MATH102 Solutions May 2005
Section A

1. The Taylor series of $f(x) = x^{-1} = (1 + (x-1))^{1/2}$ is

$$1 + \frac{1}{2}(x-1) + \frac{\frac{1}{2}.\frac{-1}{2}}{2!}(x-1)^2 \cdots = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x-1)^n,$$

where

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}.\frac{-1}{2}.\cdots\left(\frac{1}{2}-n+1\right)}{n!}.$$

This can also be worked out by computing all derivatives of f at $x = 1$.
[3 marks]
a) When $x = 0.5$ the series is convergent and equal to $f(0.5)$.
[1 mark]
b) When $x = 3$ the series is not convergent and it does not even make sense to say that it is equal to $f(3)$.
[1 mark]
No explanation is required in a) or b).
$5 = 3 + 1 + 1$ marks

2(i) Separating the variables, we have

$$\int \frac{dy}{y^2} = \int \frac{dx}{x},$$

$$-\frac{1}{y} = \ln|x| + C,$$

or

$$y = -\frac{1}{C + \ln|x|}.$$

2(ii) Using the integrating factor method, we have

$$\frac{dy}{dx} - \frac{y}{x} = 1,$$

and the integrating factor is

$$\exp\left(-\int \frac{dx}{x}\right) = \exp\ln x^{-1} = x^{-1}.$$

So the equation becomes

$$\frac{d}{dx}\frac{y}{x} = \frac{1}{x}.$$

Integrating gives

$$\frac{y}{x} = \ln|x| + C.$$

So the general solution is

$$y = x \ln |x| + Cx.$$

3 marks for (i) 4 marks for (ii).
[7 marks]

3. Try $y = e^r x$. Then

$$r^2 + 4r - 5 = 0 \Rightarrow (r-1)(r+5) = 0 \Rightarrow r = -5 \text{ or } r = 1.$$

So the general solution is

$$y = Ae^x + Be^{-5x}.$$

[2 marks]
So $y' = Ae^x - 5Be^{-5x}$ and the initial conditions $y(0) = 1$, $y'(0) = -1$ give

$$A + B = 1, \quad A - 5B = -1 \to 6B = 2, \; A = 1 - B \Rightarrow B = \frac{1}{3}, A = \frac{2}{3}.$$

So

$$y = \frac{2}{3}e^x + \frac{1}{3}e^{-5x}.$$

[3 marks]
$2 + 3 = 5$ marks
4. We have

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+x^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+y^2} + \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 1 + 0,$$

because $|x^3| \le |x|.x^2 \le |x|(x^2+y^2)$.
[3 marks] For the second one, $x^4 + y^4 \le (x^2+y^2)^2$ and so

$$\frac{x^2+y^2}{x^4+y^4} \ge \frac{1}{x^2+y^2}$$

and $x^2 + y^2 \to 0$ as $(x, y) \to (0, 0)$. So the limit does not exist.
[2 marks]
[$3 + 2 = 5$ marks]

5.

$$\frac{\partial f}{\partial x} = 3x^2y - y^3, \; \frac{\partial f}{\partial y} = x^3 - 3xy^2,$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy,$$

$$\frac{\partial^2 f}{\partial y \partial x} = 3x^2 - 3y^2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 - 3y^2,$$

so that these last two are equal, and

$$\frac{\partial^2 f}{\partial y^2} = -6xy,$$

[4 marks] which gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

[1 mark]
[5 marks]

6. By the Chain Rule,

$$\frac{\partial f}{\partial x} = F'(x+ct) + G'(x-ct), \quad \frac{\partial f}{\partial t} = cF'(x+ct) - cG'(x-ct).$$

So

$$\frac{\partial^2 f}{\partial x^2} = F''(x+ct) + G''(x-ct), \quad \frac{\partial^2 f}{\partial t^2} = c^2 F''(x+ct) + c^2 G''(x-ct),$$

and so

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2},$$

as required.
[5 marks]

7. For

$$f(x,y,z) = x^3 y - y^2 z + 2xyz,$$

we have

$$\nabla f(x,y,z) = (3x^2 y + 2yz)\mathbf{i} + (x^3 - 2yz + 2xz)\mathbf{j} + (2xy - y^2)\mathbf{k}$$

So

$$\nabla f(x,y,z)(1,1,1) = 5\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

2 marks

The vector $(-1,1,2) = -\mathbf{i}+\mathbf{j}+2\mathbf{k}$ has length $\sqrt{6}$ So the directional derivative in the direction $(-1,1,2)$ is

$$\frac{1}{\sqrt{6}}(5\mathbf{i} + \mathbf{j} + \mathbf{k}).(-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \frac{-2}{\sqrt{6}}.$$

[3 marks]
[$2 + 3 = 5$ marks.]

