Solutions to MATH102 Practice Exam

Section A

1. The Taylor series of $f(x) = e^{2x}$ is

$$1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

This series is convergent for all x
[4 marks]
2 a)

$$\int \frac{dy}{y} = \int \frac{(x+2)}{x^2} = \int \left(\frac{1}{x} + \frac{2}{x^2}\right) dx,$$

or

$$\ln|y| = \ln|x| - \frac{2}{x} + C.$$

Another way of writing this is

$$y = Axe^{-2/x}.$$

[4 marks]
b) The integrating factor is

$$\exp(-1dx) = e^{-x}.$$

So

$$\frac{d}{dx}(ye^{-x}) = e^{-2x}.$$

So

$$ye^{-x} = -\frac{1}{2}e^{-2x} + C,$$

or

$$y = -\frac{1}{2}e^{-x} + Ce^{x}.$$

[4 marks]
It is also possible to solve this by finding particular and complementary solutions.
3. Trying $y = e^{rx}$, we get

$$r^2 - 2r - 15 = 0 \Rightarrow (r+3)(r-5) = 0 \Rightarrow r = -3 \text{ or } r = 5.$$

[2 marks]
So the general solution is $y = Ae^{-3x} + Be^{5x}$, which gives $y' = -3e^{-3x} + 5Be^{5x}$. So $A + B = 0$ and $5B - 3A = 8B = 4$. So $B = \frac{1}{2}$ and $A = -\frac{1}{2}$. So the solution is

$$-\frac{1}{2}e^{-3x} + \frac{1}{2}e^{5x}.$$

[4 marks]
4. We have

$$\lim_{(x,y)\to 0, y=0} \frac{x^2+y^2}{x^2+2y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1,$$

$$\lim_{(x,y)\to 0, x=0} \frac{x^2+y^2}{x^2+2y^2} = \lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2}.$$

So the limits along the axes are different and the overall limit does not exist.
[4 marks]
5.

$$\frac{\partial f}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial f}{\partial y} = -12x^2y + 4y^3,$$

[2 marks]

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 12y^2, \quad \frac{\partial^2 f}{\partial y \partial x} = -24xy,$$

$$\frac{\partial^2 f}{\partial y^2} = -12x^2 + 12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = -24xy.$$

[3 marks]
So we do indeed have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -24xy$$

and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} =$$

$$12x^2 - 12y^2 - 12x^2 + 12y^2 = 0.$$

[1 mark]
6.

$$\frac{\partial z}{\partial x} = y + \frac{y}{x}, \quad \frac{\partial z}{\partial y} = x + \ln x.$$

We have $x(1,2) = 1$ and $y(1,2) = 1$. Then

$$\frac{\partial z}{\partial x}(1,1) = 1 + 1 = 2, \quad \frac{\partial z}{\partial y}(1,1) = 1 + \ln 1 = 1.$$

[2 marks]
We are given that

$$\frac{\partial x}{\partial u}(1,2) = 1, \quad \frac{\partial x}{\partial v}(1,2) = 2, \quad \frac{\partial y}{\partial u} = -1, \quad \frac{\partial y}{\partial v} = 0.$$

So

$$\frac{\partial z}{\partial u}(1,2) = \frac{\partial z}{\partial x}(1,1)\frac{\partial x}{\partial u}(1,2) + \frac{\partial z}{\partial y}(1,1)\frac{\partial y}{\partial u}(1,2)$$
$$= 2 \times 1 + 1 \times -1 = 1,$$

[2 marks]

$$\frac{\partial z}{\partial v}(1,2) = \frac{\partial z}{\partial x}(1,1)\frac{\partial x}{\partial v}(1,2) + \frac{\partial z}{\partial y}(1,1)\frac{\partial y}{\partial v}(1,2)$$
$$= 2 \times 2 + 1 \times 0 = 4.$$

[2 marks]

7.

$$\nabla f(x,y,z) = (2x - z)\mathbf{i} - 2y\mathbf{j} - x\mathbf{k}.$$

So

$$\nabla f(1,-1,2) = 0\mathbf{i} + 2\mathbf{j} - \mathbf{k} = 2\mathbf{j} - \mathbf{k}.$$

Now

$$\|2\mathbf{i} + \mathbf{j} + 2\mathbf{k}\| = \sqrt{2^2 + 1 + 2^2} = \sqrt{9} = 3.$$

