On a class of new and practical performance indices for approximation of fold bifurcations of nonlinear power flow equations

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Abstract

Efficient measurement of the performance index (the distance of a loading parameter from the voltage collapse point) is one of the key problems in power system operations and planning and such an index indicates the severity of a power system with regard to voltage collapse. There exist many interesting methods and ideas to compute this index. However some successful methods are not yet mathematically justified while other mathematically sound methods are often proposed directly based on the bifurcation theory and they require the initial stationary state to be too close to the unknown turning point to make the underlying methods practical.

This paper first gives a survey of several popular methods for estimating the fold bifurcation point including the continuation methods, bifurcation methods and the test function methods (Seydel's direct solution methods, the tangent vector methods and the reduced Jacobian method) and discuss their relative advantages and problems. Test functions are usually based on scaling of the determinant of the Jacobian matrix and it is generally not clear how to determine the behaviour of such functions. As the underlying nonlinear equations are of a particular type, this allows us to do a new analysis of the determinants of the Jacobian and its submatrices in this paper. Following the analysis, we demonstrate how to construct a class of test functions with a *predictable* analytical behaviour so that a suitable index can be produced. Finally, examples of two test functions from this class are proposed. For several *standard* IEEE test systems, promising numerical results have been achieved.

Keywords: Nonlinear equations, Bifurcation, singularity, turning points, performance index, power systems, test functions.

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1 Introduction

An electrical power system can be modelled in nodal analysis by a set of ordinary differential equations. The network is usually runing at an equilibrium and a planned or an unexpected change in operating conditions can set the system to a new equilibrium. Voltage collapse in a power system is a real world problem that can occur in association with bifurcation problems of the underlying nonlinear power flow equations, particularly fold bifurcations (or sometimes called saddle-node bifurcations) [3, 4, 6, 27, 9]. We concentrate on identifying the fold bifurcation point in the nonlinear system [12, 16].

In the mathematical literature, the term 'fold bifurcation' is often called the turning point because the solution turns at a fold point [10, 24, 28]. Adopting the usual notation, denote a nonlinear system of equations by

$$\mathbf{f}(\mathbf{x}, \lambda) = \mathbf{0} \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^n$ is the state vector, $\lambda \in \mathbf{R}$ is the bifurcation parameter and $\mathbf{f} : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ with $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \ \ldots \ f_n(\mathbf{x})]^\top$. Assume that the current state variable is $\mathbf{x} = \mathbf{x}_0$ corresponding to the parameter λ_0. Away from the fold point, to find the state variable $\mathbf{x} = \mathbf{x}_1$ for any $\lambda = \lambda_1$, we can use the Newton method for $k = 1, 2, 3 \ldots$

$$\left\{ \begin{array}{rcl} J(\mathbf{x}_1^{(\ell-1)})\Delta\mathbf{x}^{(\ell)} & = & -\mathbf{f}(\mathbf{x}_1^{(\ell-1)}), \\ \mathbf{x}^{(\ell)} & = & \mathbf{x}^{(\ell-1)} + \Delta\mathbf{x}^{(\ell)} \end{array} \right. \tag{2}$$

to achieve $\lim_{\ell\to\infty} \mathbf{x}_1^{(\ell)} = \mathbf{x}_1$, provided that $det(J) \neq 0$. For power systems, $\mathbf{x}$ denotes the vector of bus voltage magnitudes and angles, λ refers to the power demand and $\mathbf{f}$ the power balance equation from Kirchhoff laws (to be specified later); see also [12, 29, 2].

At the fold point, the Jacobian matrix $J = \mathbf{f}_\mathbf{x}$ is singular and one of its real eigenvalues becomes zero. In particular, we consider the case of a normal turning point, that is, assume that the rank of the $n \times n$ Jacobian matrix at a fold point is $(n-1)$; see [18]. The obvious way of finding such a λ associated with the fold point is to solve $det(J) = 0$. Unfortunately this is not feasible numerically, because (i) it is not easy or efficient to use an analytical formula for $det(J)$ for a large n; (ii) it is not easy to locate the vanishing of the Jacobian (i.e. when $\det(J) = 0$) and a well-conditioned matrix such as $J = diag(0.1, \ldots, 0.1)$ may have a zero determinant in the numerical sense due to bad scaling; see [13]. Test functions (usually denoted by $\tau(\lambda)$) are useful alternatives to $det(J)$ because they are smoothly behaving and *zero* at the fold point. It is the purpose of this paper to introduce new test functions and show that they can give an accurate prediction of the fold parameter λ starting from any current value λ_0.