8. For

$$f(x,y) = x^2 + x^2y + 2y^2,$$

we have

$$\frac{\partial f}{\partial x} = 2x + 2xy, \quad \frac{\partial f}{\partial y} = x^2 + 4y.$$

[2 marks]

So at a stationary point,

$$x(1+y) = 0 = x^2+4y \Rightarrow x = y = 0 \text{ or } (y = -1 \text{ and } x^2 = 4) \Rightarrow (x,y) = (0,0) \text{ or } (2,-1) \text{ or } (-2,-1).$$

[2 marks]

$$A = \frac{\partial^2 f}{\partial x^2} = 2 + 2y, \ B = \frac{\partial^2 f}{\partial y \partial x} = 2x, C = \frac{\partial^2 f}{\partial y^2} = 4.$$

For $(x,y) = (0,0)$ we have $A = 2 > 0$, $B = 0$, $C = 4$ and $AC - B^2 = 8 > 0$. So $(0,0)$ is a minimum.

For $(x,y) = (\pm 2, -1))$ $A = 0$, $B = \pm 4$, $C = 4$. So $AC - B^2 = -16 < 0$, and both these points are saddles.

[4 marks]

[$2 + 2 + 4 = 8$ marks]

9. For

$$f(x,y) = \frac{1}{x^2 + y},$$

we have

$$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2+y)^2}, \quad \frac{\partial f}{\partial y} = \frac{-1}{(x^2+y)^2}.$$

So

$$f(1,1) = \frac{1}{2}, \quad \frac{\partial f}{\partial x}(1,1) = -\frac{1}{2}, \quad \frac{\partial f}{\partial y}(1,1) = -\frac{1}{4}.$$

So the linear approximation is

$$\frac{1}{2} - \frac{1}{2}(x-1) - \frac{1}{4}(y-1).$$

[It would be acceptable to realise that

$$f(x,y) = (2 + 2(x-1) + (x-1)^2 + (y-1))^{-1}$$
$$= \frac{1}{2}(1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-1))^{-1}$$

and to expand out.]

[4 marks]

10.

$$\int\int_R x^2 dydx = \int_{-2}^{2} \int_0^{4-x^2} x^2 dydx$$

$$= \int_{-2}^{2} \left[x^2 y\right]_0^{4-x^2} dx = \int_{-2}^{2} (4x^2 - x^4) dx$$

$$= \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = 2\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{128}{15}.$$

[6 marks]

Section B

11. (i) We have $f^{(2k)}(x) = (-1)^k \cos x$ and $f^{(2k-1)}(x) = (-1)^k \sin x$. So $f^{(2k)}(0) = (-1)^k$ and $f^{(2k-1)}(0) = 0$. So:

a) $P_3(x) = 1 - \frac{1}{2}x^2$ and $R_3(x) = \frac{1}{4!}x^4 \cos c$ for some c between 0 and x.

b) For any $k \geq 1$

$$P_{2k-1}(x) = \sum_{r=0}^{k-1} (-1)^r \frac{x^{2r}}{(2r)!}, \quad R_{2k-1}(x) = (-1)^k \frac{x^{2k}}{(2k)!} \cos c$$

for some c between 0 and x.

[6 marks]

Since $|\cos c| \leq 1$ for all c, and $\cos c \geq 0$ for $c \in [-1, 1]$ we have

$$0 \leq R_3(x) \leq \frac{x^4}{(4)!} \cos c \leq \frac{x^4}{(4)!}.$$

So for $x \in [-1, 1]$

$$0 \leq R_3(x) \leq \frac{x^4}{24}.$$

So, since $\cos x = P_3(x) + R_3(x)$,

$$1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24},$$

and

$$1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{11x^2}{24}.$$

[3 marks]

(ii) Using the Taylor series of $\cos x$ at 0:

a)

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{1 - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \cdots\right)}{x^2} = \lim_{x \to 0} \frac{\frac{x^2}{2} - \frac{x^4}{24} \cdots}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} - \frac{x^2}{24} \cdots}{1} = \frac{1}{2};$$

[3 marks]

b)

$$\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{2(1 - \cos x)}\right) = \lim_{x \to 0} \frac{2(1 - \cos x) - x^2}{2x^2(1 - \cos x)}$$

$$= \lim_{x\to 0} \frac{-\frac{x^4}{12} + \cdots}{x^4 - \frac{x^6}{12} \cdots} = \frac{-1}{12}.$$

[3 marks]
[$6+3+3+3 = 15$ marks.]

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 - 2r + 1 = (r-1)^2 = 0,$$

that is, $r = 1$. So the complementary solution is $Axe^x + Be^x$.
[3 marks]
(i) We try $y_p = Ce^{-x}$. Then $y_p' = -Ce^{-x}$ and $y_p'' = Ce^{-x}$. So $y_p'' - 2y_p' + y_p = 4Ce^{-x}$. So $C = 1$. So the general solution is

$$y = Axe^x + Be^x + e^{-x}$$

[3 marks]
This gives

$$y' = A(x+1)e^x + Be^x - e^{-x}.$$

So putting $x = 0$ the boundary conditions give

$$B = 0, \quad A + B = 0 \quad \Rightarrow \quad B = 0, \quad A = 0.$$

So the solution is

$$e^{-x}$$

[3 marks]
(ii) We try We try $y_p = Cx + D$. Then $y_p'(x) = C$ and $y_p'' = 0$. So $y_p'' - 2y_p' + y_p = -2C + Cx + D = x$. So $C = 1$ ad $D = 2$ So the general solution is

$$y = Axe^x + Be^x + x + 2$$

[3 marks] This gives

$$y'(x) = A(1+x)e^x + Be^x + 1$$

So $y(0) = 1$, $y'(0) = -1$ give

$$B + 2 = 1, \quad A + B = -2,$$

so $B = -1$ and $A = -1$ and

$$y = -xe^x - e^x + x + 2.$$

[3 marks]
[$5 \times 3 = 15$ marks]

