All questions are similar to homework problems.

MATH102 Solutions May 2006
Section A

1. The Taylor series of $f(x) = x^{-2} = (1 + (x-1))^{-2}$ is

$$1 - 2(x-1) + \frac{2.3}{2!}(x-1)^2 - \frac{2.3.4}{3!}(x-1)^3 \cdots = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$$

This can also be worked out by computing all derivatives of f at $x = 1$.
[3 marks]
a) When $x = 0.5$ the series is convergent and equal to $f(0.5) = 4$.
[1 mark]
b) When $x = 2$ the series is not convergent to $f(2) = \frac{1}{4}$. In fact, the series is not convergent.
[1 mark]
No explanation is required in a) or b).
$5 = 3 + 1 + 1$ marks

2(i) Separating the variables, we have

$$\int y dy = -\int \sin x dx,$$

$$\frac{y^2}{2} = \cos x + C,$$

or

$$y = \pm\sqrt{2C + 2\cos x}.$$

Putting $y(0) = 1$ gives $2C = -1$ and $y = +\sqrt{2\cos x - 1}$.
2(ii) Using the integrating factor method, and the integrating factor is

$$\exp\left(\int dx\right) = e^{-x}.$$

So the equation becomes

$$\frac{d}{dx}(ye^{-x}) = e^x.$$

Integrating gives

$$ye^{-x} = e^x + C.$$

So the general solution is

$$y = e^{2x} + Ce^x.$$

Putting $y(0) = 2$ gives $C = 1$ and $y = e^{2x} + e^x$.
3 marks for (i) 4 marks for (ii).
[7 marks]

3. Try $y = e^r x$. Then

$$r^2 + 2r - 15 = 0 \Rightarrow (r-3)(r+5) = 0 \Rightarrow r = -5 \text{ or } r = 3.$$

So the general solution is

$$y = Ae^{3x} + Be^{-5x}.$$

[2 marks]
So $y' = 3Ae^{3x} - 5Be^{-5x}$ and the initial conditions $y(0) = 2$, $y'(0) = -1$ give

$$A + B = 2, \quad 3A - 5B = -1 \rightarrow 8B = 7, \; A = 2 - B \Rightarrow B = \frac{7}{8}, \; A = \frac{9}{8}.$$

So

$$y = \frac{9}{8}e^{3x} + \frac{7}{8}e^{-5x}.$$

[3 marks]
[$2 + 3 = 5$ marks]
4. We have

$$\lim_{(x,y)\to(0,0),y=0} \frac{x^2y^2}{x^4+y^4} = \lim_{x\to 0} \frac{0}{x^4} = 0,$$
$$\lim_{(x,y)\to(0,0),y=x} \frac{x^2y^2}{x^4+y^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}.$$

So the limits along two different lines as $(x, y) \to (0, 0)$ are different, and the overall limit does not exist.
[4 marks]

5.

$$\frac{\partial f}{\partial x} = 2xy\cos(x^2y), \; \frac{\partial f}{\partial y} = x^2\cos(x^2y),$$

$$\frac{\partial^2 f}{\partial x^2} = 2y\cos(x^2y) - 4x^2y^2\sin(x^2y),$$
$$\frac{\partial^2 f}{\partial y\partial x} = 2x\cos(x^2y) - 2x^3y\sin(x^2y),$$
$$\frac{\partial^2 f}{\partial x\partial y} = 2x\cos(x^2y) - 2x^3y\sin(x^2y),,$$

so that these last two are equal, and

$$\frac{\partial^2 f}{\partial y^2} = -x^4\sin(x^2y).$$

[5 marks]

6. We have

$$\frac{\partial u}{\partial x} = 2,\ \frac{\partial u}{\partial y} = 1,\ \frac{\partial v}{\partial x} = -1,\ \frac{\partial v}{\partial y} = 2.$$

By the Chain Rule,

$$\begin{aligned}\frac{\partial g}{\partial x}(x,y) &= \frac{\partial f}{\partial u}(u,v)\frac{\partial u}{\partial x}(x,y) - \frac{\partial f}{\partial v}(u,v)\frac{\partial v}{\partial x}(x,y)\\ &= 2\frac{\partial f}{\partial u}(u,v) - \frac{\partial f}{\partial v}(u,v).\end{aligned}$$

Similarly,

$$\frac{\partial g}{\partial y}(x,y) = \frac{\partial f}{\partial u}(u,v) + 2\frac{\partial f}{\partial v}(u,v).$$

