The Resident's View Revisited

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I would like to start by saying a bit about how I got into this. My first postdoc in 1978-9, was at the Institute for Advanced Study. This in itself was because of a twist of fate which had nothing to do with Bill Thurston. But there I was, and in 1978-9, Bill was running a course in hyperbolic geometry. There were some notes from the previous year. I think he moved onto three manifolds pretty fast, and those notes were in the process of being written. One section came out during the year, and another the following year. I still have these. I was a dynamicist (by instinct and, although I had a fantastic supervisor, fairly limited training because of various circumstances) with no experience in hyperbolic geometry, although I had become very interested in homogeneous Lie group actions. So although I attended many of Bill's lectures, I felt pretty much out of it. I was struggling to understand early chapters of the notes and when Bill realised during one session (for example) that he had completed a proof of the Smith Conjecture, I did not really notice, or more basically, had no idea what he was talking about (as usual).

For me it was a very tough year. However, at some point I realised that this was mathematics as I had never known it before, a real struggle, and the highly visual approach was completely new to me, but it did make perfect sense when I actually got it, or part of it. I was 25, and it felt like the start of my education. It changed the way I worked on mathematics, permanently.

I only had part of the picture that year, and although of course my understanding remains very partial, I subsequently learnt more about Bill's approach, and other aspects of it. Two years later, I was in Paris for a year, and before I went there I read most of the "Travaux de Thurston" on the isotopy classsification of surface homeomorphisms, which was written (mostly) by Fathi, Laudenbach and Poénaru. I already knew Fathi, and that year I got to know Laudenbach quite well, and also Poénaru. As I have mentioned in my abstract, it is now impossible to study dynamics in two dimensions without an understanding of this work of Bill Thurston. This work also illustrates the way in which dynamical ideas enter into Bill's work constantly. And, of course, the work brought to the fore important dynamical systems of which there has been intense study in subsequent decades, with this study contributing to studies of geometry and vice versa. (Dynamicists in general love trespassing into other areas.) Other mathematics was emerging into the limelight at this time, of course, and while I was in Paris, Dennis Sullivan, Adrien Douady, John Hubbard and Michel Herman were all making connections with complex dynamics. Sullivan formulated the dictionary translating ideas between hyperbolic geometry and complex dynamics. At some point, Bill Thurston got interested, and in 1982 he proved the classification result about critically finite branched coverings. This result has had a profound effect on work in complex dynamics ever since. There are also multiple connections with Bill's work in hyperbolic geometry and surface homeomorphisms which are intriguing. I find these connections particularly interesting and I think we still only have some facets of them. Although I this talk is announced as a "revisit", I am not going to say much about new proof of results which appeared in Astérisque 288 (2003) in the light of later developments. There are some very open-ended open problems which I will try to expound/promote.

Branched coverings

- The sphere and the torus are the only compact surfaces to admit self-branched coverings of degree greater than one.
- Branched coverings of the sphere which are not homeomorphisms necessarily have branch points, also called critical values.
- Rational maps are examples of branched coverings.
- The dynamics of rational maps is one of the main areas of complex dynamics.

(Transcendental dynamics has grown in importance in recent decades, and dynamics of Fuchsian and Kleinian groups is part of the larger study of these, with parallels pointed out by Sullivan.)

Critically finite branched coverings

A branched covering f is said to be *critically finite* if the *postcritical set*

$$X(f) = \{f^n(c) : c \text{ critical }, n > 0\}$$

is finite.

Two critically finite branched coverings f_0 and f_1 are usually said to be *Thurston equivalent* if there is a homotopy f_t ($t \in [0,1]$) through critically finite branched coverings such that $X(f_t)$ varies isotopically for $t \in [0,1]$.

In this talk a slightly stronger notion of Thurston equivalence will be used. We consider branched coverings of the Riemann sphere for which the critical values are numbered, and then f_0 and f_1 are said to be Thurston equivalent if the isotopy of $X(f_t)$ preserves the numbering of critical values.

Thurston's Theorem for critically finite branched coverings (1982)

The quotient by Möbius conjugation of a Thurston equivalence class is contractible to the Möbius conjugacy class of a unique rational map, if and only if a certain orbifold is hyperbolic, and a certain combinatorial condition holds, which can be described as the non-existence of a Thurston obstruction.

Thurston's theorem has been deliberately formulated as a result about the topology of a space of maps. It is a geometrisation result in two ways.

It gives a condition under which a map is holomorphic, modulo the appropriate type of homotopy equivalence (Thurston equivalence).

It also shows that the corresponding space of maps is contractible to a space with a geometric structure – although there is little to say about geometries on a space consisting of a single point. But a point is just the simplest case...