So the derivative in the direction $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is

$$\frac{1}{3}\nabla \mathbf{f}(1,-1,2).(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

$$= \frac{1}{3}(2\mathbf{j} - \mathbf{k}).(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \frac{1}{3} \times (2 - 2) = 0.$$

[4 marks]

8.

$$\frac{\partial f}{\partial x} = 5y^2 - 16x, \quad \frac{\partial f}{\partial y} = 10xy - 18y = 2y(5x - 9).$$

So for a stationary point we must have $y = 0$ or $x = 9/5$. If $y = 0$ then $\partial f/\partial x = 0$ gives $x = 0$. If $x = 9/5$ then $\partial f/\partial x = 0$ gives $y = \pm 12/5$. So the critical points are $(0,0)$ and $(9/5, \pm 12/5)$.

[4 marks]

Now

$$\frac{\partial^2 f}{\partial x^2} = -16, \quad \frac{\partial^2 f}{\partial y \partial x} = 10y, \quad \frac{\partial^2 f}{\partial y^2} = 10x - 18.$$

So at $(x,y) = (0,0)$ we have $A = -16$, $B = 0$, $C = -18$. So $AC - B^2 > 0$ and $A < 0$ and $(0,0)$ is a maximum. At $(x,y) = (9/5, \pm 12/5)$ we have $C = 0$, $B = \pm 24$ and $AC - B^2 < 0$. So both these points are saddles.

[4 marks]

9. We have $f(x,y) = (x^2 + y^2)^{1/2}$.

$$\frac{\partial f}{\partial x} = x(x^2 + y^2)^{-1/2}, \quad \frac{\partial f}{\partial y} = y(x^2 + y^2)^{-1/2}.$$

So

$$f(1,0) = 1, \ \frac{\partial f}{\partial x}(1,0) = 1, \ \frac{\partial f}{\partial y}(1,0) = 0.$$

So for (x,y) near $(1,0)$ we have

$$f(x,y) \approx 1 + (x - 1) = x.$$

[4 marks]
10. The region $D = \{(x, y) : x^2 + y^2 \leq 1\}$ is

$$\{(r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$$

in polar coordinates. Also $dxdy = rdrd\theta$ and $1 + x^2 + y^2 = 1 + r^2$
[3 marks]
So

$$\int\int_D \frac{1}{x^2 + y^2 + 1} dxdy = \int_0^{2\pi} \int_0^1 \frac{r}{1 + r^2} drd\theta$$

$$= 2\pi \left[\frac{\ln(1 + r^2)}{2} \right]_0^1 = \pi \ln 2.$$

[3 marks]

Section B

11a) We have $f'(y) = e^y = f(y)$. So $f^{(k)}(y) = e^y$ for all k and $f^k(0) = 1$ for all k. So

$$P_2(y, 0) = 1 + y + \frac{y^2}{2},$$

$$R_2(y, 0) = e^c \frac{y^3}{6}$$

for some c between 0 and y,

$$P_9(y, 0) = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{4!} + \frac{y^5}{5!}$$

$$+ \frac{y^6}{6!} + \frac{y^7}{7!} + \frac{y^8}{8!} + \frac{y^9}{9!}$$

and

$$R_9(y, 0) = e^c \frac{y^{10}}{10!}$$

for some c between 0 and y.
[6 marks]
If $y = -x$ for $x \geq 0$ and c is between 0 and $-x$ then $c \leq 0$ and $0 < e^c \leq 1$. So since

$$e^{-x} - P_2(-x, 0) = R_2(-x, 0),$$

$$|e^{-x} - P_2(-x, 0)| \leq \frac{x^3}{6}$$

and similarly

$$|e^{-x} - P_9(-x, 0)| \leq \frac{x^{10}}{10!}$$

If $0 \leq x \leq \frac{1}{2}$ we obtain

$$|e^{-x} - P_2(-x, 0)| \leq \frac{(\frac{1}{2})^3}{6} = \frac{1}{48}$$

and if $0 \leq x \leq 2$ we obtain

$$|e^{-x} - P_9(-x,0)| \leq \frac{2^{10}}{10!} = \frac{2^{10}}{10!} = \frac{1024}{5040 \times 720} = \frac{1024}{3628800} < .0003$$