Similarly,

$$\begin{aligned}\frac{\partial^2 g}{\partial x^2}(x,y) &= 2\left(2\frac{\partial^2 f}{\partial u^2}(u,v) - \frac{\partial^2 f}{\partial u \partial v}(u,v)\right) - \left(2\frac{\partial^2 f}{\partial v \partial u}(u,v) - \frac{\partial^2 f}{\partial v^2}(u,v)\right)\\ &= 4\frac{\partial^2 f}{\partial u^2}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v) - 4\frac{\partial^2 f}{\partial u \partial v}(u,v).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^2 g}{\partial y^2}(x,y) &= \left(\frac{\partial^2 f}{\partial u^2}(u,v) + 2\frac{\partial^2 f}{\partial u \partial v}(u,v)\right) + 2\left(\frac{\partial^2 f}{\partial v \partial u}(u,v) + 2\frac{\partial^2 f}{\partial v^2}(u,v)\right)\\ &= \frac{\partial^2 f}{\partial u^2}(u,v) + 4\frac{\partial^2 f}{\partial v^2}(u,v) + 4\frac{\partial^2 f}{\partial u \partial v}(u,v).\end{aligned}$$

Adding, we obtain

$$\frac{\partial^2 g}{\partial x^2}(x,y) + \frac{\partial^2 g}{\partial y^2}(x,y) = 5\left(\frac{\partial^2 f}{\partial u^2}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v)\right).$$

[6 marks]

7. For

$$f(x,y,z) = \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz}$$

we have

$$\nabla f(x,y,z) = (-x^{-2}(y^{-1} + z^{-1})\mathbf{i} + (-y^{-2}(x^{-1} + z^{-1})\mathbf{j} + (-z^{-2}(x^{-1} + y^{-1})\mathbf{k}.$$

So

$$\nabla f(1,1,1) = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

3 marks

The tangent plane at $(1,1,1)$ is

$$\nabla f(1,1,1).((x-1)\mathbf{i} + (y-1)\mathbf{j} + (z-1)\mathbf{k} = 0,$$

or

$$-2(x-1)-2(y-1)-2(z-1)=0,$$

or

$$x+y+z-3=0.$$

[2 marks]
[$3+2=5$ marks.]

8. For

$$f(x,y)=y^2x+2yx+2x^2-3x,$$

we have

$$\frac{\partial f}{\partial x}=y^2+2y+4x-3, \quad \frac{\partial f}{\partial y}=x(2y+2).$$

[2 marks]
So at a stationary point,

$$x(2y+2)=0=y^2+2y+4x-3 \Leftrightarrow x=0=(y+3)(y-1) \text{ or } y+1=0=4x-4$$

$$\Leftrightarrow (x,y)=(0,1) \text{ or } (0,-3) \text{ or } (1,-1).$$

[2 marks]

$$A=\frac{\partial^2 f}{\partial x^2}=4, \ B=\frac{\partial^2 f}{\partial y \partial x}=2y+2, C=\frac{\partial^2 f}{\partial y^2}=2x.$$

For $(x,y)=(0,1)$ or (0,-3) we have $C=0$ and $B\neq 0$. So $AC-B^2<0$ and these points are saddle points.
For $(x,y)=(1,-1)$, we have $A=4$, $B=0$, $C=2$. So $A>0$, $AC-B^2>0$ and $(1,-1)$ is a minimum.
[4 marks]
[$2+2+4=8$ marks]

9. For

$$f(x,y)=\frac{1}{x^2+y^2},$$

we have

$$\frac{\partial f}{\partial x}=\frac{-2x}{(x^2+y^2)^2}, \quad \frac{\partial f}{\partial y}=\frac{-2y}{(x^2+y^2}.$$

So

$$f(1,1)=\frac{1}{2}, \quad \frac{\partial f}{\partial x}(1,1)=-\frac{1}{2}, \quad \frac{\partial f}{\partial y}(1,1)=-\frac{1}{2}.$$

So the linear approximation is

$$\frac{1}{2}-\frac{1}{2}(x-1)-\frac{1}{2}(y-1).$$

[It would be acceptable to realise that

$$f(x,y) = (2 + 2(x-1) + (x-1)^2 + 2(y-1) + (y-1)^2)^{-1}$$
$$= \frac{1}{2}(1 + (x-1) + (y-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-1)^2)^{-1}$$

and to expand out.]
[4 marks]

10. In polar coordinates (r,θ), D is the set where $r \leq 1$ and $0 \leq \theta \leq 2\pi$ (by choice of argument). Also, $x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$. So

$$\int\int_D e^{x^2+y^2}\,dxdy = \int_0^{2\pi}\int_0^1 re^{r^2}\,dr = \int_0^{2\pi}\left[\frac{1}{2}e^{r^2}\right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi}\frac{1}{2}(e-1)d\theta = \pi(e-1).$$

[6 marks]