A generalisation

The elements of a connected topological space $B = B(f_0, Y(f_0))$ are $[f, Y(f)]$, and include $[f_0, Y(f_0)]$, where:

- f is a branched covering of the Riemann sphere;
- $Y(f)$ is a finite set which contains all the critical values, which are numbered, and is the union of the set of critical values and $Z(f)$, where $f(Z(f)) \subset Z(f)$;
- $Y(f)$ and$Z(f)$ vary isotopically with f, for $[f, Y(f)] \in B$;
- $[f, Y(f)]$ denotes the conjugacy class of $(f, Y(f))$ by Möbius transformations, using only Möbius transformations which preserve the numbering of critical values.

For example, B could be the Thurston equivalence class of a critically finite branched covering f_0, with $Y(f_0) = Z(f_0) = X(f_0)$.

Why?

The homotopy class of a map, suitably defined, in many cases, gives a surprising amount of information about the dynamics of the map, via a semiconjugacy. Such results have a long history, with earlier folklore results predating John Franks' thesis in 1968 (on homeomorphisms which are homotopic to hyperbolic toral automorphisms). A result of this type holds for critically finite branched coverings which are Thurston equivalent to rational maps.

Dynamicists are always interested in families of dynamical systems, not just single ones. This is especially true of researchers in complex dynamics. Probably this is partly because it is relatively easy to see variation of dynamics in families of complex dynamical systems, at least in a simplistic sense. One can see how dynamics of critical values vary and, roughly speaking, in many cases, a set of cases which is conjecturally dense, the dynamics of the (finitely many) critical values of a map control the dynamics of all points. But variation of the dynamics of critical values is related to the topology of the space of maps. So that is why.

The connection with Teichmüller space

If $[f_t, Y(f_t)]$ is a path in B then $f_t = \varphi_t \circ f_0 \circ \psi_t^{-1}$ where:

- φ_0 and ψ_0 are the identity is the identity and $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ are isotopies;
- $\varphi_t(Y(f_0)) = Y(f_t)$, $\psi_t(Z(f_0) = Z(f_t)$ and φ_t and ψ_t are isotopic homeomorphisms through an isotopy which is constant on $Z(f_0)$;
- φ_t is determined up to isotopy constant on $Y(f_0)$, and post-composition by a Möbius transformation, by $[f_t, Y(f_t)]$: that is, as an element $[\varphi_t]$ of the Teichmüller space $\mathcal{T}(Y(f_0))$ of the sphere with marked set bijective to $Y(f_0)$.

This correspondence maps the universal cover of B to $\mathcal{T}(Y(f_0))$ with contractible fibres, so that B is a $K(\pi, 1)$. The fundamental group maps to a subgroup of the pure mapping class group $PMG(\overline{\mathbb{C}}, Y(f_0))$. Since B is a $K(\pi, 1)$, the Topographer's View is a result about the structure of its fundamental group.

The Topographer's View

I obtained a result, or sequence of results, which I called *The Topographer's View* – which I do not particularly want to revisit at this juncture, but it is not possible to separate the Topographer and Resident's views completely, because they complement each other.

The Topographer's View of $B = B(f_0, Y(f_0))$ is a homotopy equivalence to an ordered graph of countably many topological spaces of maps, where these topological spaces have some geometric structure.

The "base" geometric pieces are the rational maps in B quotiented by Möbius conjugation.

An important part of the result, not easy, is that the inclusion of each component V of rational maps in the larger space B is injective on π_1.

This result was only obtained for B consisting of degree two maps, or maps "of polynomial type", $Z(f_0)$ contained in the full orbit of periodic critical points.

An open question

If a space is made up of simpler pieces, where these pieces might have some geometric structure, then how these pieces are glued together is obviously an important part of the definition of the topology. Obtaining results about the gluing was a major task in the proof of the Topologist's View. But there were some important questions that I was unable to answer. The main one was the order in which handles were glued. I initially thought that all handles would attach just one geometric piece to one other, but I was unable to prove this, obtaining instead an ordering of pieces so that each piece was attached by handles to a union of lower order pieces.

Of course, this question has a group-theoretic interpretation, about the structure of the fundamental group.

Question: nature of the embedding

Having seen that the universal cover $\widetilde{V}$ of V embeds in $\mathcal{T}(Y(f_0))$, one can obviously ask about the nature of the embedding. From the group-theoretic point of view, this is a question about a subgroup of the Pure Mapping Class Group $PMG(\overline{\mathbb{C}}, Y(f_0))$ which identifies with the fundamental group of the space of rational maps. Of course the embedding is Lipschitz, but I know nothing about the inverse map.

A projection of the embedding

In the cases that we consider, f_0 is of degree two, and $Y(f_0) \setminus Z(f_0) = Y \setminus Z$ is a single point, a critical value of f_0, denoted by $v_2 = v_2(f_0)$.

