A Discontinuous Boundary Element Method for Solving the Three Dimensional Exterior Helmholtz Problem

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Abstract

We first discuss a class of high order piecewise collocation methods for solving the boundary integral reformulation of the exterior Helmholtz equation in three dimensions, and then develop some sparse preconditioners for this high order case. This research has improved on previous work on using the collocation method based on piecewise constants, arising from discretization of a hyper-singular integral operator. We present some preliminary results.

1 Introduction

In this paper we develop high order collocation methods and suitable preconditioners for the boundary integral formulation of the three dimensional Helmholtz equation in an infinite domain. Such an equation arises from modelling time harmonic acoustic radiation or scattering by a three dimensional structure immersed in an infinite homogeneous acoustic medium [2].

For Neumann's boundary condition, we adopt the Burton and Miller reformulation method [3] for obtaining an integral equation which has a unique solution for all frequencies (including resonance).

This is a hyper-singular formulation. There have been several papers on developing suitable numerical schemes for this formulation:

1. In [2] and many references therein, the piecewise constant collocation method was used for the Helmholtz equation. Chen and Harris [6] considered effective preconditioners for iterative solution with this collocation method.

2. Using finite part integration (i.e. ignoring hyper-singularities), Harris [10] considered a high order collocation method for the Helmholtz equation in the axisymmetric boundary case. Similar work for Laplace equation were also considered by [14, 15].

3. Also for the Laplace equation, several papers have discussed the high order Galerkin method. Both Giroire and Nedelec [8] and Hackbusch [12] use a method for transforming the hyper-singular integral appearing in the boundary integral formulation of Laplace's equation into one which is at worst weakly singular. However, in neither case do they present any numerical results of the formulation and the analysis was not extended to include the Helmholtz problem.

4. Recently, for the exterior Helmholtz equation, Harris and Chen [11] developed the high order Galerkin formulation by reformulating to weakly-singular integrals and considered the associated iterative solution while [1] considered also a high order Galerkin method but they use finite part integration.

This paper continues the work of Harris [10] for the three dimensional axisymmetric boundary case and generalises it to fully three dimensional case. We have also generalised our previous work [6] on sparse preconditioners based on piecewise constant elements to this high order case. It should be remarked that it will be of interest to consider how to develop a hybrid high order Galerkin-collocation method for this problem following the idea in [9]. However our results on high order methods as well as preconditioning might be directly applied to the efficient Fast Multipole Method. Numerical experiments confirm that our proposed collocation methods are accurate and the iterative solution techniques are efficient.

2 BIM Formulation and the collocation method

Let D_+ denote the unbounded region exterior to some structure with surface S which is filled with an acoustic medium. Consider the problem of solving the Helmholtz equation

$$\nabla^2 \phi(p) + k^2 \phi(p) = 0, \qquad p \in D_+ \cup S \tag{1}$$

in some unbounded three dimensional region D_+ exterior to a closed surface S, where $k > 0$ is the wavenumber, subject to a Neumann boundary condition on S and the Sommerfeld radiation condition $\lim_{r\to\infty} r\left(\frac{\partial \phi}{\partial r} - ik\phi\right) = 0$.

A simple application of Green's second theorem leads to

$$\int_S \phi(q)\frac{\partial G_k(p,q)}{\partial n_q} - G_k(p,q)\frac{\partial \phi(q)}{\partial n_q}\, dS_q = \begin{cases} \frac{1}{2}\phi(p) & p \in S \\ \phi(p) & p \in D_+ \end{cases} \tag{2}$$

where $G_k(p,q) = \dfrac{e^{ik|p-q|}}{4\pi|p-q|}$ is the free-space Green's function, or fundamental solution, for Helmholtz equation and n_q is the unit outward normal to S at q. If the normal derivative of the acoustic field is given on the surface S then (3) for $p \in S$ gives a Fredholm integral equation of the second kind which can be solved for the surface pressure ϕ. The acoustic pressure can then be computed at any point in D_+ using (3). However, it is well known that (3) does not possess a unique solution for certain values of the wavenumber, called characteristic wavenumbers. This is a manifestation of the integral equation formulation as it can be shown that the underlying differential problem has a unique solution for all real and positive values of k [3].

