On efficient preconditioners for iterative solution of a Galerkin boundary element equation for the three dimensional exterior Helmholtz problem

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Abstract

The paper presents a Galerkin numerical method for solving the hyper-singular boundary integral equations for the exterior Helmholtz problem in three dimensions with a Neumann's boundary condition. Previous work in the topic has often dealt with the collocation method with a piecewise constant approximation because high order collocation and Galerkin methods are not available due to the presence of a hypersingular integral operator. This paper proposes a high order Galerkin method by using singularity subtraction technique to reduce the hyper-singular operator to a weakly-singular one. Moreover, we show here how to extend the previous work (Chen and Harris [13]) on sparse preconditioners to the Galerkin case leading to fast convergence of two iterative solvers: the conjugate gradient normal method (CGN) and the generalised minimal residual method (GMRES). A comparison with the collocation method is also presented for the Helmholtz problem with several wavenumbers.

Keywords:
Exterior Helmholtz, Boundary integral equation, Burton-Miller, Hyper-singular operators, Galerkin, Preconditioners, CGN, GMRES.

1 Introduction

This paper is concerned with using the boundary integral method (BIM) to solve the exterior Helmholtz problem. Such problems arise in the mathematical modelling of the steady-state single frequency acoustic field outside an object immersed in a fluid; see [2, 14]. We are mainly concerned with two problems in this paper: (1) how to apply a high order Galerkin method; (2) how to design an effective sparse preconditioner for the Galerkin method. A comparison with the collocation method as done in our previous work [13] is presented.

There are two methods for reformulating the underlying differential equation as an integral equation, the direct and indirect method. However, for either method, it is well known that for certain frequencies the solution to integral equation formulations either does not exist or is non-unique. Over the years a number of methods have been proposed for overcoming these problems, and these are surveyed in Amini et al [2]. This paper will employ a direct method for obtaining an integral equation formulation of the problem, and use the Burton and Miller method [4] for obtaining an integral equation which has a unique solution for all frequencies.

We remark that for the hyper-singular formulation for the Laplace equation, several papers have discussed the Galerkin method. Both Giroire and Nedelec [20] and Hackbusch [23] use a method similar to the one presented here for transforming the hyper-singular integral appearing in the boundary integral formulation of Laplace's equation into one which is at worst weakly singular. However, in neither case do they present any numerical results of using the formulation and the analysis was not extended to include the Helmholtz problem. In fact, there does not appear to exist a Galerkin method for the Helmholtz equation as general as the one presented in this paper, as far as we are aware.

In what follows, we shall first introduce the exterior Helmholtz problem and its reformulations. Then we discuss the numerical solution techniques leading up to our proposed Galerkin method. Finally we address the fast iterative solution of the resulting linear system by constructing effective preconditioners.

Consider the problem of solving the Helmholtz equation

$$\nabla^2 \phi(p) + k^2 \phi(p) = 0, \qquad p \in D_+ \cup S \tag{1}$$

in some unbounded three dimensional region D_+ exterior to a closed surface S, where $k > 0$ is the wavenumber, subject to a Neumann boundary condition on S and the Sommerfeld radiation condition

$$\lim_{r\to\infty} r \left(\frac{\partial \phi}{\partial r} - ik\phi \right) = 0. \tag{2}$$

Since the above exterior problem cannot be easily solved by discretising the infinite domain using finite differences or finite elements, we choose to reformulate the problem as an integral equation over the surface S; see [2, 14, 13]. However, domain methods based on infinite elements or coupled finite and infinite elements have been devised; see [18, 19] for example.

2 BIM Formulation and the collocation method

The reformulation as an integral equation can be done by either representing the solution in terms of a surface potential function (the indirect method) or by an application of Green's theorem to get a relationship between the acoustic pressure and its normal derivative (the direct method). It is the direct method which will be used here.

A simple application of Green's second theorem leads to

$$\int_S \phi(q)\frac{\partial G_k(p,q)}{\partial n_q} - G_k(p,q)\frac{\partial \phi(q)}{\partial n_q}\, dS_q = \begin{cases} \frac{1}{2}\phi(p) & p \in S \\ \phi(p) & p \in D_+ \end{cases} \tag{3}$$

where

$$G_k(p,q) = \frac{e^{ik|p-q|}}{4\pi|p-q|} \tag{4}$$

is the free-space Green's function, or fundamental solution, for Helmholtz equation and n_q is the unit outward normal to S at q. If the normal derivative of the acoustic field is given on the surface S then (3) for $p \in S$ gives a Fredholm integral equation of the second kind which can be solved for the surface pressure ϕ. The acoustic pressure can then be computed at any point in D_+ using (3). However, it is well known that (3) does not possess a unique solution for certain values of the wavenumber, called characteristic wavenumbers. This is a manifestation of the integral equation formulation as it can be shown that the underlying differential problem has a unique solution for all real and positive values of k [5].