There is a natural projection from $\mathcal{T}(Y)$ to the universal cover $\widetilde{\overline{\mathbb{C}} \setminus Z}$ of $\overline{\mathbb{C}} \setminus Z$, defined as follows.

Let $\pi_Z : \mathcal{T}(Y) \to \mathcal{T}(Z)$ denote the natural projection and let d_Y and d_Z denote the Tiechmüller metrics on $\mathcal{T}(Y)$ and $\mathcal{T}(Z)$ If $[\varphi_t]$ is a path in $\mathcal{T}(Y)$ from a basepoint $[\varphi_0] = x_0$, then let $\chi_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the unique homeomorphism minimizing qc-distortion such that

$$\pi_Z([\varphi_t]) = \pi_Z([\chi_t \circ \varphi_0])$$

Then $t \mapsto \chi_t^{-1} \circ \varphi_t(v_2)$ defines an element of $\widetilde{\overline{\mathbb{C}} \setminus Z}$.

The composition with the embedding gives a map

$$\rho : \widetilde{V} \to \widetilde{\overline{\mathbb{C}} \setminus Z}.$$

The Resident's View Theorem (or part of it)

In the cases considered, V is a finite type Riemann surface of negative Euler characteristic, and so, of course is $\overline{\mathbb{C}} \setminus Z$. So the universal cover of each of these Riemann surfaces is the unit disc up to conformal equivalence, with boundary the unit circle.

$\rho : \widetilde{V} \to \widetilde{\overline{\mathbb{C}} \setminus Z}$ *extends continuously and monotonically to map the boundary* $\partial\widetilde{V}$ *into* $\partial\widetilde{\overline{\mathbb{C}} \setminus Z}$ *with just countably many discontinuities which can be naturally characterised, and where right and left limits exist.*

Once the continuity is proved, monotonicity is straightforward.

The limit of ρ along geodesics

We use d_P to denote the Poincaré (or hyperbolic) metric on the unit disc (the universal cover of $\overline{\mathbb{C}} \setminus Z$).

Theorem 1. $\lim_{x\to\infty} \rho(x)$ *exists along any half geodesic segment* ℓ *in* $\mathcal{T}(Y)$ *such that* $\lim_{x\to\infty} d_P(0, \rho(x)) = +\infty$*. In fact, if* ℓ *starts at* x_0 *and* $d_P(0, \rho(z)) \geq n$ *for all* $z \in \ell$ *between* x *and* y *then for a suitable constant* C*,*

$$|\rho(x) - \rho(y)| \leq Cne^{-n}.$$

It therefore seems natural to consider geodesic segment with endpoints in $\widetilde{V}$ (which is a subset of $\mathcal{T}(Y)$) and to compare ρ on the geodesic with a path with the same endpoints.

A key idea in the proof of this theorem is that geodesic segments on $\overline{\mathbb{C}} \setminus Z$ tend to have many self-intersections.

More precisely, given any geodesic segment γ of length Δ there is a constant C_1 such if we consider lifted geodesics in the unit disc starting from 0, every such geodesic segment of length n ends within Euclidean distance $C_1 e^{\Delta - n}$ of a geodesic segment which has endpoints within a bounded Poincaré distance of the endpoints of γ.

So we can choose γ to do anything we like by choosing Δ sufficiently large, e.g. to cut $\overline{\mathbb{C}} \setminus Z$ into topological discs with at most one puncture.

Continuity at punctures

It is relatively easy to check that $\lim_{x \to \infty} \rho(x)$ exists along half-geodesics in V ending at any puncture corresponding to a rational map f at which $v_2(f) \in Z(f)$.

Note that $f \notin V$ because $Z(g) \cup \{v_2(g)\} = Y(g)$ varies isotopically for $g \in V$.

In these cases, $\lim_{x \to \infty} \rho(x)$ is a "lift" of a point in Z – the endpoint of a geodesic in $\overline{\mathbb{C}} \setminus Z$ which ends at a point of Z.

It is also quite easy to show that, for a lift x of any other puncture of V, either $\lim_{y \to x} \rho(y)$ exists, or left and right limits exist outside a horosphere.

Discontinuities

The discontinuities $x \in \partial \widetilde{V}$ are the points such that $\liminf_{y \to x} d_P(0, \rho(y)) < \infty$. These are quite easily characterised and also it is quite easy to show that right and left limits exist outside a horosphere or Stoltz angle at such a point. However I have never managed to find such a point of *pseudo-Anosov type*. It is possible (although unlikely) that they do not exist.

Continuity along one path is sufficient

The points at which limits exist are sufficiently dense that the proof of the Resident's View is completed by showing that for each $x \in \widetilde{V}$, and for a choice of basepoint x_0, there is just one path x_t in $\widetilde{V}$ from x_0 to x on which the same uniform continuity for limits holds, that is, if $d_P(\rho(x_u), 0) \geq n$ for all $u \in [s, t]$ then $|\rho(x_s) - \rho(x_t)| \leq C_1 n e^{-n}$.