The Burton and Miller method [3] for overcoming the non-uniqueness problem consists of differentiating (3) along the normal at p to give

$$\int_S \phi(q)\frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} - \frac{\partial G_k(p,q)}{\partial n_p}\frac{\partial \phi(q)}{\partial n_q}\, dS_q = \frac{1}{2}\frac{\partial \phi(p)}{\partial n_p} \tag{3}$$

and then taking a linear combination of (3) and (5) in the form

$$\begin{aligned} -\frac{1}{2}\phi(p) + \int_S \phi(q)\left(\frac{\partial G_k(p,q)}{\partial n_q} + \alpha\frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q}\right) dS_q = \\ \frac{\alpha}{2}\frac{\partial \phi(p)}{\partial n_p} + \int_S \frac{\partial \phi(q)}{\partial n_q}\left(G_k(p,q) + \alpha\frac{\partial G_k(p,q)}{\partial n_p}\right) dS_q \end{aligned} \tag{4}$$

where α is a coupling constant. It can be shown that provided that the imaginary part of α is non-zero then (6) has a unique solution for all real and positive k. However, this formulation has introduced the integral operator with kernel function $\frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q}$ which contains a $\frac{1}{r^3}$ singularity and hence cannot be integrated in the usual way. Here we shall refer to this operator as the hyper-singular operator. At this point we note that all the remaining integral operators have kernel functions which are at worst weakly singular and so can be evaluated using an appropriate quadrature rule.

As discussed, previous work for the collocation approach was mainly based on the use of piecewise constants. To use high order elements, we have to consider how to overcome the problem of integrating the above hyper-singular operator.

We first review how to compute a finite part integral. Consider the problem of evaluating an integral of the form for a suitably smooth function $f(s)$

$$\int_a^b \frac{f(s)}{(s-a)^2}\, ds. \tag{5}$$

If $F(s)$ is the anti-derivative of $\frac{f(s)}{(s-a)^2}$ then the finite part of (5) is defined as $F(b)$. In order to approximate (5) we need to construct a quadrature rule of the form

$$\int_a^b \frac{f(s)}{(s-a)^2}\, ds = \sum_{j=1}^{m} w_j f(s_j). \tag{6}$$

The simplest way of doing this is to use the method of undetermined coefficients, where the quadrature points $s_1, s_2, \ldots, s_m$ are assigned values and then (6) is made exact for $f(s) = (s-a)^i$, $i = 0, 1, \ldots, m-1$. The resulting equations can be written in matrix form as $A\underline{w} = \underline{g}$ where

$$A_{ij} = (s_j - a)^{i-1} \quad \text{and} \quad g_i = \int_a^b (s-a)^{i-3}\, ds, \quad 1 \le i,\ j \le m, \tag{7}$$

and, to compute g_i for $i = 1,\ 2$, we can define the following finite part integrals

$$\int_a^b \frac{1}{(s-a)^2}\, ds = -\frac{1}{b-a} \quad \text{and} \quad \int_a^b \frac{1}{s-a}\, ds = \ln(b-a). \tag{8}$$

Then to use finite part integration, one way of interpreting the hyper-singular operator is finding out an appropriate change of variables so that *the interested integral is effectively reduced to one* *dimensional integral with a hyper-singular integrand and the singularity located at an end point.* Suppose that the surface S is approximated by N non-overlapping triangular quadratic surface elements $S_1, S_2, \ldots, S_N$. If p_i, $i = 1, \ldots 6$, denote the position vectors of the six nodes used to define a given element, then that element can be mapped into a reference element in the (u, v) plane

$$p(u,v) = \sum_{j=1}^{6} \psi_j(u,v) p_j \qquad 0 \leq u \leq 1,\ 0 \leq v \leq 1-u. \tag{9}$$

Now suppose that the singular point corresponds to the point (u_1, v_1) in the (u, v) plane. The reference element is divided into three triangular sub-elements by connecting the point (u_1, v_1) to each of the vertices of the reference triangle. We need to decide on a new coordinate transform in which the singularity is only present in one variable. Since the singularity is in the radial direction (away from point (u_1, v_1)), a suitable transform must be polar like.