Section B

11. (i) a) $f'(z) = \frac{1}{2}(4+z)^{-1/2}$, $f''(z) = -\frac{1}{4}(4+z)^{-3/2}$. So $f(0) = 2$, $f'(0) = \frac{1}{4}$ and

$$P_1(z,0) = 2 + \frac{1}{4}z, \quad R_1(z,0) = -\frac{1}{8}(4+c)^{-3/2}z^2$$

for some c between 0 and z.
[3 marks]
If $|z| \leq 2$ and c is between 0 and z then $(4+c)^{-3/2} \leq 2^{-3/2}$ and

$$|R_1(z,0)| \leq \frac{1}{8.2\sqrt{2}}.2^2 = \frac{1}{4\sqrt{2}}.$$

[3 marks]
(i)b) $g'(x) = -\sin x$, $g''(x) = -\cos x$, $g^{(3)}(x) = \sin x$, $g^{(4)}(x) = \cos x$. So $g(0) = 1$, $g'(0) = 0$, $g''(0) = -1$, $g^{(3)}(0) = 0$, and

$$P_3(x,0) = 1 - \frac{x^2}{2}, \quad R_3(x,0) = \frac{\cos c}{4!}x^4$$

for some c between 0 and x.
[4 marks]
Since $|\cos c| \leq 1$ we have

$$|R_3(x,0)| \leq \frac{x^4}{24}.$$

[1 mark]

(ii) $y^2 = 8 + 2(\cos x - 1)$. Then $y = \sqrt{2} f(\cos x - 1)$. So for $P_1(z, 0)$ and $R_1(z, 0)$ as in (i)a),

$$y = \sqrt{2} P_1(\cos x - 1, 0) + \sqrt{2} R_1(c, 0))$$
$$= \sqrt{2}\left(\frac{7}{4} + \frac{1}{4}\cos x\right) + R_1(c, 0)$$

for some c between 0 and $\cos x - 1$. Since $-2 \leq \cos x - 1 \leq 0$ for all x, we have $-2 \leq c \leq 0$, and by (i)a) $\sqrt{2}|R_1(\cos x - 1, 0)| \leq \frac{1}{4}$.
[4 marks.]

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 + 4r + 3 = (r + 1)(r + 3) = 0,$$

that is, $r = -1$ or -3. So the complementary solution is $Ae^{-x} + Be^{-3x}$.
[3 marks]
(i) We try $y_p = Cx + D$. Then $y_p' = C$ and $y_p'' = 0$. So $y_p'' + 4y_p' + 3y_p = 4C + 3Cx + 3D = 3x + 1$. So $C = 1$ and $3D = 1 - 4C$, that is, $D = -1$. So the general solution is

$$y = Ae^{-x} + Be^{-3x} + x - 1.$$

[3 marks]
This gives

$$y' = -Ae^{-x} - 3Be^{-3x} + 1.$$

So putting $x = 0$, the boundary conditions give

$$A + B - 1 = 1, \quad -A - 3B + 1 = 2 \;\Rightarrow\; -2B = 3, \;\; A = 2 - B \Rightarrow B = -\frac{3}{2}, \;\; A = \frac{7}{2}.$$

So the solution is

$$y = \frac{7}{2}e^{-x} - \frac{3}{2}e^{-3x} + x - 1.$$

[3 marks]
(ii) We try $y_p = C\sin x + D\cos x$. Then $y_p'(x) = C\cos x - D\sin x$ and $y_p'' = -C\sin x - D\cos x$. So

$$y_p'' + 4y_p' + 3y_p = (2C - 4D)\sin x + (4C + 2D)\cos x = 5\sin x.$$

So $D = -2C$ and $10C = 5$ So the general solution is

$$y = Ae^{-x} + Be^{-3x} + \frac{1}{2}\sin x - \cos x.$$

[3 marks] This gives

$$y'(x) = -Ae^{-x} - 3Be^{-3x} + \frac{1}{2}\cos x + \sin x$$

So putting $x = 0$, the boundary conditions give

$$A + B - 1 = 1, \quad -A - 3B + \frac{1}{2} = 2 \Rightarrow -2B = \frac{7}{2}, A = 2 - B \Rightarrow B = -\frac{7}{4}, A = \frac{15}{4}.$$

So

$$y = \frac{15}{4}e^{-x} - \frac{7}{4}e^{-3x} + \frac{1}{2}\sin x - \cos x.$$

[3 marks]
[$5 \times 3 = 15$ marks]

13a) The area is $A(x, y) = 4xy$ if the vertices of the rectangle are at $(\pm x, \pm y)$ with $x \geq 0$, $y \geq 0$.
[1 mark]

