A Nonlinear Multigrid Method For Total Variation Minimization From Image Restoration

Ke Chen* and Xue-Cheng Tai†

Abstract

Image restoration has been an active research topic and variational formulations are particularly effective in high quality recovery. Although there exist many modelling and theoretical results, available iterative solvers are not yet robust in solving such modelling equations. Recent attempts on developing optimisation multigrid methods have been based on first order conditions. Different from this idea, this paper proposes to use piecewise linear function spanned subspace correction to design a multilevel method for directly solving the total variation minimisation. Our method appears to be more robust than the primal-dual method [14] previously found reliable. Supporting numerical results are presented.

Keywords: Image restoration, total variation, regularisation, subspace correction, multilevel solvers.
AMS subject class: 68U10, 65F10, 65K10.

1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^d,\ d = 1, 2, 3, \cdots$, we often need to solve problems which can be written in the following general form

$$\min_u \left(\int_\Omega |\nabla u| dx + \int_\Omega f(u) dx \right). \tag{1}$$

The equivalent problem to the above minimization is the Euler-Lagrange equation

$$-\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) + f'(u) = 0, \tag{2}$$

which is a nonlinear partial differential equation (PDE), also known as a curvature equation [28]. The application of problems (1) and (2) ranges from image processing including noise removal [31, 25, 26, 8], segmentation [17, 22], deblurring [4], inverse problems [16] to interface motion driven by mean curvature [30, 28]. Owing to huge number of applications involved with models (1) and (2), the demand for new and fast solvers for these problems is equally huge. In this paper, we present a nonlinear multigrid method for efficiently solving (1).

In the literature the following methods have been used to solve the equation (2):

(i) The fixed point iteration method [1, 40, 43, 44, 41, 42]: Once the coefficients $1/|\nabla \bar{u}|$ are fixed at a previous iteration $\bar{u}$, various iterative solver techniques have been considered [43, 44, 12, 10, 9, 24]. There exist excellent inner solvers but the outer solver can be slow. Further improvements are still useful.

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(ii) The explicit time marching scheme [31, 29]: It turns the nonlinear PDE into a parabolic equation before using an explicit Euler method to march in time to convergence. The method is quite reliable but often slow.

(iii) The primal-dual (PD) method [14, 15, 4]: It solves for both the primal and dual variable together in order to achieve faster convergence with the Newton method (and a constrained optimisation with the dual variable). There does not appear to exist any multilevel version and also the inner solvers can have a convergence problem if the problem dimension gets large and β gets small.

As shown in [32], a converging multigrid method (MGM) can be much faster than methods of type (i) and (ii). In some cases, the MGM is also faster than the PD method (iii). The algorithms proposed in this paper behave similarly to [32] but, unlike [32], are not parameter dependant.

The MGM is one of the most powerful numerical methods for solving linear and nonlinear elliptic problems [46, 38, 39], although the method is known to be less robust for either case with highly discontinuous coefficients [45]. As for the curvature equation (2), several attempts have been made to develop MGM to solve it, c.f. [41, 9, 32, 2]. However, the success so far is limited. The main problem is that the nonlinear diffusion coefficient $1/|\nabla u|$ can be highly oscillatory or degenerate (e.g. having large values close to infinity). Recently in [18, 23], the linear algebraic multigrid method [33] was adapted for solving the above PDE in each (outer) step of a fixed iteration while [32] attempted to use the standard multigrid methods with a non-standard and somewhat global smoother. As for solving (1) directly by MGM, the main obstacle to address is how to design the crucial coarse grid minimization problems for correction as no operator equations are directly available. Several related approaches, c.f. [3, 34, 5, 27], tried to design such coarse grid problems by using first order conditions (similar to using (2) to measure residuals). However, the convergence of this kind of methods for certain nonlinear problems is not as good as for linear problems [18, 23, 32, 20]. Several authors [36, 38, 37, 35] have studied the combined approach of MGM ideas and domain decomposition methods (DDM) (interpreted as subspace correction techniques) for some optimisation problems. Although it is proved that efficiency of DDM and MGM for a class of nonlinear problems is as good as for linear elliptic problems, there do not appear to exist any extensive uses of this approach for optimisation problems. Moreover, problem (1) is beyond the class of problems that were previously studied.

In this paper, we shall propose a nonlinear multigrid method for solving (1) based on subspace correction techniques. The essential idea is to use nonlinear smoothers for the subproblems which respect the minimization problem in order to reduce the energy functional. For the nonlinear problem (1), we shall demonstrate numerically that the efficiency of the schemes can be as good as for linear problems. Thus we may summarise our contributions of this work as follows:

a) we apply the subspace correction idea to design a nonlinear multigrid, as opposed to the geometric multigrid methods that were proposed [3, 34, 5, 27] based on regularising (2).

b) the efficiency of the proposed algorithms is high: $O(N \log N)$ where N is the total degree of freedom.

c) the proposed algorithms are not parameter dependant.

d) as an inner-outer iteration procedure, our methods respect the nonlinear nature in the outer iteration in contrast to linearisation techniques.

The rest of the paper is organised as follows. In Section 2, we introduce the general subspace correction methods for convex functional minimisation. However we note that the theory does not cover the problem type (2). In Section 3, we detail our proposed multilevel algorithms for problem (2) and present some preliminary analysis. We present numerical experiments in Section 4 for solving both the one-dimensional and two-dimensional image denoising problems. Finally in section 5, we discuss some conclusions and future work.

2 The space decomposition algorithms

Consider a general minimization problem over a reflexive Banach space V:

$$\min_{v\in V} F(v) \tag{3}$$

where F is a strongly convex cost functional. Assume that the space V has been decomposed into a sum of smaller subspaces, i.e.

$$V = V_1 + V_2 + \cdots + V_m. \tag{4}$$

This means that for any $v \in V$, there exists $v_i \in V_i$ such that $v = \sum_{i=1}^m v_i$. Then the idea employed by [35, 36, 37, 38] is to repeatedly solve the subspace minimisation of the type

$$\min_{v\in V_i} F(v^{(0)} + v)$$

where $v^{(0)}$ denotes a current approximation. The convergence of such methods requires $F(v)$ to be Lipschitz which is not satisfied by (1); it was not clear whether this kind of methodology will work for (1). Also this methodology differs from [3, 34, 5, 27] on one major aspect: here we minimize the same functional on all levels while the other methods minimize a modified functional on coarse levels.

Following previous studies [35, 36, 37, 38], two types of subspace correction methods can be derived based on (4), namely the parallel subspace correction (PSC) method and the successive subspace correction (SSC) method, as simple generalisations of the methods for operator equations [46]. The parallel subspace correction method can be described as follows.

Algorithm 1
Choose an initial value $u^{(0)} \in V$ and relaxation parameters $\gamma_i > 0$ such that $\sum_{i=1}^m \gamma_i \le 1$.

1. *For $\ell \ge 0$, if $u^{(\ell)} \in V$ is defined, then find $p_i^{(\ell)} \in V_i$ in parallel for $i = 1, 2, \cdots, m$ such that*

$$F\left(u^{(\ell)} + p_i^{(\ell)}\right) \le F\left(u^{(\ell)} + v_i\right) \ , \quad \forall v_i \in V_i. \tag{5}$$

2. *Set*

$$u^{(\ell+1)} = u^{(\ell)} + \sum_{i=1}^m \gamma_i p_i^{(\ell)} \ , \tag{6}$$

and go to the next iteration.

The successive subspace correction method can be described as follows:

Algorithm 2 *Choose an initial value $u^{(0)} \in V$.*

1. *For $\ell \ge 0$, since $u^{(\ell)} \in V$ is defined, find $u^{(\ell+i/m)} = u^{(\ell+(i-1)/m)} + p_i^{(\ell)}$ with $p_i^{(\ell)} \in V_i$ sequentially for $i = 1, 2, \cdots, m$ such that*

$$F\left(u^{(\ell+(i-1)/m)} + p_i^{(\ell)}\right) \le F\left(u^{(\ell+(i-1)/m)} + v_i\right) \ , \quad \forall v_i \in V_i. \tag{7}$$

2. *Go to the next iteration.*

The classical Gauss-Seidel and Jacobi relaxation methods and the modern DDM and MGM can all be interpreted as space decomposition algorithms. In order to reveal the relation between MGM and space decomposition, one can use finite element spaces. Similar explanations can also be given for finite difference approximations. For a given domain Ω, we assume that the finite element partition $\mathcal{T}$ of Ω is constructed by a successive refinement process. More precisely, $\mathcal{T} = \mathcal{T}_J$ for some $J > 1$, and $\mathcal{T}_j$ for $j \le J$ are a nested sequence of quasi-uniform finite element partitions, i.e. $\mathcal{T}_j$ consist of finite elements

