MATH102 Solutions May 2004

Section A

1. The Taylor series of $f(x) = x^{-1} = (1 + (x-1))^{-1}$ is

$$1 - (x-1) + (x-1)^2 \cdots = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

This can also be worked out by computing all derivatives of f at $x = 1$.
[3 marks]
a) When $x = 2$ the series is not convergent and so it does not make sense to say that it is equal to $f(2)$.
[1 mark]
b) When $x = 1.5$ the series is convergent and equal to $f(1.5) = \frac{2}{3}$.
[1 mark]

No explanation is required in a) or b).

$5 = 3 + 1 + 1$ marks

2. Solving both by the Integrating factor method, we write the equations as:
(i) $\frac{dy}{dx} - \frac{y}{x} = 0$,
(ii) $\frac{dy}{dx} - \frac{y}{x} = \frac{1}{x}$.

Then the integrating factor is

$$\exp\left(\int \frac{-1}{x} dx\right) = \exp(-\ln x) = \exp(\ln(1/x)) = \frac{1}{x}$$

Then multiplying by the integral factor we have:
(i)

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = 0,$$

$$\frac{d}{dx}\left(\frac{1}{x} y\right) = 0,$$

$$\frac{y}{x} = C \Rightarrow y = Cx,$$

(ii)

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = \frac{1}{x^2},$$

$$\frac{d}{dx}\left(\frac{1}{x} y\right) = \frac{1}{x^2},$$

$$\frac{y}{x} = -\frac{1}{x} + C \Rightarrow y = Cx - 1.$$

3 marks for (i) 4 marks for (ii). Other methods are possible: separation of variables for (i) and complementary and particular solutions for linear o.d.e.'s with constant coefficients

[7 marks]
3. Try $y = e^r x$. Then

$$r^2 + 3r - 4 = 0 \Rightarrow (r-1)(r+4) = 0 \Rightarrow r = -4 \text{ or } r = 1.$$

So the general solution is

$$y = Ae^x + Be^{-4x}.$$

[2 marks]
So $y' = Ae^x - 4Be^{-4x}$ and the initial conditions $y(0) = 1$, $y'(0) = 2$ give

$$A + B = 1, \quad A - 4B = 2 \rightarrow 5B = -1, \ A = 1 - B \Rightarrow B = \frac{-1}{5}, A = \frac{6}{5}.$$

So

$$y = \frac{6}{5}e^x - \frac{1}{5}e^{-4x}.$$

[3 marks]
$2 + 3 = 5$ marks
4. We have

$$\lim_{x \to 0, y=0} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{0}{x^2} = 0,$$

[2 marks] and

$$\lim_{x \to 0, x=y} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

[2 marks]
$2 + 2 = 4$ marks
5.

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}, \ \frac{\partial f}{\partial y} = \frac{2x}{x^2 + y^2},$$

[2 marks]

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\frac{4yx}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{4yx}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2},$$

which gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{2(y^2 - x^2 + x^2 - y^2)}{(x^2 + y^2)^2} = 0.$$

[4 marks]
[2 + 4 = 6 marks]
6

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 3y^2,$$

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial u} = 1, \quad \frac{\partial y}{\partial v} = -1,$$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 2x + 3y^2 = 2(u+v) + 3(u-v)^2 = 3u^2 + 3v^2 - 6uv + 2u + 2v,$$

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 2x - 3y^2 = 2(u+v) - 3(u-v)^2 = -3u^2 - 3v^2 + 6uv + 2u + 2v.$$

[5 marks]
7. Write

$$f(x, y, z) = 3x^2 - 2xyz + z^2y - 2.$$

Then

$$\nabla f(x, y, z) = (6x - 2yz)\mathbf{i} + (-2xz + z^2)\mathbf{j} + (-2xy + 2yz)\mathbf{k} = 4\mathbf{i} - \mathbf{j} \text{ at } (x, y, z) = (1, 1, 1).$$

[3 marks] So a normal to the surface at the point $(1, 1, 1)$ is given by $4\mathbf{i} - \mathbf{j}$ and the tangent plane at this point is given by

$$4(x-1) - (y-1) = 4x - y - 3 = 0.$$

[2 marks]
[3 + 2 = 5 marks.]
8.

$$\frac{\partial f}{\partial x} = 6y + 2x, \quad \frac{\partial f}{\partial y} = 6y^2 + 6x.$$

[2 marks]

So at a stationary point,

$$6y + 2x = 0 = 6y^2 + 6x \Rightarrow x = -3y, \ 6y^2 - 18y = 0 \Rightarrow (y = 0, \ x = 0) \text{ or } (y = 3, \ x = -9.$$

[2 marks]

$$A = \frac{\partial^2 f}{\partial x^2} = 2, \ B = \frac{\partial^2 f}{\partial y \partial x} = 6, C = \frac{\partial^2 f}{\partial y^2} = 12y.$$

For $(x, y) = (0, 0)$ we have $C = 0$, and $AC - B^2 = -36 < 0$. So $(0, 0)$ is a saddle point.

