Solutions to MATH102 May 2003

1. $f'(x) = -\sin x$ and $f''(x) = -\cos x$. So $f(0) = 1$, $f'(0) = 0$ and $f''(0) = -1$. So the Taylor polynomial $P_2(x, 0)$ is

$$1 - \frac{1}{2}x^2.$$

[4 marks]

2. In standard form

$$\frac{dy}{dx} + \frac{2}{x}y = 8x.$$

So the integrating factor is

$$\exp\left(\int \frac{2}{x}dx\right) = \exp(\log x^2) = x^2.$$

[2 marks]

So the equation becomes

$$x^2\frac{dy}{dx} + 2xy = 8x^3$$

or

$$\frac{d}{dx}(x^2y) = 8x^3$$

Integrating gives

$$x^2y = 2x^4 + C$$

So

$$y = 2x^2 + Cx^{-2}.$$

[2 marks]

3. Try $y = e^{rx}$. We have

$$r^2 + 6r + 5 = (r+1)(r+3) = 0 \Leftrightarrow r = -1 \text{ or } r = -5.$$

[2 marks]

So

$$y = Ae^{-x} + Be^{-5x}, y' = -Ae^{-x} - 5Be^{-5x}.$$

So

$$A + B = 4, \quad -A - 5B = 0.$$

So $B = -1$ and $A = 5$. So

$$y = 5e^{-x} - e^{-5x}.$$

[4 marks]

4.

$$\lim_{(x,y)\to 0, y=0} \frac{x^2 - y^2}{x^2 + 4y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1,$$

$$\lim_{(x,y)\to 0, x=0} \frac{x^2 - y^2}{x^2 + 4y^2} = \lim_{y\to 0} \frac{-y^2}{4y^2} = -\frac{1}{4}.$$

So the limits along different lines through $(0,0)$ are different, and the overall limit as $(x,y) \to (0,0)$ does not exist.

[4 marks]

5.

$$\frac{\partial z}{\partial u} = 2u, \quad \frac{\partial z}{\partial v} = -2v,$$

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial x} = \frac{1}{y}, \quad \frac{\partial v}{\partial y} = \frac{-x}{y^2}.$$

[2 marks]

So

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2uy - 2\frac{v}{y} = 2xy^2 - 2\frac{x}{y^2},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 2ux + 2\frac{vx}{y^2} = 2x^2y + 2\frac{x^2}{y^3}.$$

[2 marks]

6.

$$\nabla f = z\mathbf{i} - 2y\mathbf{j} + (x + 3z^2)\mathbf{k}.$$

So

$$\nabla f(1, -1, 1) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

[2 marks]

So the derivative at P in the direction $(1, 2, 2)$ is

$$\frac{1}{\|\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\|}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}).\nabla f(1, -1, 1) = \frac{1}{\sqrt{1+4+4}}((\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}).(\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = \frac{13}{3}.$$

[2 marks]

The equation of the tangent plane is

$$\nabla f(1, -1, 1).((x-1)\mathbf{i} + (y+1)\mathbf{j} + (z-1)\mathbf{k}) = 0,$$

that is,

$$(\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}).((x-1)\mathbf{i} + (y+1)\mathbf{j} + (z-1)\mathbf{k})$$
$$= (x-1) + 2(y+1) + 4(z-1) = x + 2y + 4z - 3 = 0.$$

[2 marks]

7.

$$\frac{\partial f}{\partial x} = -16x, \quad \frac{\partial f}{\partial y} = 12y + y^2 - y^3.$$

So at critial points we have

$$-16x = 0 = -y(y^2 - y - 12) = y(y+3)(y-4).$$

So the critical points are $(0, 0)$, $(0, -3)$ and $(0, 4)$. [3 marks]

We have

$$A = \frac{\partial^2 f}{\partial x^2} = -16, \quad B = \frac{\partial^2 f}{\partial y \partial x} = 0, \quad C = \frac{\partial^2 f}{\partial y^2} = 12 + 2y - 3y^2.$$

[2 marks]

So at $(0, 0)$ we have $AC - B^2 = -16 \times 12 < 0$, so $(0, 0)$ is a saddle.