The Burton and Miller method [4] for overcoming the non-uniqueness problem consists of differentiating (3) along the normal at p to give

$$\int_S \phi(q)\frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} - \frac{\partial G_k(p,q)}{\partial n_p}\frac{\partial \phi(q)}{\partial n_q}\, dS_q = \frac{1}{2}\frac{\partial \phi(p)}{\partial n_p} \tag{5}$$

and then taking a linear combination of (3) and (5) in the form

$$\begin{aligned} -\frac{1}{2}\phi(p) + \int_S \phi(q)\left(\frac{\partial G_k(p,q)}{\partial n_q} + \alpha\frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q}\right) dS_q = \\ \frac{\alpha}{2}\frac{\partial \phi(p)}{\partial n_p} + \int_S \frac{\partial \phi(q)}{\partial n_q}\left(G_k(p,q) + \alpha\frac{\partial G_k(p,q)}{\partial n_p}\right) dS_q \end{aligned} \tag{6}$$

where α is a coupling constant. It can be shown that provided that the imaginary part of α is non-zero then (6) has a unique solution for all real and positive k, and it has been further shown by Amini and Harris [1, 2] that by choosing $\alpha = \frac{i}{k}$ then the condition number of the integral operator is close to its minimum value. However, this formulation has introduced the kernel function $\frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q}$ which has a $\frac{1}{|p-q|^3}$ singularity. Using finite part interpretation of hyper-singular integrals, some workers have generated special quadrature rules for interpretating integrals involving the second derivative of the Green's function, but such rules will not be considered here.

To implement the collocation method, it is possible to use the result (refer to [2, 25])

$$\int_S \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q}\, dS_q = k^2 \int_S G_k(p,q) n_p \cdot n_q \, dS_q, \tag{7}$$

in order to write

$$\begin{aligned} \int_S \phi(q) \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q}\, dS_q = & \int_S (\phi(q) - \phi(p)) \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q}\, dS_q \\ & + k^2 \phi(p) \int_S G_k(p,q) n_p \cdot n_q \, dS_q. \end{aligned} \tag{8}$$

Then ϕ and $\frac{\partial \phi}{\partial n}$ are approximated by

$$\phi(q) = \sum_{j=1}^{m} \phi_j \psi_j(q), \qquad \frac{\partial \phi(q)}{\partial n} = \sum_{j=1}^{m} v_j \psi_j(q) \tag{9}$$

where $\{\psi_1, \psi_2, \ldots, \psi_m\}$ are a set of linearly independent piecewise polynomial basis functions of degree τ; $\tau \geq 0$. Substituting (9) into (6) and (8), and applying the collocation method at points $\{p_1, p_2, \ldots, p_m\}$ yields

$$\begin{aligned} \sum_{j=1}^{m} \phi_j \Bigg[-\frac{1}{2} \psi_j(p_i) + \int_S \left(\psi_j(q) \frac{\partial G_k(p_i,q)}{\partial n_q} + \alpha (\psi_j(q) - \psi_j(p_i)) \frac{\partial^2 G_k(p_i,q)}{\partial n_p \partial n_q} \right) dS_q \\ + \psi_j(p_i) \alpha k^2 \int_S G_k(p_i,q) n_p \cdot n_q \, dS_q \Bigg] = \\ \sum_{j=1}^{m} v_j \left[\frac{\alpha}{2} \psi_j(p_i) + \int_S \psi_j(q) \left(G_k(p_i,q) + \alpha \frac{\partial G_k(p_i,q)}{\partial n_p} \right) dS_q \right]. \end{aligned} \tag{10}$$

If the basis functions are chosen to give a piecewise constant approximation to ϕ (and $\frac{\partial \phi}{\partial n}$) then all the integrals appearing in (10) are at worst weakly singular. However, if a higher order approximation is used then the term involving the second derivative becomes a Cauchy Principal Value integral which is difficult to evaluate. Hence, the piecewise constant approximation collocation method has been widely used in practise.

3 The high order Galerkin method

We now present an alternative way of using high order methods via the Galerkin formulation to solve the integral equation (6). Essentially our method reformulates the hyper-singular integral equation into a weakly-singular one. As the operator in (6) is not Hermitian, we don't have to construct complex basis functions because the resulting linear system will not be Hermitian. Therefore in our Galerkin method, we shall use the well-known real trial basis functions $\psi_j(p) \in H^{\tau - 1/2}(S)$ and construct interpolation functions with complex coefficients as in (9).

Taking the inner product of (6) with each basis function in turn gives

$$\begin{aligned}&\sum_{j=1}^{m}\phi_j\int_S\psi_i(p)\left[\frac{1}{2}\psi_j(p)+\int_S\left(\psi_j(q)\frac{\partial G_k(p,q)}{\partial n_q}+\alpha\psi_j(q)\frac{\partial^2 G_k(p,q)}{\partial n_p\partial n_q}\right)dS_q\right]dS_p\\&=\sum_{j=1}^{m}v_j\int_S\psi_i(p)\left[\frac{\alpha}{2}\psi_j(p)+\int_S\psi_j(q)\left(G_k(p,q)+\alpha\frac{\partial G_k(p,q)}{\partial n_p}\right)dS_q\right]dS_p\end{aligned}\tag{11}$$

where the last integral on the first line

$$\int_S\int_S\psi_i(p)\;\psi_j(q)\;\frac{\partial^2 G_k(p,q)}{\partial n_p\,\partial n_q}\,dS_q\,dS_p\tag{12}$$

is hyper-singular while all other integrals are weakly singular.

The main result of this section, which gives a formulation where we only have to evaluate integrals which are at worst weakly singular, is now stated as follows.