[4 marks]
b) The Taylor series for e^x is

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \cdots$$

The Taylor series for e^{-x} is

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots$$

The Taylor series for e^x is convergent for all x, and equal to e^x, respectively for all x. [3 marks] Since the Taylor series for e^{-x} is simply obtained by substituting $-x$ for x, the same is true for e^{-x}. So

$$\lim_{x \to 0} \frac{1 - x + \frac{x^2}{2} - e^{-x}}{1 + x + \frac{x^2}{2} - e^x} =$$

$$\lim_{x \to 0} \frac{1 - x + \frac{x^2}{2} - (1 - x + \frac{x^2}{2} - \frac{x^3}{3!} \cdots)}{1 + x + \frac{x^2}{2} - (1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \cdots)}$$

$$= \lim_{x \to 0} \frac{\frac{x^3}{3!}}{-\frac{x^3}{3!}} = -1.$$

[2 marks]

It is also possible to prove this using l'Hopital's Rule.

12a) Making the substitution $y = vx$, we have

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

So

$$v + x\frac{dv}{dx} = \frac{xy}{x^2 - y^2} = \frac{v}{1 - v^2}.$$

So

$$x\frac{dv}{dx} = \frac{v - (v - v^3)}{1 - v^2} = \frac{v^3}{1 - v^2}.$$

So

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{dx}{x}$$

and

$$-\frac{v^{-2}}{2} - \ln|v| = \ln|x| + C.$$

So

$$-\frac{x^2}{2y^2} - \ln|y| + \ln|x| = \ln|x| + C.$$

So

$$-\frac{x^2}{2y^2} - \ln|y| = C.$$

[6 marks]
12b) Trying $y = e^{rx}$ for the complementary solution, we have

$$(r^2 + 2r - 3 = 0 \Leftrightarrow (r+3)(r-1) = 0 \Leftrightarrow r = 1 \text{ or } r = -3.$$

So the complementary solution is $y = Ae^x + Be^{-3x}$.
[2 marks]

For the particular soution we try $y = C\cos x + D\sin x$. So $y' = -C\sin x + D\cos x$ and $y'' = -C\cos x - D\sin x$ and

$$y'' + 2y' - 3y = (-4C + 2D)\cos x + (-4D - 2C)\sin x = \cos x.$$

So equating coeffiients of $\sin x$, $C = -2D$ and equating coefficients of $\cos x$, $D = \frac{1}{10}$. So

$$y(x) = -\frac{1}{5}\cos x + \frac{1}{10}\sin x + Ae^x + Be^{-3x}.$$

So

$$y'(x) = \frac{1}{5}\sin x + \frac{1}{10}\cos x + Ae^x - 3Be^{-3x}.$$

[4 marks]

So

$$1 = \frac{-1}{5} + A + B, \quad -1 = \frac{1}{10} + A - 3B.$$

Subtracting these,

$$2 = -\frac{3}{10} + 4B \Rightarrow B = \frac{23}{40}$$

So

$$A = \frac{6}{5} - \frac{23}{40} = \frac{25}{40} = \frac{5}{8}.$$

So

$$y = \frac{5}{8}e^x + \frac{23}{40}e^{-3x} - \frac{1}{5}\cos x + \frac{1}{10}\sin x.$$

[3 marks]
13.

We have

$$f(x,y,t) = (x - 2 + t)^2 + (y - t)^2$$

and

$$g(x,y) = x^2 + 2y^2 = 1.$$

So

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial t}\mathbf{k}$$

$$= 2(x-2+t)\mathbf{i} + 2(y-t)\mathbf{j} + (2(x-2+t) - 2(y-t))\mathbf{k}$$

and

$$\nabla \mathbf{g} = 2x\mathbf{i} + 4y\mathbf{j}.$$

[4 marks]

At a minimum, we must have

$$\nabla \mathbf{f} = \lambda \nabla \mathbf{g}.$$

[1 mark] So we have

$$\begin{array}{lcl} 2(x-2+t) & = & 2\lambda x, \\ 2(y-t) & = & 4\lambda y, \\ x-2+t-y+t=0. & & \end{array}$$

So we have

$$2t = y + 2 - x.$$

So substituting this in the first two equations gives:

$$\begin{array}{lllll} 2x-4+y+2-x & = x+y-2 & = & 2\lambda x, & \\ 2y-y-2+x & = & x+y-2 & = & 4\lambda y. \end{array}$$

