## SECTION A

**1.** Write down the Taylor series about $x = 1$ for the function

$$f(x) = x^{-2}.$$

State whether this Taylor series converges to $f(x)$ for:

a) $x = 0.5$, b) $x = 2$. [5 marks]

**2.** Find the solutions of the following differential equations:

(i) $y\dfrac{dy}{dx} + \sin x = 0$ with $y(0) = 1$,

(ii) $\dfrac{dy}{dx} - y = e^{2x}$ with $y(0) = 2$.

[7 marks]

**3.** Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

with the initial conditions $y(0) = 2$, $y'(0) = -1$.

[5 marks]

**4.** Show, by taking limits along two different paths to the origin $(0, 0)$, that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + y^4}$$

does not exist.

[4 marks]

**5.** Work out all first and second partial derivatives of

$$f(x, y) = \sin(x^2y),$$

and verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

[5 marks]

**6.** Suppose that $f$ is a function with continuous first and second partial derivatives, and that $g(x, y) = f(u, v)$, where $u(x, y) = 2x + y$ and $v(x, y) = -x + 2y$. Use the Chain Rule to verify that

$$\frac{\partial^2 g}{\partial x^2}(x, y) + \frac{\partial^2 g}{\partial y^2}(x, y) = 5\left(\frac{\partial^2 f}{\partial u^2}(u, v) + \frac{\partial^2 f}{\partial v^2}(u, v)\right).$$

[6 marks]

**7.** Find the gradient vector $\nabla f(1, 1, 1)$, where

$$f(x, y, z) = \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz}.$$

Find the tangent plane at $(1, 1, 1)$ of the surface $f(x, y, z) = 3$. [5 marks]

**8.** Locate and classify all stationary points of the function

$$f(x, y) = y^2 x + 2yx + 2x^2 - 3x.$$

[8 marks]

**9.** Find the linear approximation near $(x, y) = (1, 1)$ to the function

$$f(x, y) = \frac{1}{x^2 + y^2}.$$

[4 marks]

**10.** By using polar coordinates, evaluate the double integral

$$\int\int_D e^{x^2+y^2}\, dx dy,$$

where $D$ is the unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$. [6 marks]

SECTION B

**11.**

(i) a) Find the degree 1 Taylor polynomial $P_1(z,0)$ and the remainder term $R_1(z,0)$ for $f(z) = \sqrt{4+z}$. Show that if $|z| \leq 2$ then

$$|R_1(z,0)| \leq \frac{1}{4\sqrt{2}}.$$

b) Find the Taylor polynomial $P_3(x,0)$ and the remainder term $R_3(x,0)$ for $g(x) = \cos x$. Show that

$$|R_3(x,0)| \leq \frac{x^4}{24}.$$

(ii) Now suppose that

$$y^2 = 8 + 2(\cos x - 1).$$

Show that if $y > 0$,

$$\left| y - \frac{\sqrt{2}}{4}(7 + \cos x) \right| \leq \frac{1}{4}.$$

*Hint*: $y = \sqrt{2}f(z)$ for $f$ as in (i)a) and $z = \cos x - 1$. Use the Taylor polynomial $P_1(z,0)$ and the estimate on the remainder term $R_1(z,0)$

[15 marks]

**12.** Solve the following differential equations with the given boundary conditions:

(i)

$$y'' + 4y' + 3y = 3x + 1$$

with $y(0) = 1$, $y'(0) = 2$.

(ii)

$$y'' + 4y' + 3y = 5\sin x$$

with $y(0) = 1$, $y'(0) = 2$.

[15 marks]

**13.** Consider the function

$$g(x, y) = 3x^2 + 5y^2.$$

a) Find the maximum area of a rectangle with horizontal and vertical sides and inscribed in the ellipse $g(x, y) = 10$.

b) Find the maximum of the function $f(x, y) = x^2 - y^2$ in the region $R$ bounded by the ellipse $g(x, y) = 10$, including the interior.

Any valid methods may be used.

[15 marks]

**14.**

a) Find the area of the region $R$ bounded by the parabola $x = y^2$ and the line $y = x - 2$.

b) Find the centre of mass $(\overline{x}, \overline{y})$ of $R$, assuming uniform density.

[15 marks]