Lemma 1 *For piecewise polynomial basis functions $\{\psi_j\}$ of any order τ,*

$$\begin{aligned}&\int_S\int_S\psi_i(p)\;\psi_j(q)\;\frac{\partial^2 G_k(p,q)}{\partial n_p\,\partial n_q}\,dS_q\,dS_p\\&\qquad=\frac{1}{2}\int_S\int_S\left[\psi_i(p)-\psi_i(q)\right]\;\left[\psi_j(q)-\psi_j(p)\right]\;\frac{\partial^2 G_k(p,q)}{\partial n_p\,\partial n_q}\,dS_q\,dS_p\\&\qquad\quad+k^2\int_S\psi_i(p)\;\psi_j(p)\left[\int_S\;n_p\cdot n_q\;G_k(p,q)\;\;dS_q\right]dS_p,\end{aligned}\tag{13}$$

where all the integrals on the right-hand side are weakly singular.

Proof. First reformulate the integral (12) as

$$\begin{aligned}&\int_S\int_S\psi_i(p)\;\psi_j(q)\;\frac{\partial^2 G_k(p,q)}{\partial n_p\,\partial n_q}\;\;dS_q\;\;dS_p\\&\qquad=\int_S\int_S\left[\psi_i(p)-\psi_i(q)\right]\;\left[\psi_j(q)-\psi_j(p)\right]\;\frac{\partial^2 G_k(p,q)}{\partial n_p\,\partial n_q}\;\;dS_q\;\;dS_p\\&\qquad\qquad+\int_S\int_S\psi_i(p)\;\psi_j(p)\;\frac{\partial^2 G_k(p,q)}{\partial n_p\,\partial n_q}\;\;dS_q\;\;dS_p\\&\qquad\qquad+\int_S\int_S\psi_i(q)\;\psi_j(q)\;\frac{\partial^2 G_k(p,q)}{\partial n_p\,\partial n_q}\;\;dS_q\;\;dS_p\\&\qquad\qquad-\int_S\int_S\psi_i(q)\;\psi_j(p)\;\frac{\partial^2 G_k(p,q)}{\partial n_p\,\partial n_q}\;\;dS_q\;\;dS_p.\end{aligned}\tag{14}$$

Now note that the first integral on the right hand side is weakly singular and so can be evaluated using suitable quadrature rules. Using (7), the second integral on the right hand side is seen to be non-singular as

$$\begin{aligned}\int_S \int_S \psi_i(p)\psi_j(p)\frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} dS_q dS_p = & \int_S \psi_i(p)\psi_j(p)\left[\int_S \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} dS_q\right] dS_p \\ = & \ k^2 \int_S \psi_i(p)\,\psi_j(p)\int_S n_p.n_q\, G_k(p,q) dS_q dS_p.\end{aligned} \tag{15}$$

We note that $\frac{\partial^2 G_k(p,q)}{\partial n_p\, \partial n_q} = \frac{\partial^2 G_k(p,q)}{\partial n_q\, \partial n_p} = \frac{\partial^2 G_k(q,p)}{\partial n_q\, \partial n_p}$ and so it follows that by reversing the order of integration the following relationships hold:

$$\int_S \int_S \psi_i(q)\,\psi_j(p)\frac{\partial^2 G_k(p,q)}{\partial n_p\, \partial n_q}\, dS_q\, dS_p = \int_S \int_S \psi_i(p)\,\psi_j(q)\frac{\partial^2 G_k(p,q)}{\partial n_p\, \partial n_q}\, dS_q\, dS_p, \tag{16}$$

$$\int_S \int_S \psi_i(q)\,\psi_j(q)\frac{\partial^2 G_k(p,q)}{\partial n_p\, \partial n_q}\, dS_q\, dS_p = \int_S \int_S \psi_i(p)\,\psi_j(p)\frac{\partial^2 G_k(p,q)}{\partial n_p\, \partial n_q}\, dS_q\, dS_p. \tag{17}$$

Therefore equation (14) takes the interesting but elementary form:

$$\begin{aligned}&\int_S \int_S \psi_i(p)\ \psi_j(q)\ \frac{\partial^2 G_k(p,q)}{\partial n_p\, \partial n_q}\ \ dS_q\ \ dS_p \\ &= \int_S \int_S [\psi_i(p) - \psi_i(q)]\ \ [\psi_j(q) - \psi_j(p)]\ \frac{\partial^2 G_k(p,q)}{\partial n_p\, \partial n_q}\ \ dS_q\ \ dS_p \\ &+ 2k^2 \int_S \psi_i(p)\,\psi_j(p)\ \int_S n_p.n_q\, G_k(p,q)\ dS_q\, dS_p \\ &- \int_S \int_S \psi_i(p)\ \psi_j(q)\ \frac{\partial^2 G_k(p,q)}{\partial n_p\, \partial n_q}\ \ dS_q\ \ dS_p,\end{aligned}$$

from which the required result follows by solving for the hyper-singular integral. ■