$\mathcal{T}_j = \{\tau_j^i\}$ of size h_j such that $\Omega = \cup_i \tau_j^i$ for which the quasi-uniformity constants are independent of j and τ_{j-1}^l is a union of elements of $\{\tau_j^i\}$. We further assume that there is a constant $\gamma < 1$, independent of j, such that h_j is proportional to γ^{2j}. In Fig. 1 and Fig. 2, we plot the basis functions and the refined meshes for a domain in one and two dimensions. For the two dimensional case, a finer grid is obtained by connecting the midpoints of the edges of the triangles of the coarser grid, with $\mathcal{T}_1$ being the given coarsest initial triangulation, which is quasi-uniform. In this example, $\gamma = 1/\sqrt{2}$. We can use much smaller γ in constructing the meshes, but the convergence will be slower. Corresponding to each finite element

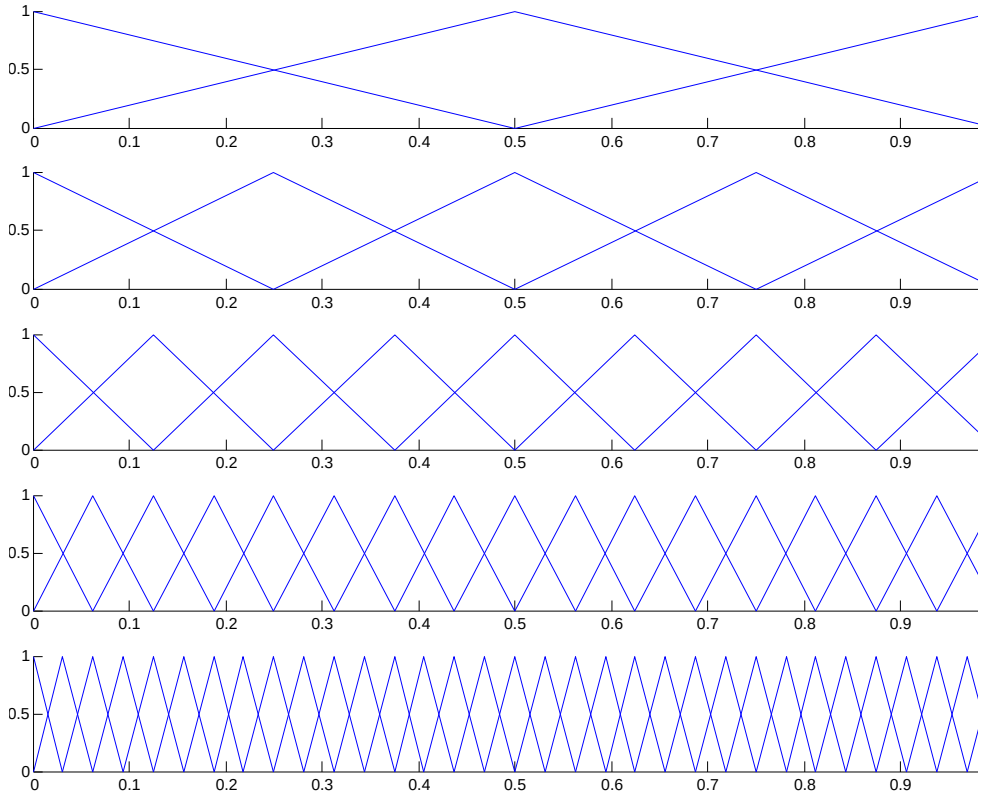


Figure 1: Basis functions and the mesh for one dimensional multigrids

partition $\mathcal{T}_j$, a finite element space $\mathcal{M}_j$ can be defined by

$$\mathcal{M}_j = \{v : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall\, \tau \in \mathcal{T}_j\},$$

where $\mathcal{P}_1$ denotes the space of all piecewise linear elements (the basis functions are as illustrated in Fig. 1

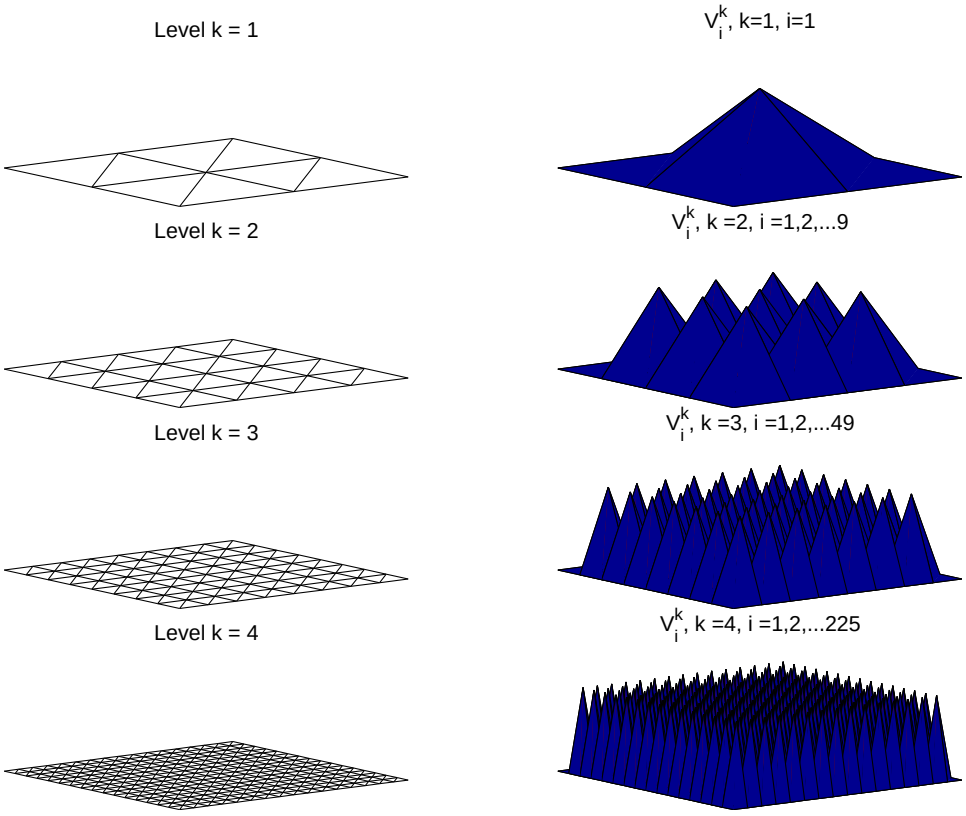


Figure 2: Basis functions and the mesh for two dimensional multigrids

and Fig. 2). Each finite element space $\mathcal{M}_j$ is associated with a nodal basis, denoted by $\{\phi_j^i\}_{i=1}^{n_j}$ satisfying

$$\phi_j^i(x_j^k) = \delta_{ik}$$

where $\{x_j^k\}_{k=1}^{n_j}$ is the set of all nodes of the elements of $\mathcal{T}_j$. Associated with each of such a nodal basis function, we define an one dimensional subspace as follows

$$V_j^k = \text{span } (\phi_j^k).$$

Letting $V = \mathcal{M}_J$, we have the following simple space decomposition:

$$V = \sum_{j=1}^{J} \sum_{k=1}^{n_j} V_j^k. \tag{8}$$

Each subspace V_j^k is one dimensional and thus the subproblem (7) is easy to solve.

We remark that the above decomposition (8) is not an orthogonal (direct) sum in general. It has more basis functions than the hierarchical basis [46, 19]. It is the hierarchical nature of such decompositions that drives a multilevel minimization algorithm. Without multilevels, the simple decomposition $V = \sum_{k=1}^{n_1} V_1^k$ defined on the single level $j = 1$ (though leading to a relaxation method) is not suitable because it is known that relaxations alone cannot reach a global minimizer [6, 13].

3 Image restoration algorithms using the total variation model

We now consider how to adapt the above methodology for solving (1).

For a given noisy image z defined on the domain $\Omega = [0,1] \times [0,1]$, one of the most well-known restoration models is the Rudin-Osher-Fatimi (ROF) total variation (TV) model [31] which is to take F in (3) to be

$$F(u) = \alpha \int_\Omega \sqrt{u_x^2 + u_y^2}\, dx + \frac{1}{2}\int_\Omega \|Ku - z\|^2\, dx, \tag{9}$$

where $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (u_x, u_y)$ and K is a known operator. As the TV term $|\nabla u|$ is non-differentiable, one often replaces the above by a regularized functional

$$F(u) = \alpha \int_\Omega \sqrt{u_x^2 + u_y^2 + \beta}\, dx + \frac{1}{2}\int_\Omega \|Ku - z\|^2\, dx, \tag{10}$$

as done in [41, 14, 32] and almost all multilevel methods. In this paper, we are concerned with fast solution of this non-regularised model (9) in the denoising case with $K = I$. The minimizer of (9) is taken as the denoised image. We note that a recent study [7] demonstrates that there are many advantages to transform (9) to a dual formulation; we expect to generalize our MGM to this dual model in the near future.