Section 2 of this paper discusses the nonlinear power flow equations and re-writes them in terms of the standard notation of nonlinear equations. Section 3 surveys several methods that may be used for locating the fold bifurcation point. Illustration of their performance will be done for a simple IEEE 9-bus system involving nonlinear equations of size 14. Section 4 introduces some new results and properties of the Jacobian matrix that lead to proposal of two new test functions with sound theoretical basis for parameter prediction. Section 5 presents more comprehensive experimental results using more and larger standard IEEE test examples that illustrate the effectiveness of our proposed method, both for a single load variation and multiple load variations. Finally section 6 gives a short summary and preliminary conclusion.

2 The power flow equations

The power flow problem involves the calculation of voltages at all nodes of an alternating current network when subjected to a specified loading condition and subject to the power and voltage constraints that are applied to the system. The essential physical laws are the KCL and KVL (Kirchhoff current and voltage laws) coupled with power balance (note that the complex power $S = P + iQ$ is linked to the voltage V and current I via $S = VI^*$, where $*$ denotes a complex conjugate). The main equations can be found in applied mathematics books e.g. [29, 2] as well as in most power system analysis books e.g. [12, 16].

Here we give a brief description to assist readers. Assume that we are dealing with an electrical power system of $m+1$ nodes (called buses): $j = 1, 2, \ldots, m, m+1$ with the last one, $m+1$, used as the reference bus. In nodal analysis, the novelty lies in converting (somewhat complicated) system control quantities into equivalent quantities in terms of admittance (the reciprocals of impedance) and equivalent circuits. At bus j, write into polar form $V_j = v_j \exp(i\delta_j)$ with v_j and δ_j denoting the voltage magnitude and phase angle. Similarly between any two connecting buses j and k, write the admittance as $Y_{jk} = y_{jk} \exp(i\theta_{jk})$.

Then at bus j, letting the combined active and reactive power (due to other buses) equal to the net injected active and reactive power yields the power flow equations as follows

$$S_j = V_j I_j^*, \tag{3}$$

that is,

$$\begin{aligned} P_j + iQ_j &= V_j \sum_{k=1}^{m+1} (Y_{jk} V_k)^* \\ &= v_j \sum_{k=1}^{m+1} y_{jk} v_k \exp(\delta_j - \delta_k - \theta_{jk}). \end{aligned} \tag{4}$$

Therefore, we can write the nonlinear equations for $Q_j,\ v_j,\ \delta_j$ as follows (P_j is usually known as discussed below):

$$\begin{cases} P_j = v_j \sum_{k=1}^{m+1} y_{jk} v_k \cos(\delta_j - \delta_k - \theta_{jk}) \\ Q_j = v_j \sum_{k=1}^{m+1} y_{jk} v_k \sin(\delta_j - \delta_k - \theta_{jk}). \end{cases} \tag{5}$$

At a network equilibrium, the net injected power $S_j = P_j + iQ_j$ is equal to the difference of the generation power $P_{G_j} + iQ_{G_j}$ and the (user consumed) load power $P_{L_j} + iQ_{L_j}$.

In iterating the power equations or in a power disturbance, the two powers (S_j and the difference) are different and hence we have the term *power mismatch*, referring to (for bus j)

$$\begin{cases} \Delta P_j = P_{G_j} - P_{L_j} - P_j = P_{G_j} - P_{L_j} - v_j \sum_{k=1}^{m+1} y_{jk} v_k \cos(\delta_j - \delta_k - \theta_{jk}) \\ \Delta Q_j = Q_{G_j} - Q_{L_j} - Q_j = Q_{G_j} - Q_{L_j} - v_j \sum_{k=1}^{m+1} y_{jk} v_k \sin(\delta_j - \delta_k - \theta_{jk}). \end{cases} \tag{6}$$

In normal circumstances, the power equations are then simply $\Delta P_j = \Delta Q_j = 0$ that are sometimes called the mismatch equations (a confusing usage!).

The precise number of equations to be solved depends upon the bus type:

Type 0: the slack bus (reference) $j = m + 1$ where $v_j = 1$ and $\delta_j = 0$ and no equation is needed;

Type 1: a PQ (load) bus at $j = 1, \ldots, m_1$ where two power equations for v_j and δ_j are needed;

Type 2: a PV (voltage control) bus at $j = m_1 + 1, \ldots, m$ where v_j is given and only one power equation is needed for δ_j.

