SECTION A

1. Write down the Taylor series about $x = 2$ for the function

$$f(x) = x^{-1}.$$

State whether this Taylor series converges to $f(x)$ for:

a) $x = 3$, b) $x = 4$. [5 marks]

2. Find the solutions of the following differential equations:

(i) $y\dfrac{dy}{dx} - e^x = 0$ with $y(0) = 1$,

(ii) $\dfrac{dy}{dx} + y = e^x$ with $y(0) = 2$.

[7 marks]

3. Solve the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = 2$.

[5 marks]

4. Show, by taking limits along two different paths to the origin $(0, 0)$, that

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4 + y^4}$$

does not exist.

[4 marks]

5. Work out all first and second partial derivatives of

$$f(x, y) = \frac{1}{x^2 + y},$$

and verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

[5 marks]

6. Suppose that f is a function with continuous first and second partial derivatives, and that $g(x, y) = f(u, v)$, where $u(x, y) = x + y$ and $v(x, y) = x - y$. Use the Chain Rule to verify that

$$\frac{\partial^2 g}{\partial x^2}(x, y) + \frac{\partial^2 g}{\partial y^2}(x, y) = 2\left(\frac{\partial^2 f}{\partial u^2}(u, v) + \frac{\partial^2 f}{\partial v^2}(u, v)\right).$$

[6 marks]

7. Find the gradient vector $\nabla f(2, 1, 1)$, where

$$f(x, y, z) = x^2 yz.$$

Find the tangent plane at $(2, 1, 1)$ of the surface $f(x, y, z) = 3$. [5 marks]

8. Locate and classify all stationary points of the function

$$f(x, y) = x^2 y - 2yx + 2y^2 - 3y.$$

[8 marks]

9. Find the linear approximation near $(x, y) = (2, 1)$ to the function

$$f(x, y) = \frac{1}{x^2 - y^2}.$$

[4 marks]

10. By using polar coordinates, evaluate the double integral

$$\int\int_D (x^2 + y^2)^{3/2} dx dy,$$

where D is the unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$. [6 marks]

SECTION B

11.

(i) Throughout this question, $f(x) = \cos x$.

a) Find the Taylor polynomial $P_3(x,0)$ and the remainder term $R_3(x,0)$ for $f(x)$. Show that

$$|R_3(x,0)| \leq \frac{x^4}{24}.$$

b) Find the Taylor polynomial $P_3(x,\pi)$ and the remainder term $R_3(x,\pi)$ for $f(x)$.

c) Find the Taylor polynomial $P_3(x,2\pi)$ and the remainder term $R_3(x,2\pi)$ for $f(x)$.

(ii) Now suppose that

$$y^2 = 0.001 + 2(\cos x - 1).$$

Show that

$$|0.001 - (x^2 + y^2)| \leq \frac{x^4}{12}. \qquad (1)$$

Hint: Use $P_3(x,0)$ and $R_3(x,0)$ from (i)a).

[15 marks]

12. Solve the following differential equations with the given boundary conditions:

(i)

$$y'' + 2y' - 15y = 3x + 2$$

with $y(0) = 1$, $y'(0) = 2$.

(ii)

$$y'' + 2y' - 15y = 13 \sin x$$

with $y(0) = 1$, $y'(0) = 2$.

[15 marks]

13. Find the maximum and minimum of the function

$$f(x, y) = 2x^2 + y^2$$

in the region bounded by the parabola $g(x, y) = 4x - x^2 - y = 0$ and the x-axis $y = 0$.
Any valid method may be used. [15 marks]

14.

a) Find the area of the region R bounded by the line $y = x$ and the parabola $y = x^2 - 2$.

b) Find the centre of mass $(\overline{x}, \overline{y})$ of R, assuming uniform density.

[15 marks]