For problem (9), we shall explain the details in using Algorithm 2 for multigrid decomposition (8). Note that all the subspaces in the multi-dimensional decomposition (8) are one dimensional. Thus, the subproblems (7) are essentially trying to solve the following one dimensional minimization problem:

$$\min_{c\in\mathbb{R}} F(w + c\phi_j^k), \tag{11}$$

where $w = u^{(\ell+(i-1)/m)} \in V$ and ϕ_j^k is the basis function over the jth level at the kth node. As F is convex, $c \in \mathbb{R}$ is a minimizer of (11) if and only if it satisfies

$$\int_\Omega \left[\alpha \frac{\nabla(w + c\phi_j^k)\cdot\nabla\phi_j^k}{\sqrt{|\nabla(w + c\phi_j^k)|^2 + \beta}} + (w + c\phi_j^k - z)\phi_j^k\right] dx = 0, \tag{12}$$

where one noticed that we have included a regularising parameter β for the local problem (11). Although we do not have to solve (11) this way, it turns out that the resulting method is not very sensitive to β unlike (10).

The key observation is that each of our local minimisation problems has only one degree of freedom (i.e. one dimensional). To solve this nonlinear equation for $c \in \mathbb{R}$, we may use the fixed point iteration (e.g. as in [1]), i.e. start with an $c^{(0)} = 0$ and recursively get $c^{(\ell)}$ from

$$\int_\Omega \left[\alpha \frac{\nabla(w + c^{(\ell+1)}\phi_j^k)\cdot\nabla\phi_j^k}{\sqrt{|\nabla(w + c^{(\ell)}\phi_j^k)|^2 + \beta}} + (w + c^{(\ell+1)}\phi_j^k - z)\phi_j^k\right] dx = 0. \tag{13}$$

It is easy to see that

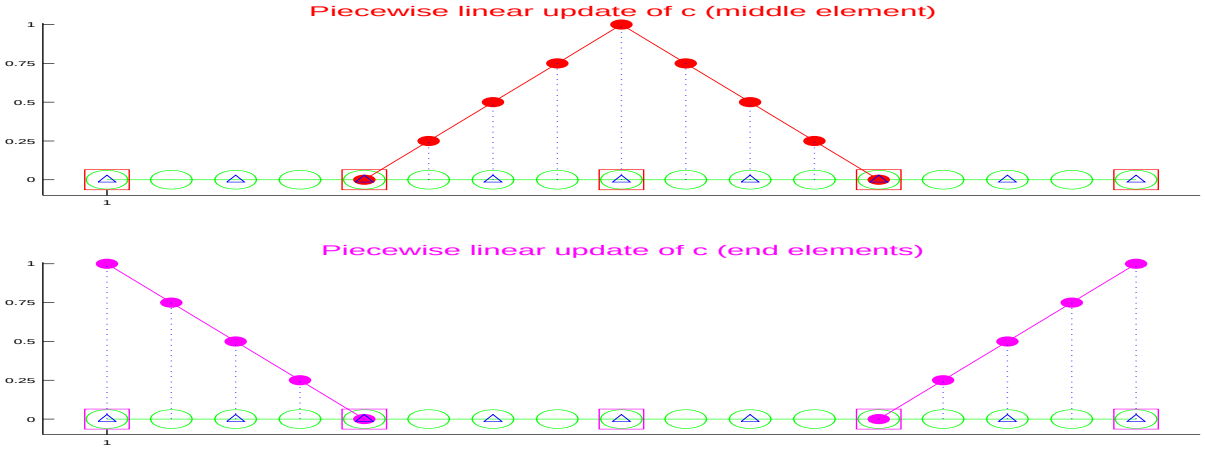
$$c^{(\ell+1)} = \frac{b_j^k - a_j^k(w)}{a_j^k(\phi_j^k)}, \quad b_j^k = \int_\Omega (z - w)\phi_j^k dx \quad \text{and}$$
$$a_j^k(v) = \int_\Omega \left[\alpha \frac{\nabla v\cdot\nabla\phi_j^k}{\sqrt{|\nabla(w + c^{(\ell)}\phi_j^k)|^2 + \beta}} + v\phi_j^k\right] dx, \tag{14}$$

where $\ell \geq 0$ (global iteration), $j = 1, \ldots, J$ (all levels) and $k = 1, \ldots, n_j$ (level j). It is easy to see that $a_j^k(\phi_j^k) > 0$ so the iteration will not break down. As w is a function over the fine mesh, much reformulation will be done in integration for efficiently obtaining $a_j^k(w)$ and $a_j^k(\phi_j^k)$ as shown below. The iteration for (14) is stopped when $|c^{(\ell+1)} - c^{(\ell)}|/|c^{(\ell)}| \leq \tau_{inner}$. Numerical experiments will show that the convergence rate is nearly independent of τ_{inner}. Normally, just carrying out one or two iterations for (12) is sufficient to obtain required results. Regarding complexity, we note that the domain integration in (13) and (14) does not present complications because the basis function ϕ_j^k is only defined locally (as with finite elements). This is addressed more precisely next.

3.1 The algorithm with $\Omega \in \mathbb{R}$

Firstly we consider the case $\Omega \in \mathbb{R}$ associated with signal processing. Note that ϕ_j^k (on level $j = 2$ and at node $k = 2$) may be illustrated by Figure 3. We wish to simplify the functional as much as possible

Figure 3: The one dimensional basis function ϕ_j^k with $\bullet$ showing its height – on the coarse level $j = 2$ and at a middle node $k = 2$ (top plot) and at end nodes (bottom plot). Here $\bigcirc$ defines the finest level, Δ refers to the first coarse level and $\square$ to the second coarse level.



by using the compact support of ϕ_j^k. As mentioned before, the parameter β is only introduced later for local minimization. (In the following, where boundary basis functions are involved, the usual adjustment in indices associated with summation is assumed.)

In the discrete setting for one dimensional problems, the cost functional (9) is (assuming α and F absorb the uniform step length $\Delta x = \Delta y = h$ from here onwards)

$$F(u) = \alpha \sum_{i=1}^{n-1} |D_x^+ u_i| + \frac{1}{2} \sum_{i=1}^{n} (u_i - z_i)^2$$

where n is the total number of nodes, D_x^+ (also later D_y^+) is the standard forward finite difference operator. Let Ω_j^k be the support set of ϕ_j^k and $\bar{\Omega}_j^k$ be its closure. Corresponding to Ω_j^k, we define $I_j^k = \{i |\ x_i \in \Omega_j^k \cap \Theta\}$ and $\bar{I}_j^k = \{i |\ x_i \in \bar{\Omega}_j^k \cap \Theta\}$ with Θ being the set of the nodal points for the discretization. It is clear that we can localize the contribution of I_j^k

$$\begin{aligned} F(u) &= \left[\alpha \sum_{i \in [1,n] \setminus \bar{I}_j^k} |D_x^+ u_i| + \frac{1}{2} \sum_{i \in [1,n] \setminus I_j^k} (u_i - z_i)^2 \right] + \left[\alpha \sum_{i \in \bar{I}_j^k} |D_x^+ u_i| + \frac{1}{2} \sum_{i \in I_j^k} (u_i - z_i)^2 \right] \\ &= \widetilde{F_j^k}(u) + \alpha \sum_{i \in \bar{I}_j^k} |D_x^+ u_i| + \frac{1}{2} \sum_{i \in I_j^k} (u_i - z_i)^2, \end{aligned} \tag{15}$$

where $\widetilde{F_j^k}$ contains all terms not overlapping with the support of ϕ_j^k. Our task now is the following: given an initial guess $w \approx u$, how to improve w.