For $(x, y) = (-9, 3)$ we have $C = 36$ and $AC - B^2 = 72 - 36 > 0$. Since also $A > 0$, $(-9, 3)$ is a minimum.
[4 marks]
$2 + 2 + 4 = 8$ marks

9. We have

$$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2+y^2)^2}, \quad \frac{\partial f}{\partial y} = \frac{-2y}{(x^2+y^2)^2}.$$

So

$$f(1,1) = \frac{1}{2}, \quad \frac{\partial f}{\partial x}(1,1) = -\frac{1}{2}, \quad \frac{\partial f}{\partial y}(1,1) = -\frac{1}{2}.$$

So the linear approximation is

$$\frac{1}{2} - \frac{1}{2}(x-1) - \frac{1}{2}(y-1).$$

[It would be acceptable to realise that

$$f(x,y) = (2 + 2(x-1) + 2(y-1) + (x-1)^2 + (y-1)^2)^{-1}$$
$$= \frac{1}{2}(1 + (x-1) + (y-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-1)^2)^{-1}$$

and to expand out.]
[4 marks]
10.

$$\int\int_T (x-y)dxdy = \int_0^1 \int_y^1 (x-y)dxdy$$
$$= \int_0^1 \left[\frac{x^2}{2} - yx\right]_y^1 dy = \int_0^1 \left(\frac{1}{2} - \frac{y^2}{2} - y + y^2\right) dy$$
$$= \int_0^1 \left(\frac{1}{2} - y + \frac{y^2}{2}\right) dy = \left[\frac{y}{2} - \frac{y^2}{2} + \frac{y^3}{6}\right]_0^1 = \frac{1}{6}.$$

[6 marks]

Section B

11. For $f(y) = (1+y)^{-1/2}$, we have

$$f'(y) = -\frac{1}{2}(1+y)^{-3/2}, \quad f''(y) = \frac{3}{4}(1+y)^{-5/2}, \quad f^{(3)}(y) = -\frac{15}{8}(1+y)^{-7/2}.$$

At $y = 0$ we have

$$f(0) = 1, \quad f'(0) = -\frac{1}{2}, \quad f''(0) = \frac{3}{4}.$$

So

$$P_2(y) = 1 - \frac{1}{2}y + \frac{3}{8}y^2$$

and

$$R_2(y) = -\frac{15}{8 \times 6}(1+c)^{-7/2}y^3$$

for some c between 0 and y.

[6 marks]

If $y = x^2$ then $y \geq 0$ and if c is betwen 0 and y we have ($c \geq 0$ and

$$0 < (1+c)^{-5/2} \leq 1$$

So

$$|f(x^2) - P_2(x^2)| = |R_2(x^2)| \leq \frac{5}{16}x^6.$$

[2 marks]

Now

$$\int_0^{1/2} P_2(x^2)dx = \int_0^{1/2} \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4\right) dx$$

$$= \left[x - \frac{x^3}{6} + \frac{3}{40}x^5\right]_0^{1/2} = \frac{1}{2} - \frac{1}{48} + \frac{3}{1280} = 0.481510419...$$

[3 marks]

on my calculator

$$\ln(\frac{1}{2} + \frac{\sqrt{5}}{2}) = 0.481211825..$$

[1 mark]

Now

$$\frac{d}{dx}\ln(x + \sqrt{1+x^2}) = \frac{1 + x(1+x^2)^{-1/2}}{x + \sqrt{1+x^2}} = (1+x^2)^{-1/2}\frac{\sqrt{1+x^2}+x}{x+\sqrt{1+x^2}}$$

$$= (1+x^2)^{-1/2}.$$

So

$$\int_0^{1/2} (1+x^2)^{-1/2}dx = \left[\ln(x+\sqrt{1+x^2})\right]_0^{1/2} = \ln(\frac{1}{2} + \frac{\sqrt{5}}{2}) - \ln 1.$$

Since $R_2(x^2)$ is small for $|x| \leq \frac{1}{2}$ (in fact $\leq \frac{5}{16}x^6$) we expect the difference of the integrals of $(1+x^2)^{-1/2}$ and $P_2(x^2)$ between limits 0 and $\frac{1}{2}$ to be small (in fact,

$$\leq \int_0^{1/2} \frac{5x^6}{16} = \frac{5}{14336}.$$

[3 marks]

$6 + 2 + 3 + 1 + 3 = 15$ marks

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 - 4 = 0,$$

that is, $r = \pm 2$. So the complementary solution is $Ae^{2x} + Be^{-2x}$. [3 marks]

(i) We try a particular solution $y_p = Cx + D$. Then $y_p'(x) = C$ and $y_p'' = 0$. So $y_p'' - 4y_p = -4Cx - 4D$. Equating coefficients we get $C = -\frac{1}{4}$ and $D = 0$. So the general solution is

$$y = Ae^{2x} + Be^{-2x} - \frac{1}{4}x.$$

[3 marks]
This gives

$$y' = 2Ae^{2x} - 2Be^{-2x} - \frac{1}{4}$$

So putting $x = 0$ the boundary conditions give

$$A + B = 1, \quad 2A - 2B - \frac{1}{4} = -1 \quad \Rightarrow \quad B = 1 - A, \quad 4A = \frac{5}{4}.$$