At $(0, -3)$ we have $AC - B^2 = -16 \times -21 > 0$ and $A < 0$, so $(0, -3)$ is a local maximum.

At $(0, 4)$ we have $AC - B^2 = -16 \times -28 > 0$ and $A < 0$, so $(0, 4)$ is also a local maximum.

[3 marks]

8.

$$f(x,y) = \frac{1}{1+(x-1)-y} = (1+(x-1-y))^{-1} \approx 1-(x-1-y)+((x-1)-y)^2$$

$$= 1-(x-1)+y+(x-1)^2-2(x-1)y+y^2,$$

because

$$(1+t)^{-1} \approx 1-t+t^2$$

if t is near 0.

Alternatively we can use Taylor's formula for two variables. We have $f(x,y) = (x-y)^{-1}$. So

$$\frac{\partial f}{\partial x} = -(x-y)^{-2}, \quad \frac{\partial f}{\partial y} = (x-y)^{-2},$$

$$\frac{\partial^2 f}{\partial x^2} = 2(x-y)^{-3}, \quad \frac{\partial^2 f}{\partial x \partial y} = -2(x-y)^{-3}, \quad \frac{\partial^2 f}{\partial y^2} = 2(x-y)^{-3}.$$

So at $(x,y) = (1,0)$ we get $f(1,0) = 1$,

$$\frac{\partial f}{\partial x}(1,0) = -1, \quad \frac{\partial f}{\partial y}(1,0) = 1,$$

$$\frac{\partial^2 f}{\partial x^2}(1,0) = 2, \quad \frac{\partial^2 f}{\partial x \partial y}(1,0) = -2, \quad \frac{\partial^2 f}{\partial y^2}(1,0) = 2.$$

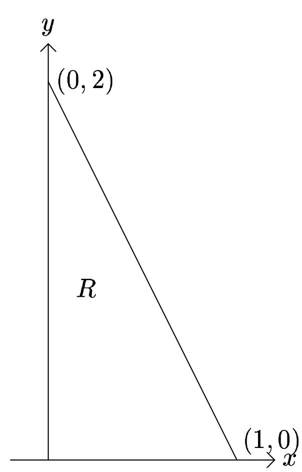
So the quadratic approximation is

$$f(x,y) \approx 1-(x-1)+y+\frac{1}{2}((2(x-1)^2-4(x-1)y+2y^2),$$

which is the same as before.

[5 marks]

9. The region R is as shown.



Then

$$\int\int_R (x+2y)dxdy = \int_0^2 \int_0^{1-y/2} (x+2y)dxdy$$

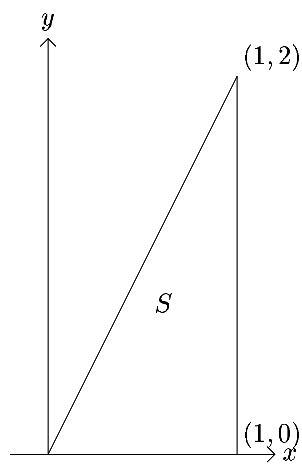
$$= \int_0^2 \left[\frac{x^2}{2} + 2xy\right]_{x=0}^{x=1-y/2} dy = \int_0^2 \left(\frac{(1-y/2)^2}{2} + 2y - y^2\right) dy$$

$$= \int_0^2 \left(\frac{1}{2} + \frac{3y}{2} - \frac{7}{8}y^2\right) dy = \left[\frac{1}{2}y + \frac{3y^2}{4} - \frac{7}{24}y^3\right]_0^2$$

$$1 + 3 - \frac{7}{3} = \frac{5}{3}.$$

[6 marks]

10. The integration is over the region S as shown.



[3 marks]

