SECTION A

1. Write down the Taylor series about $x = 0$ for the function

$$f(x) = e^{2x}.$$

State whether this Taylor series converges to $f(x)$ for:
a) $x = 1$, b) $x = 100$. [5 marks]

2. Find the general solution of the following differential equation:

$$y\frac{dy}{dx} + x = 0,$$

and sketch some of the solution curves. Also, find the solution with $y(1) = 1$.

[6 marks]

3. Solve the differential equation

$$\frac{d^2y}{dx^2} - 4y = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = -1$.

[5 marks]

4. Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 + 2y^2}$$

does not exist, by considering limits along two different lines of approach.

[4 marks]

5. Let

$$f(x,y) = \frac{x}{x^2+y^2},$$

and work out

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}.$$

Verify that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

[6 marks]

6. Let $f(x,y) = F(u,v)$ where $u = x+y$ and $v = x-y$ and suppose that all the first and second partial derivatives of f and F exist and are continuous. Work out

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}$$

in terms of partial derivatives of F and verify that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2\left(\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2}\right).$$

[6 marks]

7. Find the gradient vector $\nabla f(x,y,z)$ at (x,y,z), where

$$f(x,y,z) = xy^2 + yz^2 - 2xyz.$$

Also find the tangent plane at $(1,2,1)$ to the surface $f(x,y,z) = 2$. [5 marks]

8. Locate and classify all stationary points of the function

$$f(x,y) = 2x^2 - 2x^2y + y^2.$$

[8 marks]

9. Find the linear approximation near $(x,y) = (2,1)$ to the function

$$f(x,y) = \frac{1}{x-y^2}.$$

[4 marks]

10. Let R be the region in the plane bounded by the x-axis and the parabola $y = x^2 - 1$. Work out the double integral

$$\int\int_R x^2 dxdy.$$

You may change the order of integration if you prefer.

[6 marks]

SECTION B

11.

(i) Find an expression for the Taylor polynomial $P_n(x)$ at 0 for the function $f(x) = \sin x$, and for the remainder term $R_n(x)$ in the cases

a) $n = 4$,

b) $n = 2k$, any $k \geq 0$.

Hence, or otherwise, show that if $0 \leq x \leq 1$, then

$$0 \leq R_4(x) \leq \frac{x^5}{120},$$

and

$$x - \frac{1}{6}x^3 \leq \sin x \leq x - \frac{19}{120}x^3.$$

(ii) By using the Taylor series of $\cos x$ at 0 or otherwise, work out the limits

a) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$,

b) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

In the second one you may wish to rewrite the expression with a common denominator

[15 marks]

12. Solve the following differential equations with the given boundary conditions:

(i)

$$y'' + y = x$$

with $y(0) = 1$, $y'(0) = -1$.

(ii)

$$y'' + y = 2e^{-x}$$

with $y(0) = 1$, $y'(0) = -1$,

[15 marks]

13. Find the maximum and minimum values of the function $f(x, y)$ in the region where $g(x, y) \leq 1$, where $f(x, y)$ and $g(x, y)$ are defined by

$$f(x, y) = x^2 + 3x^2y - y^2,$$
$$g(x, y) = 2x^2 + y^2.$$

[15 marks]

14. Find the centre of mass of the plane triangle bounded by the lines

$$x = 0, \quad y = 0, \quad x + y = 1,$$

where mass is distributed with density function $\rho(x, y)$ and

$$\rho(x, y) = y.$$

[15 marks]