We have

$$\nabla A = 4y\mathbf{i} + 4x\mathbf{j}$$

$$\nabla g = 6x\mathbf{i} + 10y\mathbf{j}.$$

[2 marks]

At a stationary point of A on $g = 10$, we have $\nabla A = \lambda g$, that is,

$$4y = 6x\lambda, \quad 4x = 10y\lambda \Rightarrow (6x^2 - 10y^2)\lambda = 0.$$

If $\lambda = 0$ then $x = y = 0$, which is inconsistent with $g = 10$. So $6x^2 - 10y^2 = 0$ and $x = \pm\sqrt{5/3}y$. The condition $g = 10$ then gives $10y^2 = 10$ and $y = \pm 1$. So the maximum area is $4\sqrt{5/3}$.
[5 marks]
b) We have

$$\nabla f = 2x\mathbf{i} - 2y\mathbf{j}.$$

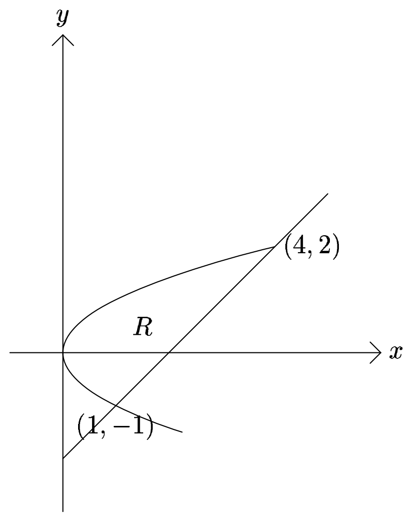
[1 mark]

The only stationary point of f is $2x = -2y = 0$, that is, $(x, y) = (0, 0)$, which is inside the ellipse and $f(0, 0) = 0$. [One can note that this is a saddle point and therefore not a local maximum of minimum but this turns out to be unnecessary.]
2 marks.

For a stationary point on the ellipse, we have $\nabla A = \lambda g$, that is $2x = 6x\lambda$ and $-2y = 10y\lambda$. So either $x = 0$ and $\lambda = -\frac{1}{5}$; or $\lambda = \frac{1}{3}$ and $y = 0$. We cannot have $x = y = 0$ since $g = 10$. If $x = 0$ then the condition $g = 10$ gives $y^2 = 2$ and $f = -2$. If $y = 0$ then the condition $g = 10$ gives $3x^2 = 10$ and $f = \frac{10}{3}$. So the maximum and minimum values of f are $\frac{10}{3}$ and -2.

[$1 + 2 + 5 + 1 + 2 + 4 = 15$ marks.]

14a). The line $y = x - 2$ meets the parabola $x = y^2$ when $y^2 - y - 2 = 0$, that is, $(y + 1)(y - 2) = 0$. When $y = -1$ then $x = 1$ and when $y = 2$ $x = 4$ The parabola is to the left of the line. The region R is as shown.



[3 marks]

The area A is

$$\int_{-1}^{2}\int_{y^2}^{y+2} dxdy = \int_{-1}^{2}(y+2-y^2)dy$$

$$= \left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^{2} = 2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}=\frac{9}{2}$$

[4 marks]

14b) Then

$$\overline{x} = \frac{1}{A}\int_{-1}^{2}\int_{y^2}^{y+2} xdxdy$$

$$= \frac{2}{9}\int_{-1}^{2}\left[\frac{x^2}{2}\right]_{y^2}^{y+2} dy = \frac{1}{9}\int_{-1}^{2}(y^2+4y+4-y^4)dy$$

$$= \frac{1}{9}\left[\frac{y^3}{3}+2y^2+4y-\frac{y^5}{5}\right]_{-1}^{2} = \frac{1}{9}\left(\frac{8}{3}+8+8-\frac{32}{5}+\frac{1}{3}-2+4-\frac{1}{5}\right)$$

$$= \frac{1}{9}.\frac{72}{5}=\frac{8}{5}$$

[4 marks]

$$\overline{y} = \frac{1}{A}\int_{-1}^{2}\int_{y^2}^{y+2} ydxdy$$

$$= \frac{2}{9}\int_{-1}^{2}(y^2+2y-y^3)dy = \frac{2}{9}\left[\frac{y^3}{3}+y^2-\frac{y^4}{4}\right]_{-1}^{2}$$

$$= \frac{2}{9}\left(\frac{8}{3} + 4 - 4 + \frac{1}{3} - 1 + \frac{1}{4}\right) = \frac{2}{9}.\frac{9}{4} = \frac{1}{2}.$$

So

$$(\overline{x}, \overline{y}) = \left(\frac{8}{5}, \frac{1}{2}\right).$$

[4 marks]

[$3 + 4 + 4 + 4 = 15$ marks.]