So in S, $0 \le x \le 1$ and for each fixed $0 \le x \le 1$, $(x,y) \in S \Leftrightarrow 0 \le y \le 2x$. So

$$I = \int_0^2 \left(\int_{y/2}^1 ye^{x^3}\,dx\right) dy = \int\int_S ye^{x^3}\,dxdy$$

$$= \int_0^1 \left(\int_0^{2x} ye^{x^3}\,dy\right) dx = \int_0^1 \left(\left[e^{x^3}\frac{y^2}{2}\right]_{y=0}^{y=2x}\right) dx$$

$$= \int_0^1 2x^2 e^{x^3}\,dx = \left[\frac{2}{3}e^{x^3}\right]_0^1 = \frac{2(e-1)}{3}.$$

[5 marks]

Solutions to Section B

11a). This is a $y = vx$ equation because it can be written in the form $F(y/x)\dfrac{dy}{dx} = G(y/x)$. Putting $y = xv$, we have

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

and

$$(x^2 + x^2 v)\left(v + x\frac{dv}{dx}\right) = x^2 v - x^2 v^2.$$

So dividing through by x^2, we have

$$(1+v)v + (1+v)x\frac{dv}{dx} = v - v^2 = v - v^2.$$

[3 marks]

So

$$(1+v)x\frac{dv}{dx} = v - v^2 - v - v^2 = -2v^2.$$

Separating variables gives

$$\int \frac{1+v}{v^2}dv = \int \frac{-2}{x}dx,$$

or

$$-v^{-1} + \ln v = -2\ln x + C,$$

or

$$\ln v + 2\ln x = \ln(vx^2) = \ln(xy) = v^{-1} = x/y + C.$$

So

$$\ln(xy) = x/y + C.$$

[4 marks]

Putting $x = y = 1$ gives $0 = 1 + C$, and $C = -1$.

[1 marks]

b) This is a linear second order o.d.e. with constant coefficients. For a complementary soution, we try $y = e^{rx}$. Then we have

$$r^2 + 2r + 5 = 0 \Leftrightarrow r = -1 \pm 2i.$$

So a complementary solution is

$$Ae^{-x}\cos 2x + Be^{-x}\sin 2x.$$

[3 marks]

For a particular solution, since we have $10x$ on the righthand side, we try $y = Cx + D$. Then $y' = C$ and $y'' = 0$. So

$$y'' + 2y' + 5y = 2C + 5Cx + 5D = 10x.$$

Equating coefficients, we have $2C + 5D = 0$ and $5C = 10$. So $C = 2$ and $D = -\frac{4}{5}$. So the general solution is

$$y = -\frac{4}{5} + 2x + Ae^{-x}\cos 2x + Be^{-x}\sin 2x.$$

[4 marks]

12. We have $f(x,y,z) = x^2y^2z^2$ and write $g(x,y,z) = x^2 + 4y^2 + 9z^2$. To find the maximum and minimum values of f subject to $g = 27$ we need to first find all solutions of $\nabla f = \lambda \nabla g$ subject to $g = 27$. Now

$$\nabla f(x,y,z) = 2xy^2z^2\mathbf{i} + 2yx^2z^2\mathbf{j} + 2zx^2y^2\mathbf{k},$$

$$\nabla \mathbf{g}(x,y,z) = 2x\mathbf{i} + 8y\mathbf{j} + 18z\mathbf{k}.$$

[5 marks]

So we have to solve

$$xy^2z^2 = \lambda x, \quad yx^2z^2 = 4\lambda y, \quad zx^2y^2 = 9\lambda z.$$

So

$$x = 0 \text{ or } y^2z^2 = \lambda,$$
$$y = 0 \text{ or } x^2z^2 = 4\lambda,$$
$$z = 0 \text{ or} x^2y^2 = 9\lambda.$$

[3 marks]

If one of x, y or $z = 0$ then $f = 0$, which is clearly a minimum of f since $x^2y^2z^2 \geq 0$. So now suppose that all of x, y, $z \neq 0$. Then eliminating λ from the remaining equations, we have

$$4y^2z^2 - x^2z^2 = 0 = x^2y^2 - 9y^2z^2.$$

Since $z \neq 0$ and $y \neq 0$ we have

$$x^2 = 4y^2 = 9z^2.$$

Then $g = 27$ gives

$$27z^2 = 27.$$

So $z^2 = 1$, $y^2 = 9/4$, $x^2 = 9$ and $f(x, y, z) = 81/4$, which must be the maximum of f subject to $g = 27$.