Within each sub-element we now propose the following transformation

$$\left.\begin{aligned} u(s,t) &= (1-s)u_1 + stu_2 + s(1-t)u_3 \\ v(s,t) &= (1-s)v_1 + stv_2 + s(1-t)v_3 \end{aligned}\right\} 0 \leq s, t \leq 1 \tag{10}$$

where (u_2, v_2) and (u_3, v_3) are the other vertices of the current sub-triangle. Now clearly the only way for $(u(s,t), v(s,t)) = (u_1, v_1)$ is for $s = 0$ as these are bi-linear functions of s and t. Further, the mapping (9) is bijective as its Jacobian is non-zero for all (u, v) of interest. Hence the only way that p can equal the singular point is if $s = 0$. After some manipulation it is possible to show that $r(s,t) = |p(s,t) - q| = s\tilde{r}(s,t)$ where $\tilde{r}(s,t) \neq 0$ for $0 \leq s, t \leq 1$. The Jacobian of the transformations (9) and (10) can be written as

$$J = s\sqrt{D_1^2 + D_2^2 + D_3^2}\, |(u_2 - u_1)(v_3 - v_1) - (u_3 - u_1)(v_2 - v_1)| = J_s s \tag{11}$$

where

$$D_1 = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial u} \end{vmatrix} \quad D_2 = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial u} \end{vmatrix} \quad D_3 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial u} \end{vmatrix} \tag{12}$$

Hence, denoting S_e the current sub-element, we can write

$$\int_{S_e} \frac{f(q)}{r^3}\, dS = \int_0^1 \frac{1}{s^2} \left[\int_0^1 \frac{f(q(u(s,t), v(s,t)))}{(\tilde{r}(s,t))^3} J_s\, dt \right] ds. \tag{13}$$

We note that the inner integration (with respect to t) is non-singular and can be approximated by an appropriate quadrature rule. However, the outer integral needs to be interpretated as a Hadamad finite part in a (desirably) single variable s. Thus equation (4) is tractable.

3 Solution of the dense linear system

Denote the dense linear system resulting from discretizing (4), i.e. $\mathcal{A}\phi = f$, by $Ax = b$. We seek suitable a preconditioning matrix P^{-1} such that the following system $AP^{-1}y = b$ can be solved efficiently by iterative methods (with either P or P^{-1} sparse), where $x = P^{-1}y$. For unsymmetric conjugate gradient solvers, fast convergence is often seen with a clustering distribution of the eigenvalues and singular values [16]. As in [6], using the operator splitting idea [4], we hope to split the operator $\mathcal{A}$ as $\mathcal{A} = \mathcal{D} + \mathcal{C}$ by domain decomposition such that $\mathcal{D}$ is a suitable bounded operator and $\mathcal{C}$ is a compact operator [6]. To proceed, as in [6], we use the surface domain partition $S = \bigcup_{j=1}^{N} S_j$ to decompose the operator

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} & \mathcal{A}_{1,3} & \cdots & \mathcal{A}_{1,N} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} & \mathcal{A}_{2,3} & \cdots & \mathcal{A}_{2,N} \\ \mathcal{A}_{3,1} & \mathcal{A}_{3,2} & \mathcal{A}_{3,3} & \cdots & \mathcal{A}_{3,N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathcal{A}_{N,1} & \mathcal{A}_{N,2} & \mathcal{A}_{N,3} & \cdots & \mathcal{A}_{N,N} \end{pmatrix} \tag{14}$$

where in element S_i operator $\mathcal{A}_{i,\ell}$ is the restriction of $\mathcal{A}$ over surface S_ℓ.