It is natural to start with the geodesic segment in $\mathcal{T}(Y(f_0))$ and to try to modify this to a path in $\widetilde{V}$.

It is only this part of the proof that I have been revisiting.

Use of a chain of geodesic segments to get the path in $\widetilde{V}$

Suppose that $[x_0, x]$ is a geodesic segment in $\mathcal{T}(Y)$ with endpoints x_0 and x in $\widetilde{V}$.

The path in $\widetilde{V}$ with these endpoints is obtained by taking $\lim_{n\to\infty} x_n(x_1) \in \widetilde{V}$ corresponding to $x_1 \in [x_0, x]$, where:

$$d_Y(x_n, x_{n+1}) - d_Z(x_n, x_{n+1}) \leq C;$$

a chain of *long thick and dominants* (α_i, ℓ_i) with $\ell_i \subset [x_i, x_{i+1}]$ and $\alpha_i \cap \alpha_{i+1} \neq \emptyset$ is non-cancelling, so that ℓ_i is a bounded d_{α_i}-distance from $[x_1, x_n]$ for $1 \leq i < n$.

From this we can deduce properties of the difference between $\rho(x_1)$ and $\rho(x_n)$. In revisiting this proof, my hope was to use a result about chains of long thick and dominants on Teichmüller geodesics which I found rather hard to prove, and useful in another situation.

I am not quite there yet, but have found an application to the analysis of the relation between $\tau([x, \tau(x)])$ and $[\tau(x), \tau^2(x)]$ where τ is the appropriate analogue for $B(f_0, Y(f_0))$ of the *Thurston pullback*.

The use of Teichmüller space

The proof of Thurston's Theorem for critically finite branched coverings used a distance-non-increasing map $\tau : \mathcal{T}(X(f_0)) \to \mathcal{T}(X(f_0))$ known as the *Thurston pullback*. For a suitable integer k, τ^k is a uniform contraction on any set $x : d(x, \tau(x) < M\}$. So there is a unique fixed point.

The map τ is given by $\tau([\varphi]) = [\psi]$ where $\varphi \circ f_0 = s \circ \psi$ where s is a holomorphic branched covering and ψ is a homeomorphism.

This equation determines s and $[\psi]$ uniquely.

If $\tau([\varphi]) = [\varphi]$ then $\varphi \circ f_0 \circ \varphi^{-1} = s$, and s is critically finite and Thurston equivalent to f_0. Since τ is a contraction, (s, φ) is unique, up to Möbius conjugation of s and post-composition of φ by this same Möbius transformation.

The iteration in the generalised case

in this more general case, the Teichm|'uller space used is $\mathcal{T}(Y)$, where Y contains all the critical values of f_0 but may not be forward invariant. Only $Z \subset Y$ is forward invariant, and Z does not contain all the critical values.

If we use the same formula as before and define $\tau([\varphi]) = [\psi]$ then we can consider $[\psi]$ as an element of $\mathcal{T}(Z)$. But we want an iteration on $\mathcal{T}(Y)$.

There is a natural way to do this, simply by defining $\tau([\varphi])$ by the two conditions

$$\pi_Z(\tau([\varphi]) = \pi_Z([\psi])$$

$$d_Y([\varphi], \tau([\varphi])) = dZ([\varphi], \tau([\varphi])),$$

where d_Z denotes Teichmüller distance in $\mathcal{T}(Z)$.

Properties of τ: decreasing along orbits

$\tau^k : \mathcal{T}(Y) \to \mathcal{T}(Y)$ is not a global contraction for any k However

$$d_Y(\tau([\varphi]), \tau^2([\varphi])) \leq d_Y([\varphi], \tau([\varphi])),$$

and for a suitable n depending only on $\#(Y)$,

$$d_Y(\tau^n([\varphi]), \tau^{n+1}([\varphi])) < d_Y([\varphi], \tau([\varphi])).$$

Properties of τ: the fixed set

The most important property is the characterisation of the fixed set of τ.

If $\tau([\varphi]) = [\varphi]$ then $s(\varphi(Z) = \varphi(Z)$ where s is the holomorphic map such that $\varphi \circ f_0 = s \circ \psi$. Also, φ and ψ are isotopic via an isotopy which is constant on Z.

This means that the fixed set of τ is the union of all the sets $\widetilde{V} \subset \mathcal{T}(Y)$, where V runs over the components of rational maps in B.

Definition of the sequence x_n

The sequence $x_n = x_n(x_1)$ is then obtained by defining x_{n+1} to be a modification of $\tau(x_n)$.