[7 marks]

13. We have

$$f(x, y) = (4 - x^2 - y^2)^{1/2}.$$

So

$$\frac{\partial f}{\partial x} = -x(4 - x^2 - y^2)^{-1/2}, \quad \frac{\partial f}{\partial y} = -y(4 - x^2 - y^2)^{-1/2}.$$

[3 marks]

So

$$\frac{\partial^2 f}{\partial x^2} = -(4 - x^2 - y^2)^{-1/2} - x^2(4 - x^2 - y^2)^{-3/2},$$
$$\frac{\partial^2 f}{\partial x \partial y} = -xy(4 - x^2 - y^2)^{-3/2},$$
$$\frac{\partial^2 f}{\partial y^2} = -(4 - x^2 - y^2)^{-1/2} - y^2(4 - x^2 - y^2)^{-3/2}.$$

[4 marks]

So at $(0, 0)$ all the terms are 0 except that

$$f(0, 0) = 2,$$

$$\frac{\partial^2 f}{\partial x^2}(0, 0) = -4^{-1/2} = -\frac{1}{2} = \frac{\partial^2 f}{\partial y^2}(0, 0).$$

So the second order approximation to $f(x, y)$ is

$$2 - \frac{1}{4}(x^2 + y^2).$$

[4 marks]

The binomial expansion of $(1-z)^{1/2}$ starts

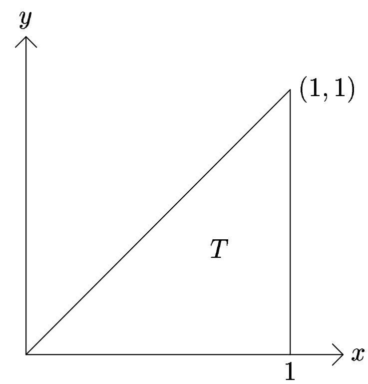
$$1 - \frac{1}{2}z \cdots$$

So

$$(4 - x^2 - y^2)^{1/2} = 4^{1/2}\left(1 - \frac{x^2+y^2}{4}\right)^{1/2} = 2\left(1 - \frac{x^2+y^2}{8}\cdots\right)$$
$$= 2 - \frac{1}{4}(x^2+y^2)\cdots$$

as required. [4 marks]

14. The integration is over the region T as shown.



[3 marks]

Since the density is $x+y$ the mass M satisfies

$$M = \int_0^1 \int_0^x (x+y)dydx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$
$$= \int_0^1 \frac{3x^2}{2}dx = \left[\frac{x^3}{2}\right]_0^1 = \frac{1}{2}$$

as required.

[4 marks]

Then the centre of mass $(\overline{x}, \overline{y})$ satisfies

$$\overline{x} = M^{-1}\int_0^1 \int_0^x x(x+y)dydx = 2\int_0^1 \left[x^2y + x\frac{y^2}{2}\right]_{y=0}^{y=x} dx$$
$$= 2\int_0^1 \frac{3x^3}{2}dx = 2\left[\frac{3x^4}{8}\right]_0^1 = \frac{3}{4},$$
$$\overline{y} = M^{-1}\int_0^1 \int_0^x y(x+y)dydx = 2\int_0^1 \left[x\frac{y^2}{2} + x\frac{y^3}{3}\right]_{y=0}^{y=x} dx$$
$$= 2\int_0^1 \frac{5x^3}{6}dx = 2\left[\frac{5x^4}{24}\right]_0^1 = \frac{5}{12}.$$

So

$$(\overline{x}, \overline{y}) = \left(\frac{3}{4}, \frac{5}{12}\right).$$

[8 marks]