Our idea is to look for $u = w + c\phi_j^k$ for the best $c \in \mathbb{R}$. Recall that the above functional $F(u)$ is defined on the finest level so it is necessary to localize the formulation by collecting terms involving $c \in \mathbb{R}$ only (see (12)). Let $w = [w_1, \ldots, w_n]^T$ and $v_i = \phi_j^k(x_i)$. In the above functional (15), substitute u by

$w + c\phi_j^k$ and then combine terms involving c:

$$\begin{aligned} F(w + c\phi_j^k) &= \widetilde{F_j^k}(w) + \alpha \sum_{i\in\bar{I}_j^k} |D_x^+ w_i + cD_x^+ v_i| + \frac{1}{2}\sum_{i\in I_j^k} (w_i - z_i + cv_i)^2 \\ &= \widetilde{F_j^k}(w) + \alpha \sum_{i\in\bar{I}_j^k} |D_x^+ w_i + cD_x^+ v_i| \ + \frac{1}{2}\Big[s\left(c^2 - 2cz^* + z^{*2}\right) + \sum_{i\in I_j^k} \bar{z}_i^2 - z^{*2}s\Big] \\ &= \overline{F_j^k}(w) + \alpha \sum_{i\in\bar{I}_j^k} |D_x^+ w_i + cD_x^+ v_i| \ + \frac{s}{2}(c - z^*)^2, \quad (16) \end{aligned}$$

where $\bar{z} = z - w$, $\overline{F_j^k}(w) = \widetilde{F_j^k}(w) + \big[\sum_{i\in I_j^k} \bar{z}_i^2 - z^{*2}s\big]/2$ does not involve c (ignored in subsequent minimisation),

$$s = \sum_{i\in I_j^k} v_i^2 \qquad \text{and} \qquad z^* = \sum_{i\in I_j^k} v_i \bar{z}_i / s.$$

Therefore in 1D, solving (11) for $c \in \mathbb{R}$ is equivalent to solving

$$\min_{c\in\mathbb{R}} \Big[\alpha \sum_{i\in\bar{I}_j^k} |D_x^+ w_i + cD_x^+ v_i| \ + \frac{s}{2}(c - z^*)^2 \Big],$$

and (with β added locally) the following

$$\min_{c\in\mathbb{R}} J(c), \qquad J(c) = \Big[\alpha \sum_{i\in\bar{I}_j^k} \sqrt{\left(D_x^+ w_i + cD_x^+ v_i\right)^2 + \beta} + \frac{s}{2}(c - z^*)^2 \Big]. \quad (17)$$

Further with $c^{(0)} = 0$, implementing (13) and (14) for equation (17) leads to the iterations

$$\begin{aligned} &\Big[\alpha \sum_{i\in\bar{I}_j^k} \frac{|D_x^+ v_i|^2}{\sqrt{\left(D_x^+ w_i + c^{(\ell)} D_x^+ v_i\right)^2 + \beta}} + s \Big] c^{(\ell+1)} = \\ &\qquad \Big[sz^* - \alpha \sum_{i\in I_j^k} \frac{D_x^+ w_i D_x^+ v_i}{\sqrt{\left(D_x^+ w_i + c^{(\ell)} D_x^+ v_i\right)^2 + \beta}} \Big], \quad \text{for } \ell = 0, 1, 2, \cdots. \quad (18) \end{aligned}$$

In summary, our algorithm proceeds as follows.

Algorithm 3 *Let the signal domain $\Omega = [0, 1]$ be discretized with J levels. Start from the finest level $j = 1$ with the initial guess $w = z$:*

(1) On level j, compute $\bar{z} = z - w$ first.

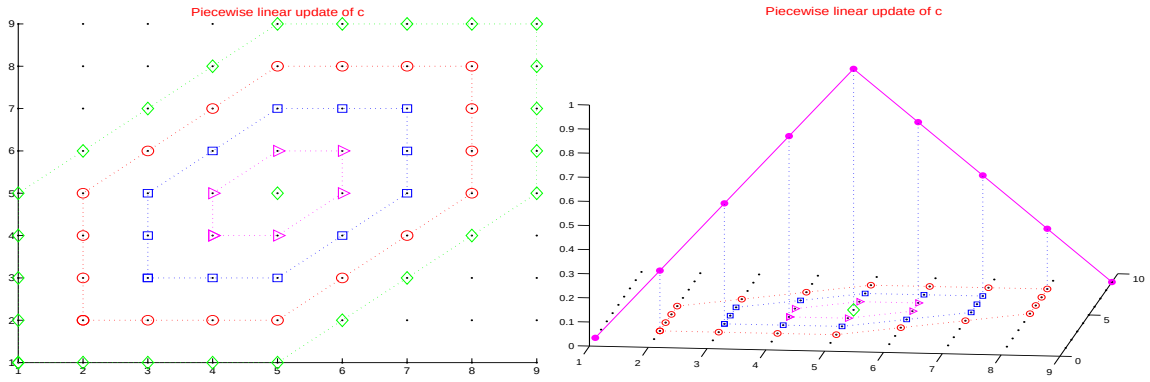
(2) For each $k = 1, \ldots, n_j$:
First work out s and z^* and then solve the local coarse problem by iterating (18) until the relative (dynamic) residual is less than τ_{inner}.
Add the correction in the (built-in) interpolation step: $w = w + c\phi_j^k(x)$.

(3) If $j < J$, set $j := j + 1$ and continue with Step (1). If $j = J$, check whether the relative (dynamic) residual is less than τ_{outer}; if yes, exit with $u = w$ as the solution or otherwise continue with Step (1) with $j = 1$.

3.2 The algorithm with $\Omega \in \mathbb{R}^2$

Secondly we can apply the same argument of simplification to the image case with $\Omega \in \mathbb{R}^2$, where we note that a 2D basis function ϕ_j^k (similar to Figure 3) may be illustrated by Figure 4. That is, the terms in the

Figure 4: The two dimensional basis function ϕ_j^k (on the coarse level $j = 3$ and at the center node k. Note on the right plot, only the weights v_ℓ along a diagonal, as in (19), are shown.) Here $\Diamond$ defines the outer boundary of the 2D basis function, $\bigcirc$ shows the nodes where the corresponding weights are 1/4, $\square$ shows the nodes where the corresponding weights are 1/2, $\triangleright$ shows the nodes where the corresponding weights are 3/4 and the central node $\Diamond$ defines the weight of 1.



functional $F(w + c\phi_j^k)$, $c \in \mathbb{R}$, from (11) may again be grouped and simplified according to the compact support of ϕ_j^k. Similar to the 1D case in (16), the values of the 2D basis function may be denoted by matrix v, which takes the values

$$v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

for the example of $j = 3$ and $b = b_j = 4$ (as in Figure 4) and when we zoom in the neighborhood of index k (as v is actually a global quantity with a compact support). Let the quantities v, Ω_j^k, I_j^k and $\bar{I}_j^k$ be defined in a similar way as for 1D problems with $k = (k_1, k_2)$. In the discretized setting, we have

$$\begin{aligned} F(u) &= \alpha \sum_{\ell_1=1}^{n} \sum_{\ell_2=1}^{m} \sqrt{(D_x^+ u_{\ell_1,\ell_2})^2 + (D_y^+ u_{\ell_1,\ell_2})^2} + \frac{1}{2} \sum_{\ell_1=1}^{n} \sum_{\ell_2=1}^{m} (u_{\ell_1,\ell_2} - z_{\ell_1,\ell_2})^2 \\ &= \widetilde{F_j^k}(u) + \alpha \sum_{k_1,k_2 \in \bar{I}_j^k} \sqrt{(D_x^+ u_{k_1,k_2})^2 + (D_y^+ u_{k_1,k_2})^2} + \frac{1}{2} \sum_{(k_1,k_2) \in I_j^k} (u_{k_1,k_2} - z_{k_1,k_2})^2, \end{aligned} \tag{20}$$

where $\widetilde{F_j^k}$ contains all terms not overlapping with the support of ϕ_j^k. Similar to the 1D case, we are ready to simplify $F(w + c\phi_j^k)$ to reveal the simplified minimisation for $c \in \mathbb{R}$ by grouping other unrelated terms (to c) into $\overline{F}$. The result is the following (refer to (15)):

$$\begin{aligned} F(w + c\phi_j^k) = F(w + cv) &= \widetilde{F_j^k}(w) + \\ &\alpha \sum_{(k_1,k_2) \in \bar{I}_j^k} \sqrt{(D_x^+ w_{k_1,k_2} + cD_x^+ v_{k_1,k_2})^2 + (D_y^+ w_{k_1,k_2} + cD_y^+ v_{k_1,k_2})^2} \\ &+ \frac{1}{2} \sum_{(k_1,k_2) \in I_j^k} (\bar{z}_{k_1,k_2} - cv_{k_1,k_2})^2, \\ &= \overline{F_j^k}(w, \bar{z}, v) + \alpha \sum_{(k_1,k_2) \in \bar{I}_j^k} T_{k_1,k_2}(c) + \frac{s}{2}(c - z^*)^2, \end{aligned} \tag{21}$$

where all three quantities $\widetilde{F}$, $\bar{z} = z - w$ and $\overline{F}$ do not involve c,

$$z^* = \sum_{(k_1,k_2)\in I_j^k} \frac{\bar{z}_{k_1,k_2} v_{k_1,k_2}}{s}, \quad s = \sum_{(k_1,k_2)\in I_j^k} v_{k_1,k_2}^2, \quad \text{and}$$

$$T_{k_1,k_2}(c) = \sqrt{|D_x^+(w_{k_1,k_2} + cv_{k_1,k_2})|^2 + |D_y^+(w_{k_1,k_2} + cv_{k_1,k_2})|^2}.$$