Hence we have successfully derived a Galerkin formulation that enables us to use basis functions of any order τ without having to construct special quadrature rules to deal with the hypersingular integrals. The reformulated form of (11) is the following

$$\begin{aligned}&\sum_{j=1}^m \phi_j \int_S \left[\psi_i(p)\left(-\frac{1}{2}\psi_j(p) + \int_S \psi_j(q)\frac{\partial G_k(p,q)}{\partial n_q} dS_q\right) + \right. \\ &\qquad \frac{1}{2}\alpha \int_S \left((\psi_i(p) - \psi_i(q))\,(\psi_j(q) - \psi_j(p))\,\frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q}\right) dS_q \\ &\qquad\quad \left. +\ \ \alpha\psi_i(p)\psi_j(p)k^2 \int_S G_k(p,q) n_p \cdot n_q\, dS_q\right] dS_p \ \ = \\ &\sum_{j=1}^m v_j \int_S \psi_i(p)\left[\frac{\alpha}{2}\psi_j(p) + \int_S \psi_j(q)\left(G_k(p,q) + \alpha\frac{\partial G_k(p,q)}{\partial n_p}\right) dS_q\right] dS_p.\end{aligned} \tag{18}$$

4 Solution of the linear system of equations

Discretizing the integral equations described above using either the collocation or Galerkin method will yield a matrix equation of the form

$$A\underline{x} = \underline{b}, \tag{19}$$

with $A \in C^{n\times n}$, $\underline{x}$, $\underline{b} \in C^n$. Although it is possible to solve this system using a LU decomposition for large n, it is more efficient to use an iterative technique with a suitable preconditioner.

The idea of using operator splitting to precondition an integral equation with a singular or weakly-singular kernel was previously discussed for 2D problems in [8, 9, 10], and applied to the 3D case in [13] for a simple collocation method. Two mesh neighbours based ideas were tested in [13]; here we consider how to generalise these ideas to the Galerkin case.

Denote a bounded integral operator equation such as (6) by

$$\mathcal{A}u = f. \tag{20}$$

Then a suitable splitting of $\mathcal{A}$ as far as effective preconditioning is concerned is the following

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{C}_1, \tag{21}$$

where $\mathcal{C}_1$ is a compact operator and $\mathcal{A}_1$ is thus a bounded operator. The choice of $\mathcal{A}_1$ and $\mathcal{C}_1$ via the domain decomposition is such that $\mathcal{A}_1$ leads to a sparse matrix on discretization. In this approach, the preconditioned operator equation is

$$\mathcal{A}_1^{-1}\mathcal{A}u = (\mathcal{I} + \mathcal{A}_1^{-1}\mathcal{C}_1)u = \mathcal{A}_2 u = \mathcal{A}_1^{-1}f, \tag{22}$$

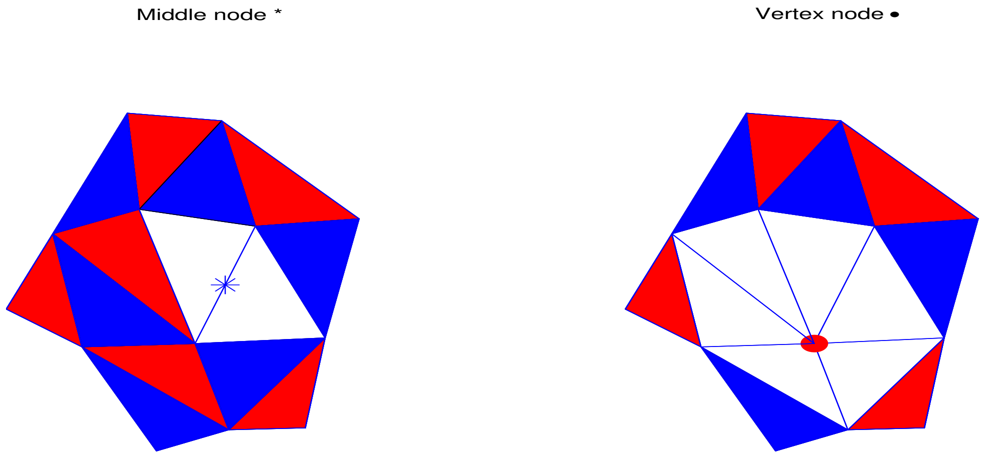
which will have a clustering distribution of eigenvalues and, more importantly, the normal operator $\mathcal{A}_2^*\mathcal{A}_2$ (and the normal matrix for the discretized case) also have a clustering distribution of eigenvalues. By 'clustering', we mean that most eigenvalues are contained in a small interval; for a precise definition, see [11]. Therefore the resulting linear system is better suited to iterative solution; in fact we have found that most conjugate gradient solvers are efficient for the preconditioned system.

To apply the operator splitting idea to the Galerkin case, we first use the surface non-overlapping domain partition $S = \bigcup_{j=1}^N S_j$ to decompose the operator

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} & \mathcal{A}_{1,3} & \cdots & \mathcal{A}_{1,N} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} & \mathcal{A}_{2,3} & \cdots & \mathcal{A}_{2,N} \\ \mathcal{A}_{3,1} & \mathcal{A}_{3,2} & \mathcal{A}_{3,3} & \cdots & \mathcal{A}_{3,N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathcal{A}_{N,1} & \mathcal{A}_{N,2} & \mathcal{A}_{N,3} & \cdots & \mathcal{A}_{N,N} \end{pmatrix} \tag{23}$$

where in element S_i operator $\mathcal{A}_{i,\ell}$ is the restriction of $\mathcal{A}$ over surface S_ℓ. We assume that the boundary S does not admit any geometric singularities; otherwise singularities