13. For $f(x,y) = 3x^2 + x^2y + y^2$, $g(x,y) = 2x^2 + y^2$, we have

$$\nabla f = (6x + 2xy)\mathbf{i} + (x^2 + 2y)\mathbf{j}$$
$$\nabla g = 4x\mathbf{i} + 2y\mathbf{j}.$$

[2 marks]
At a stationary point of f, we have

$$2x(3 + y) = x^2 + 2y = 0.$$

So $x = 0$ or $y = -3$. If $x = 0$ then $y = 0$. If $y = -3$ then $x^2 = 6$, so $x = \pm\sqrt{6}$. So the stationary points are $(0, 0)$ and $(\pm\sqrt{6}, -3)$. Only the first of these satisfies $g(x, y) \leq 1$. So $(0, 0)$ is the only stationary point which is a candidate for a maximum or minimum of f on the set where $g \leq 1$.
[4 marks]
At a maximum or minimum on the set where $g = 1$, we must have $\nabla f = \lambda\nabla g$. that is

$$6x + 2xy = 4\lambda x, \quad x^2 + 2y = 2\lambda y.$$

[1 mark]
The first equation gives $x = 0$ or $3 + y = 2\lambda$. If $x = 0$ then the equation $g = 1$ gives $y^2 = 1$ and $y = \pm 1$.
If $2\lambda = 3 + y$, then plugging into the second equation gives

$$x^2 + 2y = y(3 + y)$$

So multiplying by 2 and replacing $2x^2$ by $1 - y^2$ gives

$$1 + 4y - y^2 = 6y + 2y^2.$$

So

$$3y^2 + 2y - 1 = (3y - 1)(y + 1) = 0.$$

So $y = -1$ or $y = \frac{1}{3}$ and using $g = 1$ gives

$$(x, y) = (0, -1) \text{ or } \left(\pm\frac{2}{3}, \frac{1}{3}\right).$$

So altogether the points on $g = 1$ which can be maxima or minima of f on $g \leq 1$ are

$$(0, 0) \text{ or}(0, \pm 1) \text{ or } \left(\pm\frac{2}{3}, \frac{1}{3}\right).$$

[6 marks]
We have

$$f(0, 0) = 0, \; f(0, \pm 1) = 1, \quad f\left(\pm\frac{2}{3}, \frac{1}{3}\right) = \frac{43}{27}.$$

So the minimum value of f on $g \leq 1$ is 0, realised at $(0, 0)$, and the maximum is $\frac{43}{27}$, realised at $(\pm\frac{2}{3}, \frac{1}{3})$.
[2 marks.]

[2 + 4 + 1 + 6 + 2 = 15 marks.]

14. The line $x + 2y = 1$ meets the y-axis $x = 0$ at $y = \frac{1}{2}$ and the x-axis $y = 0$ at $x = 1$. So the mass is given by

$$M = \int_0^{1/2} \int_0^{1-2y} xy dx dy = \int_0^{1/2} \frac{y(1-2y)^2}{2} dy$$

$$= \left[\frac{y^2}{4} - 2\frac{y^3}{3} + \frac{y^4}{2} \right]_0^{1/2} = \frac{1}{16} - \frac{1}{12} + \frac{1}{32} = \frac{1}{96}.$$

[5 marks]

Then the centre of mass is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{1}{M} \int_0^{1/2} \int_0^{1-2y} x^2 y dx dy = 96 \int_0^{1/2} \frac{y}{3}(1-2y)^3 dy$$

$$= 96 \int_0^{1/2} \left(\frac{y}{3} - 2y^2 + 4y^3 - 8\frac{y^4}{3} \right) dy = 96 \left[\frac{y^2}{6} - \frac{2y^3}{3} + y^4 - 8\frac{y^5}{15} \right]_0^{1/2}$$

$$= 96 \left(\frac{1}{24} - \frac{1}{12} + \frac{1}{16} - \frac{1}{60} \right) = -4 + 6 - \frac{8}{5} = \frac{2}{5},$$

[5 marks]

Changing the order of integration,

$$\overline{y} = 96 \int_0^{1/2} \frac{y^2}{2}(1 - 4y + 4y^2) dx dy$$

$$96 \left[\frac{y^3}{6} - \frac{y^4}{2} + 2\frac{y^5}{5} \right]_0^{1/2} = 2 - 3 + \frac{6}{5} = \frac{1}{5}.$$

So the centre of mass is

$$\left(\frac{2}{5}, \frac{1}{5} \right).$$

[5 marks]

[5 + 5 + 5 = 15 marks]