} \cos c \leq \frac{x^5}{120}.$$

So, since $\sin x = P_4(x) + R_4(x)$,

$$x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}.$$

If in addition $0 \leq x \leq 1$ then $x^5 \leq x^3$ and so

$$x - \frac{x^3}{6} \leq \sin x \leq x - \frac{19x^3}{120}.$$

[3 marks]
(ii) Using the Taylor series of $\sin x$ at 0:
a)

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{6} + \frac{x^5}{120} \cdots\right)}{x^3} = \lim_{x \to 0} \frac{\frac{x^3}{6} - \frac{x^5}{120} \cdots}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{1}{6} - \frac{x^2}{120} \cdots}{1} = \frac{1}{6};$$

[3 marks]
b)

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x \to 0} \frac{\sin x - x}{x \sin x}$$
$$= \lim_{x \to 0} \frac{-\frac{x^3}{6} + \cdots}{x^2 - \frac{x^4}{6} \cdots} = \lim_{x \to 0} \frac{-\frac{x}{6} \cdots}{1 - \frac{x^2}{6} \cdots} = 0.$$

[3 marks]
[$6+3+3+3=15$ marks.]

12. For the complementary solution in both cases, if we try $y=e^{rx}$ we need

$$r^2+1=0,$$

that is, $r=\pm i$. So $e^{\pm ix}$ is a complementary solution and in general form the complementary solution can be written $A\cos x+B\sin x$.
[3 marks]
(i). We try We try $y_p=Cx+D$. Then $y_p'(x)=C$ and $y_p''=0$. So $y_p''+y_p=Cx+D=x$. So $C=1$ and $D=0$ So the general solution is

$$y=A\cos x+B\sin +x$$

[3 marks]
This gives

$$y'(x)=-A\sin x+B\cos x+1$$

So $y(0)=1$, $y'(0)=-1$ give

$$A=1,\quad B+1=-1,$$

so $B=-2$ and $A=1$ and

$$y=\cos x-2\sin x+x.$$

[3 marks]
(ii) We try $y_p=Ce^{-x}$. Then $y_p'=-Ce^{-x}$ and $y_p''=Ce^{-x}$. So $y_p''+y_p=2Ce^{-x}$. So $C=1$. So the general solution is

$$y=A\cos x+B\sin x+e^{-x}$$

[3 marks]
This gives

$$y'=A\sin x+B\cos x-e^{-x}.$$

So putting $x=0$ the boundary conditions give

$$A+1=1,\quad B-1=-1\quad\Rightarrow\quad A=B=0.$$

So the solution is

$$e^{-x}$$

[3 marks]
[$5\times 3=15$ marks]
13. For

$$f(x,y)=x^2+3x^2y-y^2,\quad g(x,y)=2x^2+y^2,$$

$$\nabla f=(2x+6xy)\mathbf{i}+(3x^2-2y)\mathbf{j}$$

$$\nabla g = 4x\mathbf{i} + 2y\mathbf{j}.$$

[2 marks]

At a stationary point of f, we have

$$2x(1+3y) = 3x^2 - 2y = 0.$$

So $x = 0$ or $y = -\frac{1}{3}$. If $x = 0$ then $y = 0$. If $y = -\frac{1}{3}$ then $x^2 = -\frac{2}{9}$, so thera rae no solutions. So the only stationary point in $g < 1$ is $(0,0)$.

[4 marks]

At a maximum or minimum on the set where $g = 1$, we must have $\nabla f = \lambda \nabla g$. that is

$$2x + 6xy = 4\lambda x, \quad 3x^2 - 2y = 2\lambda y.$$

[1 mark]

The first equation gives $x = 0$ or $1 + 3y = 2\lambda$. If $x = 0$ then the equation $g = 1$ gives $y^2 = 1$ and $y = \pm 1$.

If $2\lambda = 1 + 3y$, then plugging into the second equation gives

$$3x^2 - 2y = y(1+3y)$$

So multiplying by 2 and replacing $2x^2$ by $1 - y^2$ gives

$$3 - 3y^2 - 4y = 2y + 6y^2.$$

So

$$9y^2 + 6y - 3 = 3(3y-1)(y+1) = 0.$$

So $y = -1$ or $y = \frac{1}{3}$ and using $g = 1$ gives

$$(x,y) = (0,-1) \text{ or } \left(\pm\frac{2}{3}, \frac{1}{3}\right).$$

So altogether the points on $g = 1$ which can be maxima or minima of f on $g \le 1$ are

$$(0,0) \text{ or } (0,-1) \text{ or } \left(\pm\frac{2}{3}, \frac{1}{3}\right).$$

[6 marks]

We have

$$f(0,0) = 0, \ f(0,-1) = -1, \quad f\left(\pm\frac{2}{3}, \frac{1}{3}\right) = \frac{7}{9}.$$

So the minimum value of f in $g \le 1$ is -1, realised at $(0,-1)$, and the maximum is $\frac{7}{9}$, realised at $(\pm\frac{2}{3}, \frac{1}{3})$.

[2 marks.]

[$2 + 4 + 1 + 6 + 2 = 15$ marks.]

14. The line $x + y = 1$ meets the y-axis $x = 0$ at $y = 1$ and the x-axis $y = 0$ at $x = 1$. So the mass is given by

$$M = \int_0^1 \int_0^{1-y} y\,dx\,dy = \int_0^1 (y - y^2)\,dy$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{6}.$$

[5 marks]

Then the centre of mass is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{1}{M}\int_0^1 y \int_0^{1-y} x dx dy = \frac{6}{2}\int_0^1 (y - 2y^2 + y^3) dy$$

$$= 3\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1 = 3\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{4},$$

[5 marks]

$$\overline{y} = 6\int_0^1 y^2(1-y) dx dy$$

$$6\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{2}.$$

So the centre of mass is

$$\left(\frac{1}{4}, \frac{1}{2}\right).$$

[5 marks]

[5 + 5 + 5 = 15 marks]