Figure 1: Relation of node associated basis functions and near domain neighbours. Left: Middle node $*$ shared by two domains; Right: Vertex node $\bullet$ shared by all six near neighbouring domains.



must be isolated in the above splitting and incorporated into further preconditioners as suggested in [6, 7] and [12]. Then a suitable splitting of the form (21) is the following

$$(\mathcal{A}_1)_{ij} = \begin{cases} \mathcal{A}_{ij}, & \text{if } S_j \text{ is a neighbour of } S_i, \\ 0, & \text{otherwise.} \end{cases}$$

Here operator $\mathcal{A}_1$ is sparse and the exact sparsity pattern depends on how the surface triangles are ordered; in the one dimensional case, $\mathcal{A}_1$ is of the tridiagonal form. If we assume that the original operator $\mathcal{A}$ is non-singular, then repeatedly using the result that a compact perturbation of a non-singular operator is still non-singular can establish that operator $\mathcal{A}_1$ is non-singular. Hence $\mathcal{A}_1$ is admissible.

Note that in S_i, although we can use any high order elements, the quadratic interpolation function adopted is

$$\phi_i(p) = \sum_{j=1}^{6} \phi_i^{(j)} \psi_i^{(j)}(p)$$

where $\psi_i^{(j)}(p)$ is the jth node based quadratic basis function; three nodes are vertexes and three mid-points of the underlying triangle. Therefore, on discretization using the Galerkin method, each operator $\mathcal{A}_{ij}$ involves all 6 rows and columns of matrix $A_{n\times n}$, where n is the total number of nodes (i.e. basis functions).

The above neighbouring information refers to local stiffness matrices and we need to work out the preconditioner in the globally assembled stiffness matrix A because we only need to discretize $\mathcal{A}$ in (21) — not $\mathcal{A}_1$ and $\mathcal{C}_1$ separately. To this end, we note that form Figure 1 that there are 2 types of nodes: the middle nodes (such as "*") and the vertex nodes (such as "•"). Thus the construction of preconditioning matrix M becomes essentially an identification task for neighbouring elements in matrix A:

Algorithm 1 (Operator splitting preconditioner) *Set the preconditioning matrix* $M :=$ 0. *For row* $i = 1, \cdots, n$*:*

Figure 2: Sparsity of preconditioner M_1 based on a simple node ordering for test problem 1 in Section 5 with $n = 290$ nodes.

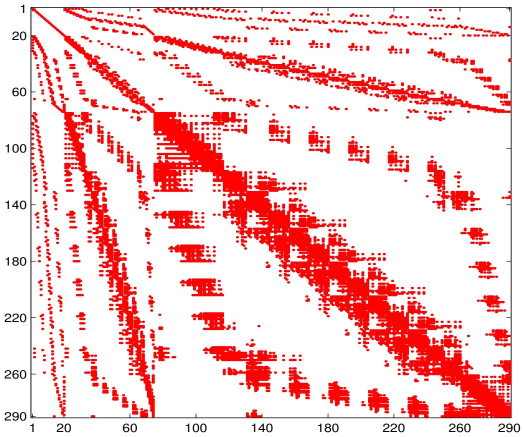
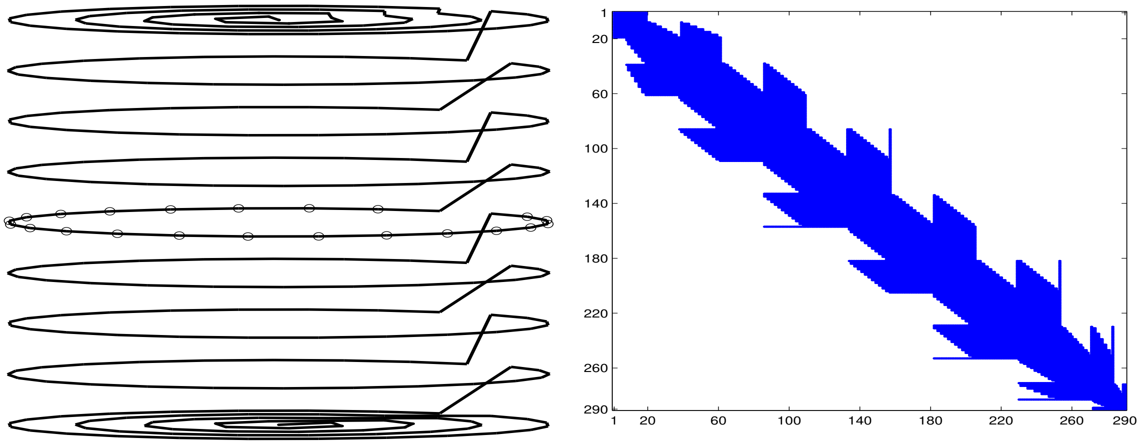


Figure 3: Duff's spiral ordering for test problem 3 in Section 5 (left plot) and Sparsity of preconditioner M_1 based on Duff's spiral ordering for test problem 1 in Section 5 (right plot).