To solve the local problem (21), we re-define

$$T_{k_1,k_2}(c) = \sqrt{|D_x^+(w_{k_1,k_2} + cv_{k_1,k_2})|^2 + |D_y^+(w_{k_1,k_2} + cv_{k_1,k_2})|^2 + \beta},$$

adding a small parameter $\beta > 0$. Then omitting all non-essential details, we find that the updating of (14) or (13) in the discretized setting for a 2D problem proceeds as follows:

$$\left[\alpha \sum_{(k_1,k_2)\in \bar{I}_j^k} \frac{|D_x^+ v_{k_1,k_2}|^2 + |D_y^+ v_{k_1,k_2}|^2}{T_{k_1,k_2}(c^{(\ell)})} + s\right] c^{(\ell+1)} =$$

$$\left[sz^* - \alpha \sum_{(k_1,k_2)\in \bar{I}_j^k} \frac{D_x^+ w_{k_1,k_2} D_x^+ v_{k_1,k_2} + D_y^+ w_{k_1,k_2} D_y^+ v_{k_1,k_2}}{T_{k_1,k_2}(c^{(\ell)})}\right], \quad \text{for } \ell = 0, 1, 2, \cdots. \quad (22)$$

Putting all the steps together, we give the following

Algorithm 4 *Let the image domain $\Omega = [0,1] \times [0,1]$ be discretized with J levels. Start from the finest level $j = 1$ with the initial guess $w = z$ over $n_1 \times m_1 = n \times m$ pixel points:*

(1) On level j, compute $\bar{z} = z - w$ first.

(2) For each index $k = (k_1, k_2)$ with $k_1 = 1, \ldots, n_j$ and $k_2 = 1, \ldots, m_j$,
First work out s and z^* and then solve the local coarse problem by iterating (22) until the relative (dynamic) residual is less than τ_{inner}.
Add the correction in the (built-in) interpolation step: $w = w + c\phi_j^k(x)$.

(3) If $j < J$, set $j := j + 1$ and continue with Step (1). If $j = J$, check whether the relative (dynamic) residual is less than τ_{outer}; if yes, exit with $u = w$ as the solution or otherwise continue with Step (1) with $j = 1$.

Here, on level j, $n_j = (n-1)/2^{j-1} + 1$ and $m_j = (m-1)/2^{j-1} + 1$ define $n_j \times m_j$ basis functions.

Finally we briefly discuss the complexity issue. For linear problems, the cost per iteration for the multigrid iteration is typically $O(DOF)$ where DOF is the total number of degrees of freedom. For our nonlinear problems, the cost per iteration by our Algorithms 3 and 4 is $O(DOF \log(DOF))$. To verify this result, we may consider the 2D case with $DOF = N = mn$. Then the size of the set $\bar{I}_j^k$ is $2^{j-1} \times 2^{j-1}$ while the size of the set I_j^k is less than that of $\bar{I}_j^k$. Computing $\bar{z}$ requires N flops (floating point operations). For each k on level j, computing z^* and s requires $4b_j$ flops (with $b_j = 4^{j-1}$) so the total number of flops for level j is $4b_j n_j m_j \approx 4N$. Let t steps be needed for a typical inner iteration which corresponds to about $32b_j t n_j m_j \approx 32tN$. Hence over all J levels, the number of flops is $(N + 4N + 32tN)J = O(N \log N)$ since $\max(J) \leq log_2 \min(m, n) = O(N \log N)$.

4 Numerical algorithms and experiments

To demonstrate the effectiveness of our Algorithms 3 and 4, denoted by MG below, we now present some experimental results. We remark that the above proposed algorithms have not been applied to the image minimisation problem (2) before. It is pleasing to see some good results for the first time.

Table 1: Residual information for Problem 3 with up to 100 MG cycles.

Problem Size	Levels	MG cycles	Residual
17×17	5	87	8.5E-10
33×33	6	100	5.9E-10
65×65	7	93	5.0E-10
129×129	8	87	1.1E-11
257×257	9	77	2.2E-11

We shall first test the algorithms' effectiveness by solving a few image denoising problems in both 1D and 2D. Then we experiment on the dependence of the convergence of the proposed multigrid algorithm on image sizes and algorithm parameters. Finally we experiment on the influence of the inner Picard fixed point iterations (12) on the overall convergence performance. As we see, the method is not sensitive to the choice of problem sizes and accuracy of the inner Picard type fixed point iterations. Finally we compare and remark on the advantages of our algorithms over the popular method of [14]. In a simple word, our algorithms (being multilevel) are fast, robust and reliable.

4.1 Test problems and results

We shall consider 4 signal denoising problems as shown in Figure 5 and another 4 image denoising problems as shown in Figure 7. The signal-to-noise ratio (SNR) is taken as 10 (for smaller SNR all iterations will be less, as expected). The iterative method will be stopped whenever the relative dynamic residual $\|u^{(\ell)} - u^{(\ell-1)}\|_2/\|u^{(\ell)}\|_2 < \tau_{outer}$ for a prescribed tolerance τ_{outer}. Then ℓ will be the number of outer iteration steps (or *cycles*). There is another prescribed tolerance τ_{inner} which is to control how accuracy the iterations should be in the solution of the local minimisation (11). Here we take $\tau_{inner} = \tau_{outer} = 10^{-3}$ and $\beta = 10^{-4}$ for the regularising parameter. The processed results by our algorithm is shown in Figure 6 (for $N = 4097$) and Figure 8 (for $N \times N = 257 \times 257$) respectively, where the symbol $\square$ refers to our algorithm while the symbol $\times$ the method of [14]. Clearly one observes that our method converges quite quickly and gives a result which is not distinguishable from the result of [14].

4.2 Test of convergence of the method

When $\beta \neq 0$, the convergence theory developed in [35, 38] may be invoked to establish a convergence result of our algorithms. Here we hope to give some numerical tests to demonstrate the convergence behaviour for the specific example of Problem 3 with $\alpha = 15$. First we show in Figure 9 the convergence history of MG residuals for $\tau = \tau_{outer} = 10^{-10}$ and $n = 129$. Second we show some more residual information in Table 1 for various n with up to 100 steps of the MG method to achieve $\|r\| \leq \tau = 10^{-10}$. Clearly convergence slows downs at we approach the machine accuracy but it is not much depending on n. Hence in the following tests, we shall restrict ourselves to a larger τ.

To test further on the sensitivity of the method on problem size n, it is of interest to investigate any dependence of the overall algorithm convergence as the problem sizes increase (n in 1D signals and $n \times n$ in 2D images). In Table 2, we fix both tolerances $\tau_{inner}, \tau_{outer}$ and vary the problem size to see how many convergence steps are needed. Clearly one observes that the convergence of our method is not much affected by n, especially for the 2D problems. For the 1D case, the convergence patterns become clear and the number of steps (i.e. *MG cycles*) approaches a constant as n increases.

4.3 Sensitivity to the inner fixed point iterations

We next address how crucial the inner nodal solver is for the overall algorithm. To this end, we fix the problem size n and the tolerance τ_{outer}. Table 3 shows the results obtained for the selected test problems in 1D and 2D from varying the inner solver tolerance τ_{inner} within the range of a value below τ_{outer} to

Table 2: Test of dependence of the problem sizes (n in 1D and $n \times n$ in 2D): 'Dim' denotes 'Dimension', 'Prob' stands for 'Problem number', 'Levels' indicates the "levels used in the multilevel algorithm" and 'Steps' the "number of multilevel cycles'. Here $\tau = 10^{-3}$, $\beta = 10^{-4}$. Clearly there is no strong dependence. Here the Problem numbers refer to Figure 5 for 1D and Figure 7 for 2D.

Dim	Prob	Size	Levels	Steps	Prob	Size	Levels	Steps
1D	1	65	6	25	2	65	6	17
		129	7	11		129	7	11
		257	8	7		257	8	5
		513	9	8		513	9	6
		1015	10	5		1015	10	4
		2049	11	4		2049	11	4
		4097	12	3		4097	12	4
		8193	13	4		8193	13	4
		16385	14	3		16385	14	4
		32769	15	3		32769	15	4
		65537	16	3		65537	16	4
1D	3	65	6	9	4	65	6	34
		129	7	7		129	7	23
		257	8	8		257	8	18
		513	9	5		513	9	12
		1015	10	5		1015	10	8
		2049	11	5		2049	11	6
		4097	12	5		4097	12	4
		8193	13	4		8193	13	4
		16385	14	4		16385	14	4
		32769	15	4		32769	15	4
		65537	16	4		65537	16	4
2D	1	33×33	5	6	2	33×33	5	5
		65×65	6	6		65×65	6	5
		129×129	7	6		129×129	7	5
		257×257	8	6		257×257	8	5
2D	3	33×33	5	6	4	33×33	5	5
		65×65	6	6		65×65	6	5
		129×129	7	6		129×129	7	5
		257×257	8	6		257×257	8	5

Table 3: Test of dependence of the accuracy of the inner nodal solver ($n = 8193$ and Levels=13 in 1D and $n \times n = 257 \times 257$ and Levels= 8 in 2D): 'Levels' indicates the "levels used in the multilevel algorithm" and 'Steps' the "number of multilevel cycles'. Here $\tau = \beta = 10^{-4}$ and τ_{inner} is the tolerance used for each nodal relaxation solver (note: the minimal number of relaxation steps is 1). Clearly there is no strong dependence. Here again, the Problem numbers refer to Figure 5 for 1D and Figure 7 for 2D.

