An *n'th order ordinary differential equation* is an equation of the form

$$F(x, y(x), y'(x), \cdots y^{(n)}(x)) = 0,$$

for a function F. Here are some examples

$$\frac{d^2y}{dx^2} + 1 = 0, \tag{1}$$

$$\frac{dy}{dx} - y = 0, \tag{2}$$

$$y\frac{dy}{dx} + \sin x = 0. \tag{3}$$

(1) is a second order equation and (2) and (3) are first order. (1) and (3) can be related to simple mechanics problems, and (2) to a simple model of population growth. (1) and (2) are *linear* diferential equations but (3) is not. An n'th order differential equation is *linear* if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + \cdots a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

The equation is *homogeneous* if $b(x) = 0$ for all x and *with constant coefficients* if the $a_i(x)$ are all constant functions. The equations (1) and (2) are *linear with constant coefficients*. (2) is homogeneous but (1) is not. The main questions about differential equations are:

. are there solutions?

. if so how many solutions are there?

. how can we find the solutions?

The answer to the first question is usually "yes" or at least there are usually local solutions $y(x)$ defined for some interval of x. If a differential equation has any solutions at all it probably has infinitely many. Very roughly, we expect a first order differential equation to have one variable constant in the solution, and we expect an n'th order differential equation to have n variable constant in the solution.

As for finding the solutions: there are a number of different methods. We shall study:

1. first order differential equations which are *separable*, that is, can be written in the form

$$f(y)\frac{dy}{dx} = g(x);$$

2. first order differential equations of the form

$$\frac{dy}{dx} = h(y/x);$$

3. linear first order differential equations, and solution by the integrating factor method;

4. the solution of linear second order differential equations with constant coefficients.

1. If

$$f(y)\frac{dy}{dx} = g(x)$$

then

$$\int f(y)dy = \int g(x)dx,$$

and integrating both sides gives and equation relating x and y, and gives y as a function of x in some cases.
2. If

$$\frac{dy}{dx} = h(y/x)$$

then we make the substitution $y = xv$. Then

$$\frac{dy}{dx} = x\frac{dv}{dx} + v = h(v).$$

So

$$x\frac{dv}{dx} = h(v) - v.$$

So

$$\int \frac{dv}{h(v) - v} = \int \frac{dx}{x}$$

and integrating both sides gives an equation relating x and v, which is also an equation relating y and x since $v = y/x$, and this gives y as a function of x in some cases.
3. If

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

then

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{4}$$

where

$$P(x) = \frac{a_0(x)}{a_1(x)}, \quad Q(x) = \frac{b(x)}{a_1(x)}.$$

The *integrating factor* is then

$$f(x) = \exp\left(\int^x P(t)dt\right).$$

Then

$$\frac{df}{dx} = P(x)f(x).$$

Multiplying (4) by $f(x)$ gives

$$f(x)\frac{dy}{dx} + f(x)P(x)y = f(x)Q(x),$$

that is

$$f(x)\frac{dy}{dx} + \frac{df}{dx}y = f(x)Q(x),$$

that is,

$$\frac{d}{dx}(f(x)y) = f(x)Q(x),$$

that is

$$f(x)y = \int^x f(t)Q(t)dt.$$

4. *Linear Homogeneous Case.*

To solve

$$a_2y'' + a_1y' + a_0y = 0, \tag{5}$$

we try a solution $y = e^{rx}$. Then $y' = re^{rx}$ and $y'' = r^2e^{rx}$. So to solve (5), we need

$$a_2r^2 + a_1r + a_0 = 0. \tag{6}$$

If $r = r_1$ is a solution of (6) then $y = Ae^{r_1x}$ is a solution of (5) for any real (or complex) number A. If $r = r_1$ and $r = r_2$ are both solutions of (6) then

$$y = Ae^{r_1x} + Be^{r_2x}$$

is a solution of (5) for any A and B. If $r_1 \neq r_2$ then this is the *general solution.* If $r = r_1$ is a *repeated* solution of (6) then

$$y = (Ax + B)e^{r_1x}$$

is a solution of (5) for any A and B, and this is the *general* solution of (5).

If $r = \alpha + i\beta$ is a complex solution of (6) with α, β real and $\beta \neq 0$ then $r = \alpha - i\beta$ is another solution of (6). Then

$$y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} = e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x})$$

is a solution of (5) for any complex A and B. So then

$$y = e^{\alpha x}(C\cos(\beta x) + D\sin(\beta x))$$

is a solution of (5) for any real (or complex) numbers C and D, and this is the *general* solution.

Linear Inhomogeneous case

To solve

$$a_2 y'' + a_1 y' + a_0 y = f(x), \tag{7}$$

note that if $y = y_p$ is a solution of (7) and $y = y_c$ is the general solution of (5) then $y = y_p + y_c$ is a solution of (7) because

$$\begin{aligned} & a_2(y_p'' + y_c'') + a_1(y_p' + y_c') + a_0(y_p + y_c) \\ = & a_2 y_p'' + a_1 y_p' + a_0 y_p + a_2 y_c'' + a_1 y_c' + a_0 y_c \\ = & f(x) + 0 = f(x). \end{aligned}$$

In fact $y_p + y_c$ is the *general solution* of (7), because if $y = y_p$ and $y = z_p$ are both solutions of (7) then $y = y_p - z_p$ is a solution of (5).

The general solution $y = y_c$ of (5) is called the *complementary solution* for (7). Any solution $y = y_p$ of (7) is called a *particular solution* of (7). So to find the general solution of equations of the form (7) we need to develop our technique for finding particular solutions. *Example* To find the general solution of

$$y'' + y' - 2y = e^{2x}$$

we need to find the complementary solution and a particular solution. To find the complementary solution we look for solutions of

$$r^2 + r - 2 = (r + 2)(r - 1) = 0 \Rightarrow r = 1 \text{ or } r = -2.$$

So the complementary solution is

$$Ae^x + Be^{-2x}.$$

To find a particular solution we try $y_p(x) = Ce^{2x}$. If we take this then

$$y_p'(x) = 2Ce^{2x}, \quad y_p''(x) = 4Ce^{2x},$$

$$y_p'' + y_p' - 2y_p = (4C + 2C - 2C)e^{2x} = 4Ce^{2x}.$$

So Ce^{2x} is a particular solution if and only if

$$C = \frac{1}{4}.$$

So the general solution is

$$Ae^x + Be^{-2x} + \frac{1}{4}e^{2x}.$$