1. *Work out an index list $L(i,:) = [i_1, \cdots, i_{r_i}]$ from the surface elements, where i_j denotes the jth neighbouring node's global index number. Define $L(i,1) = i$ i.e. the present node is in the first position of neighbouring nodes. Note that r_i is the total number of basis functions involved in the immediate neighbouring elements. This depends whether i refers to a middle basis or a vertex basis — with '*' in Figure 1 $r_i = 9$ and with '$\bullet$' $r_i = 19$ (number of nodes in neighbouring elements).*

2. *Set element $M(i, i_j) = A(i, i_j)$ for $j = 1, 2, \cdots, i_{r_i}$.*

3. *Finally solve $Ax = b$ by CGN or GMRES using the preconditioner M^{-1} i.e. solve $M^{-1}Ax = M^{-1}b$.*

Here for a typical well-balanced surface mesh (which we consider to be a mesh where either 5 or 6 elements coming together at each vertex and which is made up of triangles which are almost equilateral triangles), $r_i \leq 19$. Therefore the sparsity of M is assured. However there is no guarantee that a preconditioning step $Mx = y$ can be solved cheaply. Indeed, a naive finite element mesh will not be sufficient; see Figure 2 for a typical example of M which is difficult to invert.

One ideal situation is to order the nodes in such a way that neighbouring elements are always numbered near each other, giving rise to a global banded matrix. But in general this is not possible to achieve. Therefore it is not surprising that the use of the reverse Cuthill-McKee algorithm [16] produces a banded-like matrix with a very large band width. This suggests that a global and optimal index minimization ordering is difficult to realize. After careful testing of several ordering ideas, we have found that the spiral ordering [17] (refer also the earlier work of [15]) of elements gives an almost optimal solution in the sense that the resulting preconditioner is nearly of a doubly-bordered block diagonal form; see the right hand plot of Figure 3 for a typical example. We believe that this algorithm is suitable for this context because S is a closed 3D surface. As a general and black box algorithm for sparse matrices, the spiral ordering may not be effective. Overall, the spiral ordering will define our first preconditioner.

To present a second preconditioner, we use a ordering free preconditioning method based on [28, 13, 11]. This is the approximate inversion method advocating a cheaper inversion of M. Here this new preconditioner will be names as M_2^{-1}, approximating A, that has the same sparse pattern as M above. That is

$$M_2^{-1} = [m_1, m_2, \cdots, m_n], \tag{24}$$

where column vector m_i (for $i = 1, \cdots, n$) satisfies

$$m_i(j) = \begin{cases} m_{j,i}, & \text{if } j \in L(i,:) = \{i_1, i_2, \cdots, i_{r_i}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Minimising the usual Frobenius norm

$$\min_{M_2^{-1}} \|AM_2^{-1} - I\|_F^2 = \sum_{j=1}^{n} \min_{m_j} \|Am_j - e_j\|_2^2$$

leads to a $n \times r_j$ least squares problem for each column j. The near neighbour method from [28, 13, 11] recommends the reformulation of the least squares problem into a squares matrix problem by solving the approximate problem

$$\min_{\tilde{m}_j} \|\widetilde{A}_j \tilde{m}_j - \tilde{e}_j\|_2^2$$

where $\widetilde{A}_j$ is the $r_j \times r_j$ principal submatrix of A induced by the index set $L(j,:)$. Note $\tilde{e}_i$ is a compact vector of e_i and $\tilde{m}_j$ of m_j.

Algorithm 2 (Approximate inversion preconditioner) *Set the preconditioning matrix $M_2 := 0$. For column $i = 1, \cdots, n$:*

1. *Identify the index list $L(i,:) = [i_1, \cdots, i_{r_i}]$ for i as in Algorithm 1 (corresponding to the basis function $\psi_i(p)$).*

2. *Assign matrix $\widetilde{A}_i(k,j) = A(i_k, i_j)$ and vector $\tilde{e}_i(k) = 0$ for $k,\ j = 1, 2, \cdots, i_{r_i}$. Set $\tilde{e}_i(1) = 1$.*

3. *Solve the small linear system $\widetilde{A}_i \tilde{m}_i = \tilde{e}_i$ to obtain column i of preconditioner M_2.*

4. *Finally solve $Ax = b$ by CGN or GMRES using the preconditioner M_2 i.e. solve $M_2 A x = M_2 b$.*

We remark that the effectiveness of a full sparse approximate inverse (SPAI) preconditioners is determined by the a priori sparsity pattern used. Here with the pattern defined by geometrical information, M_2 is quite sparse so the resulting preconditioner may not be efficient. However the above adopted strategy, although an approximate solution of a least square problem is sought, can be shown to have implicitly implemented a diagonal operator splitting and, when this happens, Algorithm 2 is more effective than the full SPAI method using near neighbour information; see [11]. Here matrix $\widetilde{A}_j$ is invertible due to the presence of a hyper-singular operator in operator $\mathcal{A}$ and the assumption on the closed boundary S. For other type of problems e.g. first kind operator with geometric singularities, this matrix may not be invertiable so one may have to use the usual SPAI method.

5 Numerical Results

The Galerkin method described in Section 3 and the iterative solution methods described in Section 4 have been applied to a number of test problems. In each case the surface was approximated by six-noded quadratic triangular elements and both the surface pressure and its normal derivative were interpolated using the same quadratic basis functions (isoparametric elements). The surface data was generated by placing a number of point sources inside the surface and using these to compute $\frac{\partial \phi}{\partial n}$ on the surface. The solution ϕ is simply that due to the point sources.