We shall choose a bounded operator splitting $\mathcal{D}$ in order for it to give rise to a sparse matrix D on discretisation. Then the new operator $\mathcal{D}^{-1}\mathcal{A} = \mathcal{I} + \mathcal{D}^{-1}\mathcal{C}$ (similarly $\mathcal{A}\mathcal{D}^{-1}$) will be a compact perturbation of the identity operator because a product of a bounded operator with a compact operator is still compact. Since compact operators have all eigenvalues clustered at most at point 0, eigenvalues of operator $\mathcal{D}^{-1}\mathcal{A}$ will cluster at 1. Furthermore eigenvalues of its normal operator also cluster at 1 because $\mathcal{D}^{-1}\mathcal{A}(\mathcal{D}^{-1}\mathcal{A})^* = \mathcal{I} + \mathcal{D}^{-1}\mathcal{C} + \mathcal{C}^*\mathcal{D}^{-*} + \mathcal{D}^{-1}\mathcal{C}\mathcal{C}^*\mathcal{D}^{-*}$.

The properties of these continuous operators are inherited by the discrete operators if a consistent discretization scheme such as collocation is used. On discretization, with $P = D$, the preconditioned system $AP^{-1}y = b$ has a new matrix with clustering eigenvalues at 1. Moreover, the singular values and the eigenvalues of the normal of this new matrix AP^{-1} are also clustered at 1. Thus conjugate gradient methods will be expected to exhibit fast convergence. This differs from the idea of a bounded condition number which may not describe the convergence e.g. although matrix A with $(A)_{ij} = 1$ for $i \le j$ (0 elsewhere) has a single eigenvalue $\lambda(A) = 1$ but most iterative solvers will not converge since $\lambda(A^*A)$ has no desirable pattern.

Based on previous 3D work [6] with piecewise constants and 2D work [4], we shall consider in this study two operator splitting type preconditioners: (I) the element based block diagonal preconditioner I; (II) the edge based block non-diagonal preconditioner II. Refer to Fig. 1.

4 Numerical Results

The collocation method described in Section 2 and the iterative solution methods described in Section 3 have been applied to a number of test problems, from which a small sample is reported here. In each case the surface was approximated by six-noded quadratic triangular elements and both the surface pressure and its normal derivative were interpolated using the same quadratic basis functions (isoparametric elements). The surface data was generated by placing a number of point sources inside the surface and using these to compute $\frac{\partial \phi}{\partial n}$ on the surface. The solution ϕ is simply that due to the point sources. The individual test problems considered are outlined below

(1). A unit sphere with point sources at (0, 0, 0.5) and (0.25, 0.25, 0.25) with strengths $2+3i$ and $4-i$ respectively.

(2). A 'peanut' shaped surface defined by

$$x = \sqrt{\cos 2\theta + \sqrt{1.5 - \sin^2 2\theta}}\,\sin\theta\cos\gamma, \quad z = \sqrt{\cos 2\theta + \sqrt{1.5 - \sin^2 2\theta}}\,\cos\theta,$$

$$y = \sqrt{\cos 2\theta + \sqrt{1.5 - \sin^2 2\theta}}\,\sin\theta\sin\gamma, \quad (0 \le \theta \le \pi, \quad 0 \le \gamma < 2\pi),$$

with point sources at (0.2, 0 , 1) and (0, 0.2, -0.75) with strength $2+3i$ and $4-i$ respectively.

(3). A cylinder of length 0.537 and radius 0.2685 with point sources at (0, 0, 0.15) and (0.25, 0.25, 0.25) with strengths $2+3i$ and $4-i$ respectively.

Note that the second test problem has a non-convex surface, and that the third test problem has a non-smooth surface in the sense that it does not possess a unique normal at every point. The measure of the error is the usual relative L_2 error, defined by $E = \frac{||\phi - \tilde{\phi}||_2}{||\phi||_2}$ where $||\phi||_2 = \sqrt{\int_S |\phi(q)|^2 \; dS_q}$ and ϕ and $\tilde{\phi}$ denote the exact and approximate solution respectively. In the following experiments a mesh with $n_e = 576$ elements is used which yields linear systems of size $n = 1728,\; 3456$ respectively for linear and quadratic elements.

