Taylor's formula for functions of two variables , up to second derivatives.

Remember that the degree two Taylor polynomial at 0 for a function $g = g(t)$ of one variable is

$$g(0) + tg'(0) + \frac{t^2}{2} g''(0),$$

and if t is small and the second derivative is continuous,

$$g(t) \approx g(0) + tg'(0) + \frac{t^2}{2} g''(0).$$

Now let $f = f(x, y)$ be a function of two variables. If (x, y) is near (a, b) and the first derivatives of f are continuous then we know that

$$f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

Just as in one variable, there is a better approximation using second derivatives, if these are continuous. Write

$$A = \frac{\partial^2 f}{\partial x^2}(a, b), \quad C = \frac{\partial^2 f}{\partial y^2}(a, b),$$

$$B = \frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

Then if (x, y) is near (a, b),

$$\begin{aligned} &f(x, y) \approx f(a, b) + \tfrac{\partial f}{\partial x}(a, b)(x - a) + \tfrac{\partial f}{\partial y}(a, b)(y - b) \\ &+ \tfrac{1}{2}(A(x - a)^2 + 2B(x - a)(y - b) + C(y - b)^2). \end{aligned} \tag{1}$$

This can be derived form the 1-dimensional Taylor formula, using the chain rule. Let

$$g(t) = f(x(t), y(t)),$$

and

$$x(t) = a + t(x - a), \quad y(t) = b + t(y - b).$$

Then

$$x'(t) = x - a, \quad y'(t) = y - b,$$
$$x''(t) = y''(t) = 0.$$

Then using the chain rule,

$$g'(t) = \frac{\partial f}{\partial x}(x(t), y(t)).x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)).y'(t),$$

$$g''(t) = \frac{\partial^2}{\partial x^2}(x(t), y(t))(x'(t))^2$$

$$+2\frac{\partial^2 f}{\partial x \partial y}(x(t), y(t))x'(t)y'(t) + \frac{\partial^2 f}{\partial y^2}(x(t), y(t))(y'(t))^2.$$

So, putting $t = 0$, $x(0) - a$, $y(0) = b$ and

$$g'(0) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(y-b)$$

and

$$g''(0) = A(x-a)^2 + 2B(x-a)(y-b) + C(y-b)^2.$$

Then using

$$g(1) \approx g(0) + g'(0) + \frac{1}{2}g''(0)$$

gives (1). Also, as in one variable, we can use the second derivatives to estimate the error in the first derivative approximation. Suppose that M is any number such that, for all x' between a and x and all y' between b and y,

$$\left|\frac{\partial^2 f}{\partial x^2}(x',y')\right| \le M, \quad \left|\frac{\partial^2 f}{\partial y^2}(x',y')\right| \le M,$$

$$\left|\frac{\partial^2 f}{\partial x \partial y}(x',y')\right| \le M.$$

Then

$$\left|f(x,y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b)\right|$$

$$\le \frac{M}{2}(|x-a| + |y-b|)^2.$$

Deriving the two-variable stationary point test from Taylor's formula

If(a,b) is a stationary point then

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0.$$

So then Taylor's formula for $f(x,y)$ for (x,y) near (a,b) becomes

$$f(x,y) - f(a,b)$$

$$\approx \frac{1}{2}(A(x-a)^2 + 2B(x-a)(y-b) + C(y-b)^2)$$

$$= \frac{A}{2}\left(x - a + \frac{B}{A}(y-b)\right)^2 + \frac{AC-B^2}{2A}(y-b)^2$$

if $A \neq 0$. So

$$f(x,y) - f(a,b) \begin{array}{ll} \ge 0 & \text{if } A > 0,\ AC - B^2 > 0, \\ \le 0 & \text{if } A < 0,\ AC - B^2 > 0, \\ \text{both} & \text{if } AC - B^2 < 0. \end{array}$$

This gives the alternatives of local minimum, local maximum and saddle at (a,b).