The individual test problems considered are outlined below

1. A unit sphere with point sources at (0, 0, 0.5) and (0.25, 0.25, 0.25) with strengths $2+3i$ and $4-i$ respectively.

2. A 'peanut' shaped surface defined by

$$\begin{aligned} x &= \sqrt{\cos 2\theta + \sqrt{1.5 - \sin^2 2\theta}}\, \sin\theta \cos\gamma \\ y &= \sqrt{\cos 2\theta + \sqrt{1.5 - \sin^2 2\theta}}\, \sin\theta \sin\gamma \qquad \begin{matrix} 0 \leq \theta \leq \pi \\ 0 \leq \gamma < 2\pi \end{matrix} \\ z &= \sqrt{\cos 2\theta + \sqrt{1.5 - \sin^2 2\theta}}\, \cos\theta \end{aligned}$$

   with point sources at (0.2, 0 , 1) and (0, 0.2, -0.75) with strength $2+3i$ and $4-i$ respectively.

3. A cylinder of length 0.537 and radius 0.2685 with point sources at (0, 0, 0.15) and (0.25, 0.25, 0.25) with strengths $2+3i$ and $4-i$ respectively.

We note that second test problem is for a surface which is not convex, and that the third test problem is for a non-smooth surface in the sense that it does not possess a unique normal at every point.

The relative L_2 error, defined by

$$E = \frac{||\phi - \tilde{\phi}||_2}{||\phi||_2} \tag{25}$$

where $||\phi||_2 = \sqrt{\int_S |\phi(q)|^2 \, dS_q}$ and ϕ and $\tilde{\phi}$ denote the exact and approximate solution respectively, is the measure of the error used here. Figures 4 gives a typical comparison of the relative L_2 error (expressed as a percentage) on the surface of a sphere for different values of the wavenumber. In all cases tested, the quadratic Galerkin method performs much better than the constant collocation method, but the error increases with the wavenumber with either method as expected because solution resolutions are linked to the frequency.

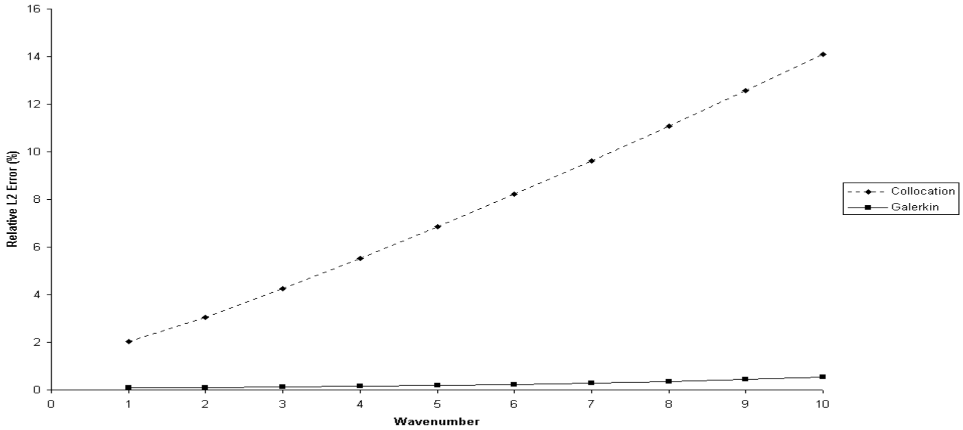
In all of the remaining experiments the following two meshes were used:

- Mesh I; 576 elements and 1154 nodes.
- Mesh II; 2304 elements and 4610 nodes.

For the range of values of k considered here the relative errors in the computed solution mesh I ranged from 0.079% to 6.1% and using mesh II the errors range from 0.039% to 2.8%. In both cases the error in the solution on the cylinder (test example 3) were worse than those on the sphere or peanut (test examples 1 and 2). This is probably due to the geometric singularities which require more specialised treatment than that given here.

The results of using both preconditioners (from Algorithms 1 and 2 and denoted by A and B respectively) and Mesh I with the CGNR method are given in Table 1, whilst the

Figure 4: The relative L_2 error in the computed solution for the test problem on the surface of a unit sphere.



corresponding results for the GMRES(25) method are given in Table 2. Here the columns headed n gives the number of iterations and those headed t/t_d gives the CPU time t relative to the time t_d needed to carry out a full LU decomposition on the matrix. The results of using Mesh II instead of Mesh I are given in Tables 3 and 4. For both meshes the stopping criterion are based on the relative residule being less than 10^{-6}. It is clear that both iterative methods are offer a considerable time-saving over the LU decomposition method. Of the two iterative methods GMRES(25) performs much better, taking less than 1/10 of the time needed for the LU decomposition. Other GMRES methods, such as GMRES(5) and GMRES(40) were tried, but offered no significant increase in performance. Although the preconditioners constructed here are for the hyper-singular integral equation on a smooth convex surface, these results suggest that they are equally effective when applied to problems involving non-convex surfaces and surfaces with geometric singularities.

6 Conclusions

The well-known approach for solving the exterior Helmholtz problem in 3D is the collocation method using piecewise constant elements. High order collocation and Galerkin methods are not available due to the presence of the hypersingular integral operator involving the second derivative of the Green's function. In this paper we have proposed a high order Galerkin method and tested it using quadratic elements. Moreover we have developed two preconditioners based on operator splitting and experimented their effectiveness, generalising the previous work in the subject. Preliminary results are successful.

