The quasi-conformal deformation space of a Kleinian group

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- If Γ_0 has a generating set with r elements, then we can identify the set of all (Γ, ρ) with a closed affine subvariety of $(PSL(2, \mathbb{C}))^r$.
- We are interested in the case when Γ is Kleinian, that is discrete.

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- This also defines a Γ_1- invariant line field, taking the the major axis or 0 depending on whether the ellipse is not, or is, a circle.
- Alternatively, $\varphi_{\overline{z}}/\varphi_z$ is a Γ_1-invariant Beltrami-differential.

Stable representations

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Definition. A group Γ, ρ is *stable* if for any representation $\rho : \Gamma_0 \to \Gamma$ and any (Γ', ρ') sufficiently close to (Γ, ρ) there is a homeomorphism $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

$$\varphi(\rho(\gamma).z) = \rho'(\gamma'.\varphi(z))$$

for all $\gamma \in \Gamma_0$ and $z \in \overline{\mathbb{C}}$. It is relatively straightforward to prove that any finitely generated Kleinian group Γ which acts hyperbolically on L_Γ is stable. The following theorem is due to Sullivan.

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Theorem

If Γ is stable then Γ acts hyperbolically on L_Γ.

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  λ-Lemma *If $\Lambda \subset \mathbb{C}^n$ is open and $X \subset C$ with $\Phi(\underline{0}, z) = z$ and $\Phi : \Lambda \times X \to \mathbb{C} : (\lambda, z) \mapsto \Phi(\lambda, z)$ is holomorphic in λ, and injective on X for each fixed λ, then the map $z \mapsto \Phi(\lambda, z)$ extends to a quasi-conformal homeomophism from $\overline{X}$ to its image.*

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- An argument due to Thurston, which shows that the representation space is bounded below by a sum of numbers, one corresponding to each topological end of the manifold. This, in turn, depends the existence, in hyperbolic 3-manifold with finitely generated fundamental group of the compact Scott core;
- The following theorem (also due to Sullivan)

Invariant line fields

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Theorem

Let Γ be a finitely generated Kleinian group. Then any Γ-invariant line field is supported a.e. on the domain of discontinuity Ω_Γ.

The analogues of Sullivan's Theorems for holomorphic maps, even for polynomials, is still unknown, although quasi-conformal rigidity is now known in some cases.

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- An eventual corollary of Sullivan's No-invariant line fields, and the Ahlfors' Finiteness Theorem is that the quasi-conformal deformation space of (Γ, ρ) is a finite-dimensional manifold, whose dimension can be computed.

Recurrent and Dissipative

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Definition. The action of Γ on L_Γ is said to be *recurrent* if for any set $U \subset L_\Gamma$ of positive Lebesgue measure, there exists $\gamma \in \Gamma$ with $\gamma \neq I$ such that $U \cap \gamma.U$ has positive measure.

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- Then the space of Beltrami differentials supported on U is infinite dimensional. To find an infinite linearly independent set we can for example choose disjoint positive measure sets U_j in U and let $\mu_j \in L^\infty(U_j)$ with $\|\mu_j\| \leq \frac{1}{2}$. Then

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  is an infinite-dimensional family of Beltrami differentials on U.
- Any Beltrami differential μ on U extends to a unique Γ-invariant Beltrami-differential on $\Gamma.U$ ($\gamma^*\mu = \mu$ for all $\gamma \in \Gamma$) and then to $\overline{\mathbb{C}}$ by taking it to be 0 on the complement of $\Gamma.U$.

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- For any $\gamma \in \Gamma$, $\varphi_\mu \circ \gamma \circ \varphi_\mu^{-1}$ is a Möbius transformation because it is quasi-conformal and

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- Since the map $\mu \mapsto \varphi_\mu$ is injective, the map $\mu \mapsto \Gamma_\mu$ is also injective.
  For if $\Gamma_{\mu_1} = \Gamma_{\mu_2}$ and $\varphi_{\mu_1}^{-1} \circ \varphi_{\mu_2} = \varphi$, then $\varphi(\gamma.z) = \gamma.\varphi(z)$ for all $z \in \mathbb{C}$. It follows that φ fixes all fixed points of hyperbolic elements of Γ and must be the identity on L_Γ. Since φ is holomorphic on Ω_Γ, it is holomorphic on $\overline{\mathbb{C}}$ and must be the identity. So $\varphi_{\mu_1} = \varphi_{\mu_2}$ and $\mu_1 = \mu_2$.

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- This gives a contradiction, completing the proof that the action of Γ on L_Γ is recurrent. There is no dissipative part.

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- By Egoroff's Theorem, there is a compact set K of strictly positive Lebesgue measure restricted to which the line field is continuous.
- By compactness, the line field is uniformly continuous restricted to K. So given $\varepsilon > 0$ there is $\delta > 0$ such that the direction of the line field varies by at most ε on the intersection of K with any ball of radius δ.

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- By a basic result in geometric measure theory, almost every point z of K is a Lebesgue density point of K, that is,

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- Let K_1 be the set of points in K where the density in $B_{r'}(z)$ is at least $1 - \varepsilon_0$ for all $r' \leq r$, choosing r so that K_1 has positive measure.

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- The aim is to show that the line field cannot vary in direction by $< \varepsilon$ on both $B_\delta(z)$ and $B_\delta(\gamma.z)$.
- Use the compact-abelian-compact decomposition

$$\gamma = \pm P \Delta Q$$

where

$$\Delta = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $0 < \lambda < 1$.

- Then for a constant C, either $|\gamma.z - P.0| < C\lambda$ or $|z - Q^{-1}.\infty| < C\lambda$.

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- We can assume λ small enough that $2C\lambda < r$.
- In the first case consider the image under γ^{-1} of

$$\{z' : |z' - \gamma.z| < C\lambda\}$$

- Then for a constant C, either $|\gamma.z - P.0| < C\lambda$ or $|z - Q^{-1}.\infty| < C\lambda$.
- We can assume λ small enough that $2C\lambda < r$.
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- The other case is similar.