Domain and Range

For a function $f(x,y)$ of real variables x, y, and taking real values, the *domain* is the set of (x,y) for which $f(x,y)$ is defined. The *range* is the set of real numbers t for which $t = f(x,y)$ for at least one (x,y).

If U is the domain of f, then we write

$$f : U \to \mathbb{R},$$

meaning that f is a function with domain U and range a subset of $\mathbb{R}$.

Continuity and Limits

If f is a function taking real values whose domain includes points (x,y) arbitrarily near (x_0,y_0) then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$$

means that $|f(x,y) - \ell|$ can be taken arbitrarily small by taking $|x - x_0|$ and $|y - y_0|$ sufficiently small.

If (x_0,y_0) is in the domain of f, we say that f *is continuous at* (x_0,y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

We say that f *is continuous* if f is continuous at all points of the domain.

For a function of two (or more) variables, it may happen that limits exist *in some directions* and not in others, or limits along different paths might be different. Limits along a line are commonly considered. For example, we write

$$\lim_{(x,y)\to(0,0),y=kx} f(x,y)$$

for

$$\lim_{x\to 0} f(x,kx),$$

and

$$\lim_{(x,y)\to(0,0),x=0} f(x,y)$$

for

$$\lim_{y\to 0} f(0,y).$$

Many functions are continuous. All polynomials are continuous. For example

$$x^2, \quad x^3y + xy^2, \quad xyz = z^4$$

are all continuous.

sin, cos, exp are continuous. The function log is continuous on its domain $\{x : x > 0\}$.

The sum $f+g$, difference $f-g$, product fg are all continuous if f and g are continuous.

If f is continuous, defined on a subset of $\mathbb{R}^n$ then $1/f$ is continuous on its domain, which is the set

$$X = \{\underline{x} : f(\underline{x}) \neq 0\}.$$

Hence if g and f are continuous, g/f is continuous on the intersection of X with the domain of g.

If f and g are continuous the composition $f \circ g$ is continuous on its domain, where

$$f \circ g(\underline{x}) = f(g(\underline{x})).$$

Partial Differentiation

Suppose that f is a function defined on a subset of $\mathbb{R}^2$. Then the *partial derivatives of f at* (x_0, y_0) are defined by

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

if these limits exist. Similar definitions are made if f is a function of n variables.

Approximation using Partial Derivatives

If $\partial f/\partial x(x_1, y_1)$ and $\partial f/\partial y(x_1, y_1)$ are both continuous for all (x_1, y_1) near (x_0, y_0), then for (x_1, y_1) near (x_0, y_0),

$$f(x_1, y_1) \approx f(x_)\text{, } y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x_1 - x_0)$$

$$+ \frac{\partial f}{\partial y}(x_0, y_0)(y_1 - y_0).$$