Dimension	Problem	τ_{inner}	Steps	Problem	τ_{inner}	Steps
1D	1	10^{-5}	8	2	10^{-5}	4
		10^{-4}	8		10^{-4}	4
		10^{-3}	8		10^{-3}	4
		10^{-2}	8		10^{-2}	4
		10^{-1}	8		10^{-1}	4
		10^{-0}	8		10^{-0}	4
		10^{+1}	8		10^{+1}	4
1D	3	10^{-5}	5	4	10^{-5}	11
		10^{-4}	5		10^{-4}	11
		10^{-3}	5		10^{-3}	11
		10^{-2}	5		10^{-2}	11
		10^{-1}	5		10^{-1}	11
		10^{-0}	5		10^{-0}	11
		10^{+1}	5		10^{+1}	11
2D	1	10^{-5}	10	2	10^{-5}	5
		10^{-4}	10		10^{-4}	5
		10^{-3}	10		10^{-3}	5
		10^{-2}	8		10^{-2}	5
		10^{-1}	8		10^{-1}	5
		10^{-0}	10		10^{-0}	5
		10^{+1}	11		10^{+1}	5
2D	3	10^{-5}	6	4	10^{-5}	7
		10^{-4}	6		10^{-4}	7
		10^{-3}	6		10^{-3}	7
		10^{-2}	6		10^{-2}	7
		10^{-1}	6		10^{-1}	7
		10^{-0}	6		10^{-0}	8
		10^{+1}	6		10^{+1}	9

another much larger value. Clearly the overall multilevel method is not much affected. Note that for the cases associated with using the largest tolerance $\tau_{inner} = 10$, the number of inner iterations is mostly one and hence the inner solver is far from convergence and yet the outer iterations can converge. This latter observation is somewhat related to the inner-outer iteration control as shown in [21] and adopted in the algorithm of [14]. It is possible to work out an appropriate formula for τ_{inner}.

4.4 Sensitivity of the parameters α and β

There are two general issues here. Firstly one cares about whether or not α and β affect the convergence of a method. Secondly for difficult choices of α and β, one desires for a remedial solution. Here we mainly test the former as our algorithms are not sensitive to such parameter changes. As for the latter question with other sensitive methods, one should consider the parameter continuation idea as used and discussed in [18, 47].

We take two test examples as shown in Figs. 10 and 11. We have done the following experiments (for the tolerance of $\tau = 10^{-3}$) in Table 4. Here we measure the restoration qualitatively by the peak

Table 4: Test of dependence of the parameters α and β

Problem	α	β	MGM cycles	PSNR
3	1.25	10^{-12}	6	20.67
	2.50		6	21.31
	5.00		6	22.53
	10.0		6	24.38
	20.0		8	24.68
3	20.0	10^{-4}	8	24.68
		10^{-8}	8	24.68
		10^{-12}	8	24.68
		10^{-16}	8	24.68
		10^{-20}	8	24.68
5	1.88	10^{-12}	6	21.12
	3.75		6	22.26
	7.50		6	24.56
	15.0		6	27.94
	30.0		9	28.75
5	30.0	10^{-4}	8	28.69
		10^{-8}	8	28.70
		10^{-12}	8	28.70
		10^{-16}	8	28.70
		10^{-20}	8	28.71

signal-to-noise ratio (PSNR) defined by (see e.g. [11])

$$\mathrm{PSNR}(u,w) = 10\log_{10}\frac{255^2}{\frac{1}{mn}\sum_{i,j}(u_{i,j}-w_{i,j})^2},$$

where $w_{i,j}$ and $u_{i,j}$ denote the pixel values of the restored and the original images respectively.

Clearly from Table 4, we observe that the convergence of Algorithm 4 is not significantly affected by parameter changes. Evidently changing α leads to different restorations and hence the PSNR values as expected.

One may wonder why our optimisation MG is less sensitive to the above parameters while the PDE MG is more sensitive for the essentially same model (1). We believe that this is due to the PDE (2) attempting to assign a normal at pixels where such geometrical information is not defined while the optimisation does not require such assignments.

4.5 Comparisons with an established method

There are many aspects that could be compared with other methods. Here we choose to compare with the well-known method (perhaps the best) of Chan-Golub-Mulet (CGM) [14] as other methods such as the fixed point iterations and time marching schemes have been shown to be slower than a multigrid method [32]. However our task of comparing with CGM becomes somewhat easier because the CGM method 'fails' in 2 cases: (i) when the image size N becomes large (due to ill-conditioning); (ii) when $\beta \leq 10^{-32}$ (due to singularity). Here (i), not (ii), may be fixable but no such work is available for the primal-dual method. (However there exists important work of β-free methods [1, 7, 13]; of these the dual method is the most well-known.) In either of these cases, our method would converge although the local solvers take a few more iterations.

It may be of interest to show some results from parameter ranges where the CGM performs well: we take $\beta = 10^{-20}$ and 2 test examples in Figs. 10 and 11. Here we mainly compare the solution's visual quality and the PSNR values. As seen from Figs. 10 and 11, $\mathrm{PSNR}(u_{CGM}) = 24.60$ and $\mathrm{PSNR}(u_{MG}) = 24.70$ for problem 3, and $\mathrm{PSNR}(u_{CGM}) = 28.27$ and $\mathrm{PSNR}(u_{MG}) = 28.74$ for problem 4. Since the

PSNR values of the results from our algorithm are quite close to the CGM results, the restored images are indeed indistinguishable.

For larger images, our MG method can solve problems 3 and 4 in a reasonable time (as seen below) on a Sun-Blade 1000 with Matlab 6.5:

Problem	N	MGM cycles	PSNR	CPU
3	513	4	31.48	3165.6
	1025	4	34.03	14478.0
4	513	4	30.16	4291.0
	1025	4	33.09	14428.0

In contrast, CGM cannot be run because the memory requirement is too large.

Therefore our algorithm is evidently more robust (without having to concern about what parameters to use) and being a multilevel method there is a scope to achieve even better performance with future parallelization.

5 Conclusions

This paper has introduced a nonlinear multigrid method for solving curvature equations related to total variation minimization. The resulting algorithms are efficient but are different from the existing optimisation multigrid methods in coarse level construction.

Numerical tests show that the convergence for the MGM algorithm is mesh independent for a large range of α; however extremely large α may be tackled by using the continuation idea based on causality as recommended by [47] and likewise extremely small β should be combined with the continuation step on β as in [18]. The parameter β is used to control the width of the jumps and the smallest width is over one mesh element. For most algorithms, β is chosen in this range but our algorithm offers mesh independent convergence which is also not sensitive to the values of β. It is known that the CGM algorithm [14] is rather robust with respect to α and β. For most of the experiments we have done, we need less than 10 outer iterations to get a result as nearly undistinguishable as the unigrid method [14]. In fact, the CGM method fails to converge when we take $\beta = 10^{-32}$ while our method converges. If we must take $\beta = 0$, all we need to do is to replace our local minimisation solvers.