So we have

$$2\lambda x = 4\lambda y.$$

So either $\lambda = 0$ or $x = 2y$. But $\lambda = 0$ gives $x + y - 2 = 0$ from the equations above. But (x, y) is a point on the ellipse, and the line and the ellipse do not intersect. So we have

$$x = 2y.$$

[5 marks] Substituting $x = 2y$ in $g(x, y) = x^2 + 2y^2 = 1$, we obtain $6y^2 = 1$. So

$$(x, y) = \pm(\sqrt{6}/3, \sqrt{6}/6)$$

So

$$(x, y, t) = (\sqrt{6}/3, \sqrt{6}/6, 1 - \sqrt{6}/12) \text{ or } (-\sqrt{6}/3, -\sqrt{6}/6, 1 + \sqrt{6}/12)$$

So

$$f(x,y,t) = (\sqrt{6}/4-1)^2+(\sqrt{6}/4-1)^2 = 2(\sqrt{6}/4-1)^2 \text{ or } (-\sqrt{6}/4-1)^2+(-\sqrt{6}/4-1)^2 = 2(-\sqrt{6}/4-1)^2$$

The first is smaller. So the minimum distance is

$$\sqrt{2}(1 - \sqrt{6}/4) = \sqrt{2} - (\sqrt{3}/2).$$

[5 marks]
14.
The centre of mass is $(\overline{xy})$ where

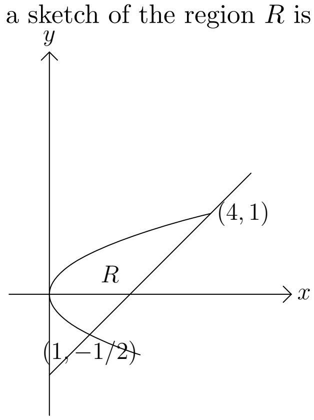
$$A = \int\int_R dxdy,$$

$$\overline{x} = \frac{1}{A}\int\int_R xdxdy, \quad \overline{y} = \frac{1}{A}\int\int_R ydxdy.$$

We have

$$4y^2 = 2y + 2 \Leftrightarrow 2y^2 - y - 1 = 0 \Leftrightarrow (2y+1)(y-1) = 0.$$

So a sketch of the region R is as shown.



[3 marks]
The area A is

$$\int_{-1/2}^{1}\int_{4y^2}^{2y+2} dxdy = \int_{-1/2}^{1}(2y + 2 - 4y^2)dy$$

$$= \left[y^2 + 2y - \frac{4y^3}{3}\right]_{-1/2}^{1} = 1 + 2 - \frac{4}{3} - \frac{1}{4} + 1 - \frac{1}{6} = \frac{9}{4}$$

[4 marks]
14b) Then

$$\overline{x} = \frac{1}{A}\int_{-1/2}^{1}\int_{4y^2}^{2y+2} xdxdy$$

$$= \frac{4}{9}\int_{-1/2}^{1}\left[\frac{x^2}{2}\right]_{4y^2}^{2y+2} dy = \frac{2}{9}\int_{-1/2}^{1}(4y^2 + 8y + 4 - 16y^4)dy$$

$$= \frac{2}{9}\left[\frac{4y^3}{3} + 4y^2 + 4y - \frac{16y^5}{5}\right]_{-1/2}^{1} = \frac{2}{9}\left(\frac{4}{3} + 4 + 4 - \frac{16}{5} + \frac{1}{6} - 1 + 2 - \frac{1}{10}\right)$$

$$= \frac{2}{9}.\frac{36}{5} = \frac{8}{5}$$

[3 marks]

$$\overline{y} = \frac{1}{A}\int_{-1/2}^{1}\int_{4y^2}^{2y+2} y dx dy$$

$$= \frac{4}{9}\int_{-1/2}^{1}(2y^2 + 2y - 4y^3)dy = \frac{4}{9}\left[\frac{2y^3}{3} + y^2 - y^4\right]_{-1/2}^{1}$$

$$= \frac{4}{9}\left(\frac{2}{3} + 1 - 1 + \frac{1}{12} - \frac{1}{4} + \frac{1}{16}\right) = \frac{4}{9}.\frac{9}{16} = \frac{1}{4}.$$

So

$$(\overline{xy}) = \left(\frac{8}{5}, \frac{1}{4}\right).$$

[5 marks]

[$3 + 4 + 3 + 5 = 15$ marks.]