Table 1: Mesh I: The number of outer iterations and relative CPU time when using the GMRES(25) method with preconditioners A and B to solve the final linear system of equations.

	SPHERE				PEANUT				CYLINDER			
	A		B		A		B		A		B	
k	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d
1	1	0.0548	2	0.0777	2	0.0915	2	0.0815	2	0.0904	2	0.0808
2	2	0.0892	2	0.0784	2	0.0914	2	0.0816	2	0.0902	2	0.0808
3	2	0.091	2	0.0809	2	0.0914	2	0.0816	2	0.0897	2	0.0804
4	1	0.0529	2	0.082	2	0.0915	2	0.0816	2	0.09	2	0.0808
5	1	0.0528	1	0.0452	2	0.0927	2	0.0822	2	0.0902	2	0.0811
6	1	0.053	1	0.0454	2	0.0911	2	0.0824	1	0.0527	1	0.0457
7	1	0.054	1	0.0463	1	0.0533	1	0.0458	1	0.0524	1	0.0456
8	1	0.054	1	0.0466	1	0.0533	1	0.0458	1	0.0525	1	0.0455
9	1	0.0537	1	0.0464	1	0.0533	1	0.0458	1	0.0525	1	0.0457
10	1	0.0532	1	0.0458	1	0.0535	1	0.0458	1	0.0527	1	0.0457

Table 2: Mesh I: The number of iterations and relative CPU time when using the CGNR method with preconditioners A and B to solve the final linear system of equations.

	SPHERE				PEANUT				CYLINDER			
	A		B		A		B		A		B	
k	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d
1	41	0.2083	65	0.2874	46	0.2296	74	0.3236	46	0.232	77	0.3354
2	27	0.1472	43	0.1966	37	0.1898	61	0.2692	32	0.1671	52	0.2326
3	23	0.1325	33	0.1556	32	0.1686	49	0.2212	27	0.1453	40	0.1849
4	19	0.1142	26	0.1272	30	0.1602	40	0.1842	23	0.1278	33	0.1547
5	18	0.1081	22	0.111	28	0.1512	36	0.168	22	0.1232	27	0.1301
6	16	0.0989	18	0.094	26	0.1448	34	0.1599	21	0.1189	23	0.114
7	15	0.0941	16	0.0857	23	0.1293	30	0.1432	20	0.1146	22	0.1099
8	14	0.0899	15	0.0829	22	0.1251	28	0.1353	20	0.1147	22	0.11
9	14	0.0896	15	0.0818	22	0.1247	25	0.1227	19	0.1101	21	0.1078
10	14	0.0897	15	0.0818	23	0.1295	25	0.1249	19	0.1109	20	0.1014

Table 3: Mesh II: The number of outer iterations and relative CPU time when using the GMRES(25) method with preconditioners A and B to solve the final linear system of equations.

	SPHERE				PEANUT				CYLINDER			
	A		B		A		B		A		B	
k	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d
1	2	0.039	3	0.053	3	0.047	4	0.06	3	0.048	4	0.067
2	2	0.036	3	0.049	3	0.047	4	0.06	3	0.048	4	0.06
3	2	0.036	3	0.05	3	0.048	3	0.046	3	0.049	3	0.047
4	2	0.037	3	0.051	3	0.048	3	0.046	3	0.048	3	0.046
5	2	0.036	2	0.034	3	0.048	3	0.046	3	0.048	3	0.046
6	2	0.036	2	0.034	2	0.034	3	0.046	2	0.032	3	0.045
7	2	0.036	2	0.034	2	0.034	3	0.046	2	0.032	2	0.031
8	2	0.036	2	0.034	2	0.034	2	0.032	2	0.035	2	0.033
9	2	0.036	2	0.034	2	0.034	2	0.032	2	0.035	2	0.033
10	1	0.02	2	0.034	2	0.034	2	0.032	2	0.035	2	0.033

Table 4: Mesh II: The number of iterations and relative CPU time when using the CGNR method with preconditioners A and B to solve the final linear system of equations.

	SPHERE				PEANUT				CYLINDER			
	A		B		A		B		A		B	
k	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d	n	t/t_d
1	79	0.072	133	0.12	95	0.069	165	0.11	95	0.088	163	0.14
2	54	0.048	89	0.073	75	0.055	130	0.09	65	0.049	110	0.077
3	43	0.039	70	0.059	61	0.046	102	0.071	49	0.04	87	0.061
4	33	0.031	53	0.045	51	0.039	86	0.061	40	0.031	70	0.05
5	29	0.027	46	0.039	47	0.036	75	0.053	36	0.029	58	0.042
6	26	0.025	39	0.034	43	0.034	68	0.048	33	0.026	50	0.036
7	23	0.022	33	0.029	39	0.031	58	0.042	30	0.024	43	0.031
8	22	0.022	30	0.026	39	0.031	57	0.041	29	0.027	39	0.033
9	20	0.02	26	0.023	37	0.03	51	0.037	27	0.025	36	0.03
10	20	0.02	24	0.021	35	0.028	48	0.035	26	0.024	32	0.028

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