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Figure 5: The 1D test examples

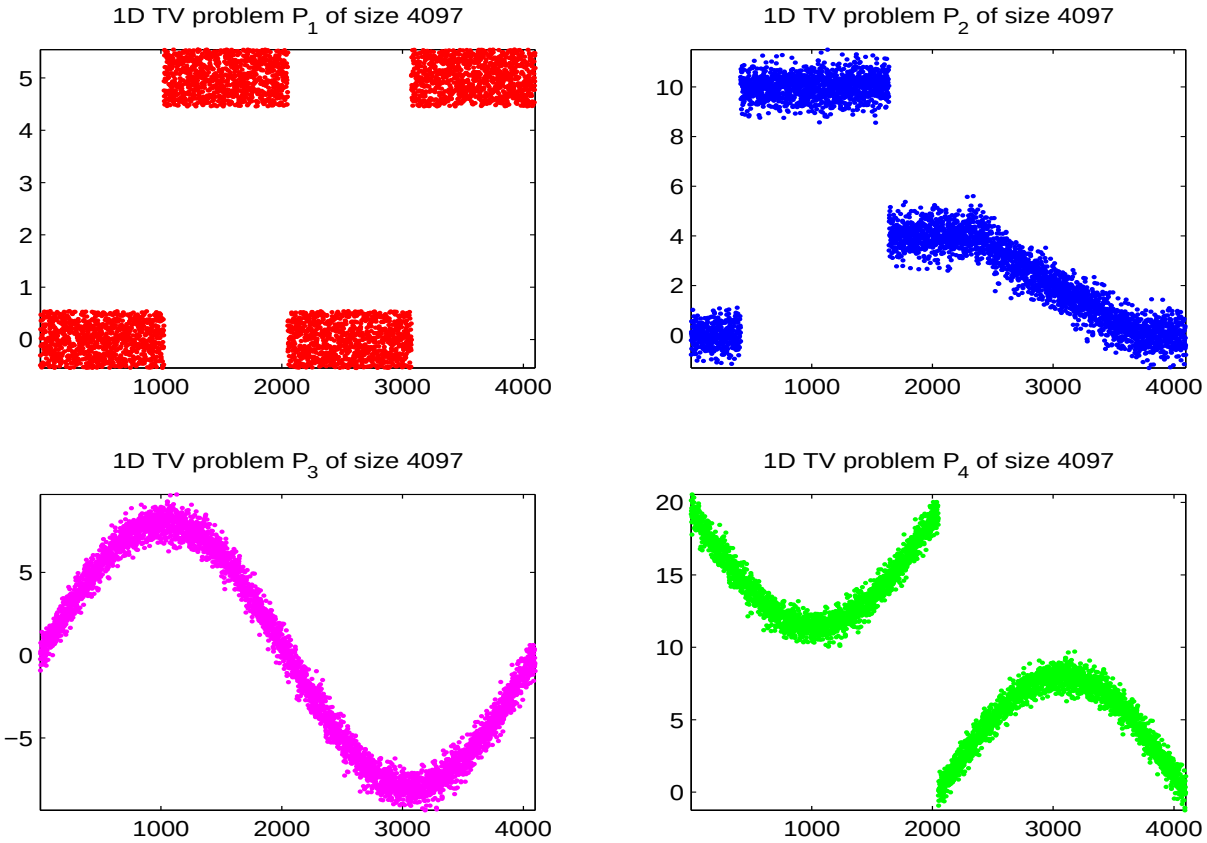

Figure 6: The 1D processed results with solutions from the two methods superimposed on each other: □ - the new multilevel algorithm and × - the primal-dual method [14].

