SECTION A

1. Write down the Taylor series about $x = 1$ for the function

$$f(x) = x^{1/2}.$$

State whether this Taylor series converges to $f(x)$ for:

a) $x = 0.5$, b) $x = 3$. [5 marks]

2. Find the general solutions of the following differential equations:

(i) $x\dfrac{dy}{dx} - y^2 = 0$,

(ii) $x\dfrac{dy}{dx} - y = x$.

[7 marks]

3. Solve the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 5y = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = -1$.

[5 marks]

4. Determine which of the following limits exist, giving the value of any limits.

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2 + x^3}{x^2 + y^2}$, b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^4 + y^4}$.

[5 marks]

5. Work out all first and second partial derivatives of

$$f(x, y) = x^3 y - x y^3,$$

and verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

[5 marks]

6. Let

$$f(x,t) = F(x+ct) + G(x-ct),$$

where F and G have continuous first and second derivatives. Find

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial t}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial t^2}$$

in terms of the first and second derivatives of F and G. Verify that

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}.$$

[5 marks]

7. Find the gradient vector $\nabla f(x,y,z)$ at (x,y,z), where

$$f(x,y,z) = x^3 y - y^2 z + 2xyz.$$

Also find the directional derivative at $(1,1,1)$ in the direction $(-1,1,2)$.

[5 marks]

8. Locate and classify all stationary points of the function

$$f(x,y) = x^2 + x^2 y + 2y^2.$$

[8 marks]

9. Find the linear approximation near $(x,y) = (1,1)$ to the function

$$f(x,y) = \frac{1}{x^2+y}.$$

[4 marks]

10. Let R be the region in the plane bounded by the x-axis and the parabola $y = 4 - x^2$. Work out the double integral

$$\int\int_R x^2 dxdy.$$

You may change the order of integration if you prefer.

[6 marks]

SECTION B

11.

(i) Find an expression for the Taylor polynomial $P_n(x)$ at 0 for the function $f(x) = \cos x$, and for the remainder term $R_n(x)$ in the cases

a) $n = 3$,

b) $n = 2k - 1$, any $k \geq 1$.

Show also that if $|x| \leq 1$, then

$$0 \leq R_3(x) \leq \frac{x^4}{24},$$

and

$$1 - \frac{11}{24}x^2 \geq \cos x \geq 1 - \frac{1}{2}x^2.$$

(ii) By using the Taylor series of $\cos x$ at 0 or otherwise, work out the limits

a) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$,

b) $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{2(1 - \cos x)} \right)$.

In the second one, you may wish to rewrite the expression with a common denominator.

[15 marks]

12. Solve the following differential equations with the given boundary conditions:

(i)

$$y'' - 2y' + y = 4e^{-x}$$

with $y(0) = 1$, $y'(0) = -1$.

(ii)

$$y'' - 2y' + y = x$$

with $y(0) = 1$, $y'(0) = -1$,

[15 marks]

13. Find the maximum and minimum values of the function $f(x, y)$ in the region where $g(x, y) \leq 1$, where $f(x, y)$ and $g(x, y)$ are defined by

$$f(x, y) = 3x^2 + x^2 y + y^2,$$
$$g(x, y) = 2x^2 + y^2.$$

[15 marks]

14. Find the centre of mass of the plane triangle bounded by the lines

$$x = 0, \quad y = 0, \quad x + 2y = 1,$$

where mass is distributed with density function $\rho(x, y)$ and

$$\rho(x, y) = xy.$$

[15 marks]