For the range of values of k considered here the relative errors in the computed solution ranged from 0.3% (example 1) to 1% (examples 2 and 3). The larger error is probably due to the special geometries which may require more specialised treatment than that given here.

The results of using both preconditioners with the GMRES(25) method are given in Table 1. Here the columns headed Σ gives the number of iterations and those headed t/t_d gives the CPU time t relative to the time t_d needed to carry out a full LU decomposition on the matrix. The stopping criterion is based on the relative residule being less than 10^{-8}. It is clear that both iterative methods are offering a considerable time-saving over the LU decomposition method, taking less than 1/10 of the time needed for the LU decomposition. Other GMRES methods, such as GMRES(5) and GMRES(40) were tried, but offered no significant increase in performance. Although the preconditioners constructed here are for the hyper-singular integral equation on a smooth convex surface, these results suggest that they are equally effective when applied to problems involving non-convex surfaces and surfaces with geometric singularities.

In other experiments, we have observed that the cases of no preconditioning or of using a simple diagonal preconditioner lead to no convergence of the iterative solver. This is quite different from the piecewise constant case, as considered in [6].

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Figure 1: Illustration of Preconditioners I (◯ nodes) and II (• and ◯ nodes) for piecewise linear elements (left plot) and piecewise quadratic elements (right plot).

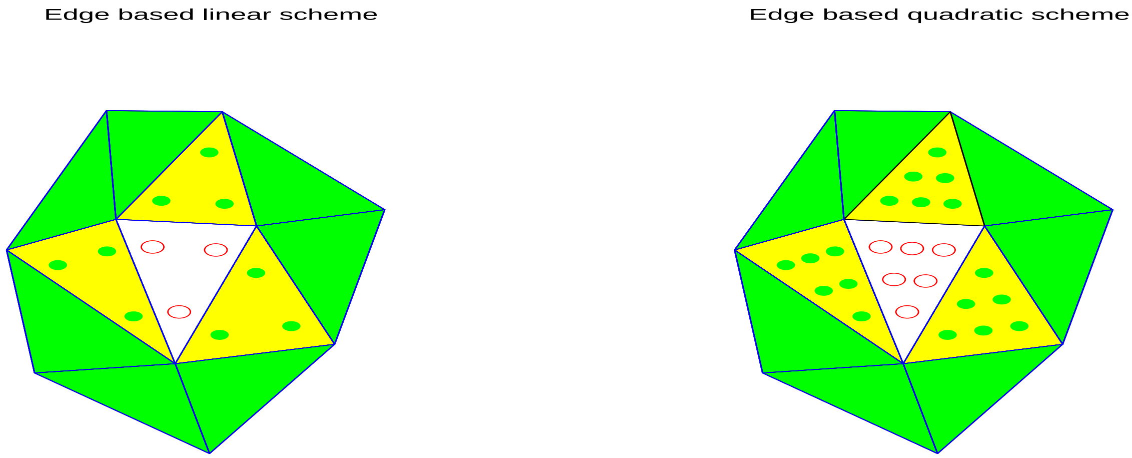

Table 1: The number of total GMRES iterations Σ and relative CPU time t/t_d with preconditioners I and II for Linear case: $n = 1728$ and Quadratic case: $n = 3456$.

	SPHERE				PEANUT				CYLINDER			
Linear	Σ_I	t_I/t_d	Σ_{II}	t_{II}/t_d	Σ_I	t_I/t_d	Σ_{II}	t_{II}/t_d	Σ_I	t_I/t_d	Σ_{II}	t_{II}/t_d
$k = 1$	14	0.03	13	0.03	30	0.07	19	0.05	21	0.05	19	0.05
$k = 2$	16	0.04	15	0.04	33	0.08	21	0.05	23	0.06	21	0.05
Quadratic												
$k = 1$	14	0.1	13	0.1	47	0.2	19	0.2	22	0.1	19	0.1
$k = 2$	16	0.1	15	0.1	48	0.2	21	0.2	24	0.2	21	0.2

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