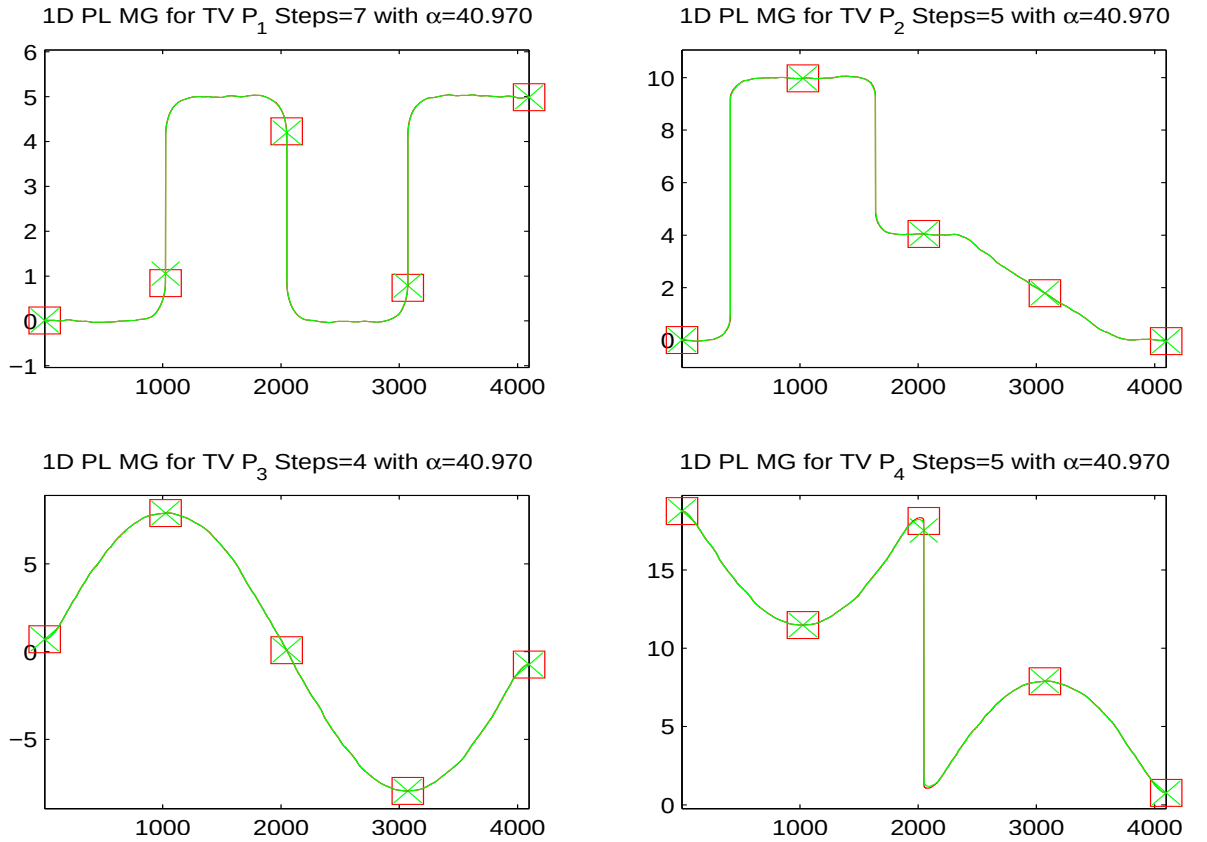

Figure 7: The 2D test examples

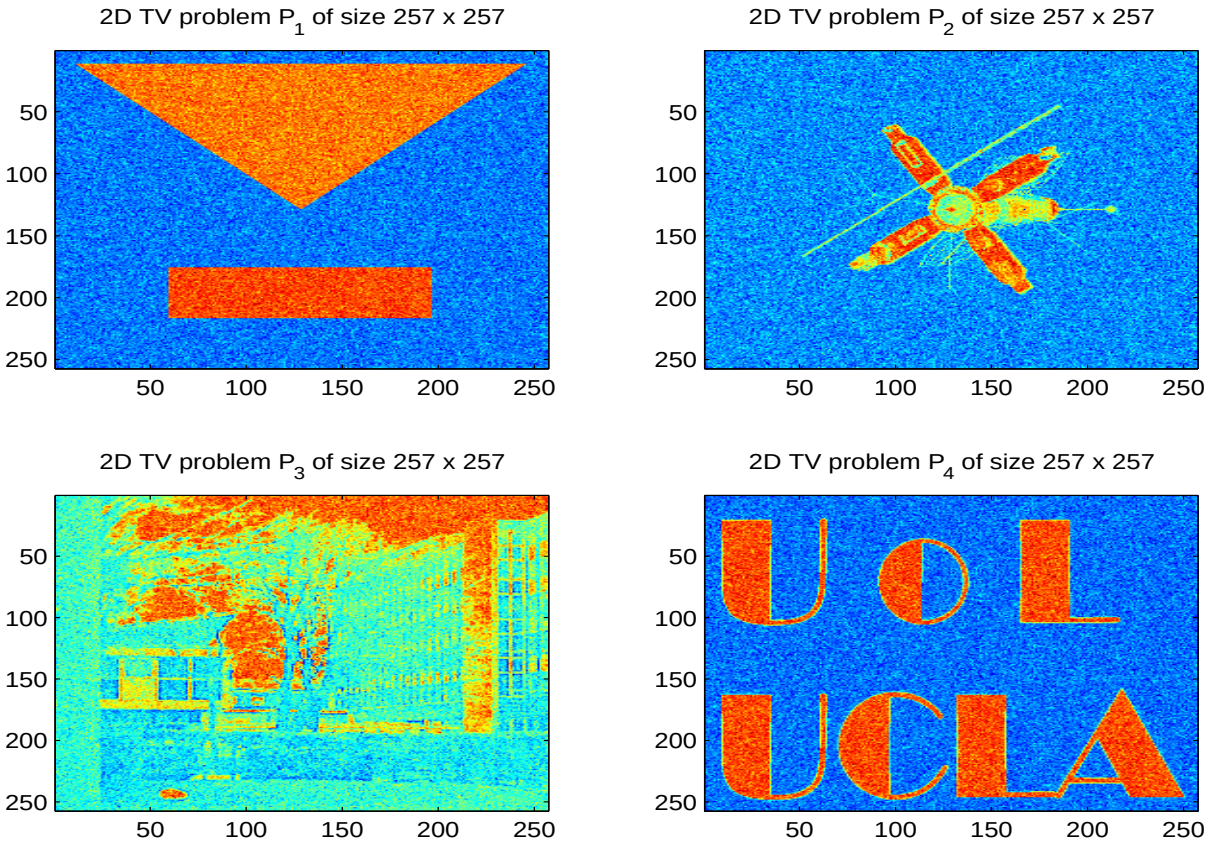

Figure 8: The 2D multigrid restored results

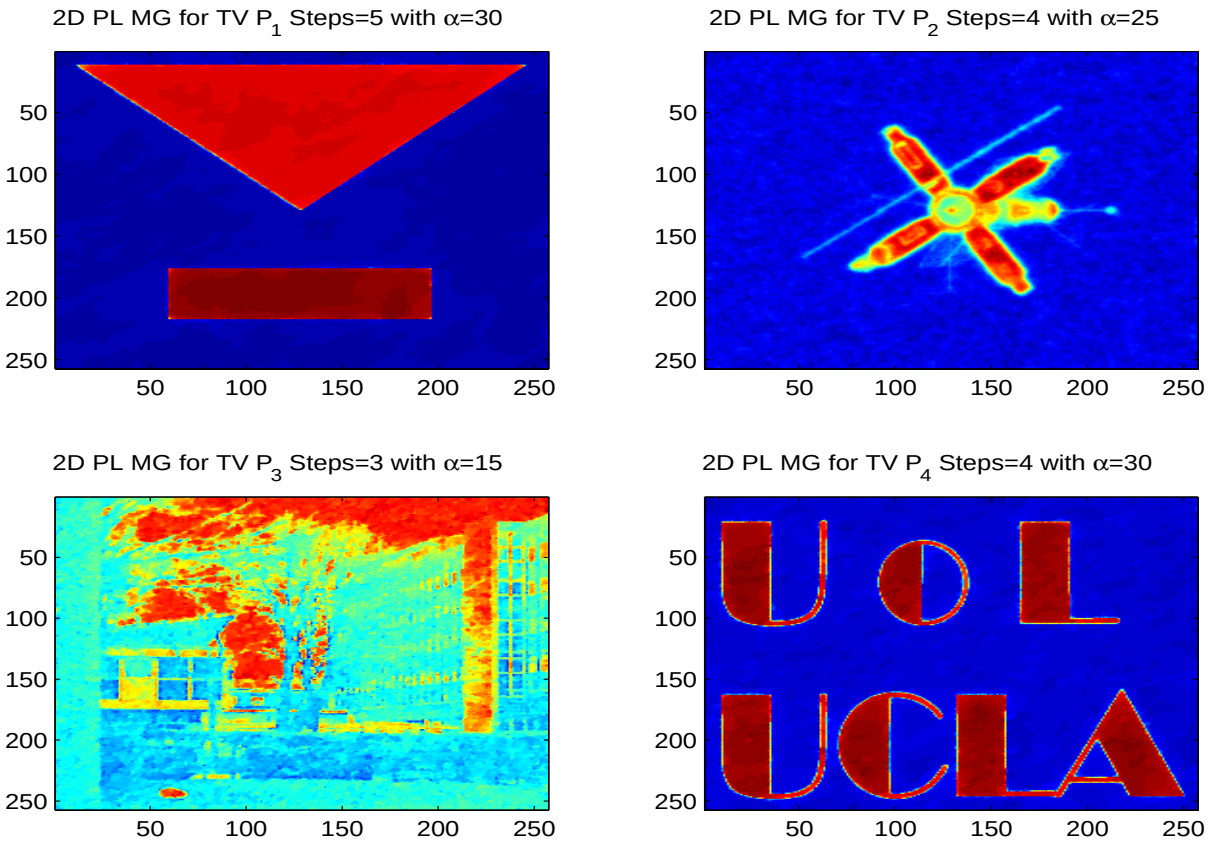

Figure 9: Residual history for Problem 3 with $n = 129$

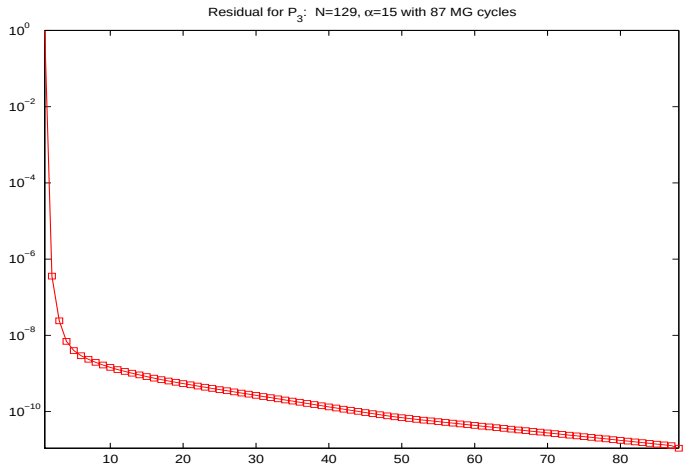

Figure 10: Comparison with the CGM method [14] for test example P_3: $\alpha = 20$ and $\beta = 10^{-20}$

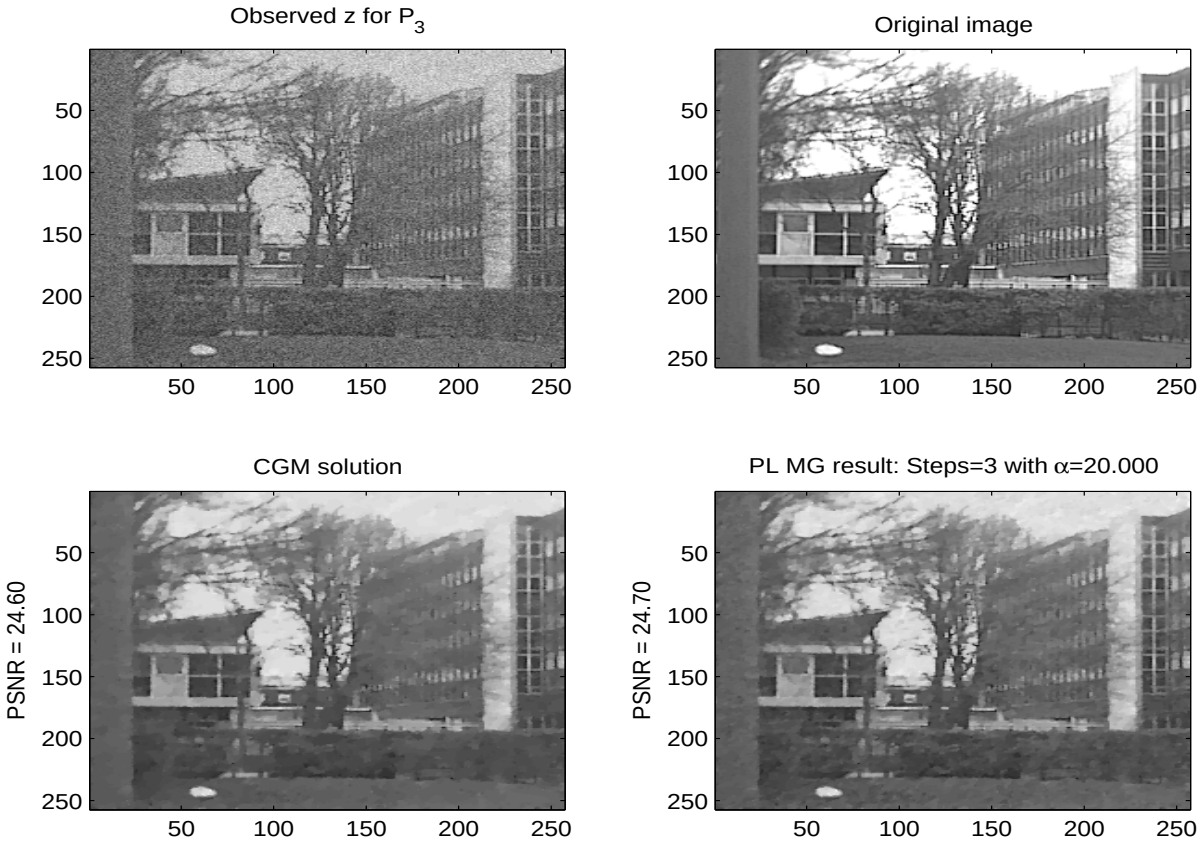

Figure 11: Comparison with the CGM method [14] for test example P_5: $\alpha = 30$ and $\beta = 10^{-20}$