So

$$A = \frac{5}{16}, \quad B = \frac{11}{16}.$$

and the solution is

$$\frac{5}{16}e^{2x} + \frac{11}{16}e^{-2x} - \frac{1}{4}x.$$

[3 marks]
(ii) We try $y_p = C\sin x + D\cos x$. Then $y_p'' = -C\sin x - D\cos x$. So $y_p'' - 4y_p = -5C\sin x - 5D\cos x$. So $D = 0$ and $C = -\frac{1}{5}$. So the general solution is

$$y = Ae^{2x} + Be^{-2x} - \frac{1}{5}\sin x$$

[3 marks]
This gives

$$y' = 2Ae^{2x} - 2Be^{-2x} - \frac{1}{5}\cos x.$$

So putting $x = 0$ the boundary conditions give

$$A + B = 1, \quad 2A - 2B - \frac{1}{5} = -1 \quad \Rightarrow \quad B = 1 - A, \quad 4A = \frac{6}{5}.$$

So

$$A = \frac{3}{10}, \quad B = \frac{7}{10}$$

and the solution is

$$\frac{3}{10}e^{2x} + \frac{7}{10}e^{-2x} - \frac{1}{5}\sin x.$$

[3 marks]
$5 \times 3 = 15$ marks
13. We want to minimise $f(x, y) = x^2 + y^2$ (the square of the distance of (x, y) from $(0, 0)$) subject to a constraint in each case a), b). [1 mark]
a)

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$$
$$\nabla g = 2x\mathbf{i} - 4y\mathbf{j}.$$

At a constrained minimum we must have

$$x = \lambda x, \quad y = -2\lambda y.$$

[3 marks]

So $\lambda = 1$ or $x = 0$,

If $\lambda = 1$ then $y = 0$ and the equation $g(x, y) = 1$ gives $x = \pm 1$. We have $f(\pm 1, 0) = 1$.

If $x = 0$ then $g(x, y) = 1$ gives $-2y^2 = 1$, which is impossible. So the minimum distance is 1.

[3 marks]

b)

$$\nabla h = 4y\mathbf{i} + (4x - 6y)\mathbf{j}.$$

At a constrained minimum we must have

$$x = 2\lambda y, \quad y = \lambda(2x - 3y).$$

[2 marks]

Multiplying the first equation by y and the second by x and subtracting we have

$$\lambda(2y^2 - 2x^2 + 3xy = (2y - x)(y + 2x)) = 0.$$

So $x = 2y$ or $y = -2x$, because $\lambda = 0$ gives $x = y = 0$ which is incompatible with $h(x, y) = 1$.

[3 marks]

Substituting the first into $h(x, y) = 1$ we obtain

$$5y^2 = 1$$

So $y = \pm 1/\sqrt{5}$ and $x = \pm 2/\sqrt{5}$. Then $f(x, y) = 1$. Substituting $y = -2x$ gives $-8x^2 - 12x^2 = 1$, which has no solutions. So the minimum distance is 1.

[3 marks]

$1 + 3 + 3 + 2 + 3 + 3 = 15$ marks.

14. The line $x + 2y = 1$ meets the y-axis $x = 0$ at $y = \frac{1}{2}$ and the x-axis $y = 0$ at $x = 1$. So the area of the triangle is given by

$$A = \int_0^{1/2} \int_0^{1-2y} dxdy = \int_0^{1/2} (1 - 2y)dy$$

$$= \left[y - y^2\right]_0^{1/2} = \frac{1}{4}.$$

[5 marks]

Then the centre of mass is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{1}{A} \int_0^{1/2} \int_0^{1-2y} xdxdy = 4 \int_0^{1/2} \left[\frac{x^2}{2}\right]_0^{1-2y} dy$$

$$= 4 \int_0^{1/2} \left(\frac{1}{2} - 2y + 2y^2\right) dy = 4 \left[\frac{y}{2} - y^2 + \frac{2y3}{3}\right]_0^{1/2}$$

$$= 4\left(\frac{1}{4} - \frac{1}{4} + \frac{1}{12}\right) = \frac{1}{3},$$

[5 marks]

$$\overline{y} = 4\int_0^{1/2}\int_0^{1-2y} y dx dy = 4\int_0^{1/2} y(1-2y) dy$$

$$4\left[\frac{y^2}{2} - 2\frac{y^3}{3}\right]_0^{1/2} = 4\left(\frac{1}{8} - \frac{1}{12}\right) = \frac{1}{6}.$$

So the centre of mass is

$$\left(\frac{1}{3}, \frac{1}{6}\right).$$

[5 marks]
[5 + 5 + 5 = 15 marks]