Here m_1 is the total number of PQ buses (which are assumed to be the *first* m_1 buses out of the network). Then the power equations for the network can be written as follows

$$\begin{bmatrix} \Delta P_1 \\ \vdots \\ \Delta P_m \\ \Delta Q_1 \\ \vdots \\ \Delta Q_{m_1} \end{bmatrix} = \begin{bmatrix} P_{G_1} - P_{L_1} - P_1 \\ \vdots \\ P_{G_m} - P_{L_m} - P_m \\ Q_{G_1} - Q_{L_1} - Q_1 \\ \vdots \\ Q_{G_{m_1}} - Q_{L_{m_1}} - Q_{m_1} \end{bmatrix} = \mathbf{0}. \tag{7}$$

Let $n = m + m_1$ be the total number of unknowns and let $\mathbf{x} \in \mathbf{R}^n$ denote the system state variables i.e.

$$\mathbf{x} = [x_1 \ \cdots \ x_n]^\top = [\delta_1 \ \delta_2 \ \ldots \ \delta_{m_1} \ \ldots \ \delta_m \ v_1 \ v_2 \ \cdots v_{m_1}]^\top.$$

Define $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ as follows

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \\ f_{m+1}(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \Delta P_1(\mathbf{x}) \\ \vdots \\ \Delta P_m(\mathbf{x}) \\ \Delta Q_1(\mathbf{x}) \\ \vdots \\ \Delta Q_{m_1}(\mathbf{x}) \end{bmatrix}. \tag{8}$$

In the above setting of nonlinear equations, the load can be a varying parameter. When the network is operating at a stationary equilibrium, we use Newton's method to solve the power flow problem to determine the voltages of the entire network [30]. Assume the current state vector is $\mathbf{x}_0$ associated with the present load $[P_{L_{10}} \ P_{L_{20}} \ldots P_{L_{m0}} \ P_{L_{m+1,0}}]$.

Voltage collapse occurs in power systems as a result of a sequence of events that accompany a loss of stability where a change in system conditions causes a progressive and uncontrollable drop in voltage in significant parts of a power system [11, 5]. The main factor causing this collapse has been shown to be the depletion of reactive load power on the power system. Mathematically, voltage collapse is associated with fold bifurcations resulting from a loss of stability in the parameterised nonlinear equations that describes the static power system [4]. Over the last few years many articles and papers have been written on the subject [5].

To describe the load increase in terms of a varying parameter λ, define the new load as

$$\begin{cases} P_{L_j} = P_{L_{j0}} + \lambda \alpha_j \\ Q_{L_j} = Q_{L_{j0}} + \lambda \beta_j \end{cases} \tag{9}$$

where P_{L_j} and Q_{L_j} are the new real and reactive loads increased after the initial state at $\lambda_0 = 0$; and α_j and β_j describe the load increase pattern at each bus j for the real and reactive loads respectively. Then we may write the parameterised power flow equations at bus j as follows:

$$\begin{cases} \Delta \mathcal{P}_j = P_{G_j} - (P_{L_{j0}} + \lambda \alpha_j) - P_j = 0 \\ \Delta \mathcal{Q}_j = Q_{G_j} - (Q_{L_{j0}} + \lambda \beta_j) - Q_j = 0 \end{cases} \tag{10}$$

where P_j and Q_j are as defined in equations (5). Combining equation (10) with (6) we obtain

$$\begin{cases} \Delta\mathcal{P}_j = \Delta P_j - \lambda\alpha_j = 0 \\ \Delta\mathcal{Q}_j = \Delta Q_j - \lambda\beta_j = 0. \end{cases} \tag{11}$$

Now we define $\mathbf{f}(.,.) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ as follows

$$\mathbf{f}(\mathbf{x},\lambda) = \begin{bmatrix} \Delta\mathcal{P}_1(\mathbf{x},\lambda) \\ \vdots \\ \Delta\mathcal{P}_m(\mathbf{x},\lambda) \\ \Delta\mathcal{Q}_1(\mathbf{x},\lambda) \\ \vdots \\ \Delta\mathcal{Q}_{m_1}(\mathbf{x},\lambda) \end{bmatrix} = \begin{bmatrix} \Delta P_1(\mathbf{x}) - \lambda\alpha_1 \\ \vdots \\ \Delta P_m(\mathbf{x}) - \lambda\alpha_m \\ \Delta Q_1(\mathbf{x}) - \lambda\beta_1 \\ \vdots \\ \Delta Q_{m_1}(\mathbf{x}) - \lambda\beta_{m_1} \end{bmatrix} = \mathbf{0} \tag{12}$$

and combining with (8) we obtain the our main system as a special case of (1)

$$\mathbf{f}(\mathbf{x},\lambda) = \mathbf{f}(\mathbf{x}) - \lambda\mathbf{b} = 0, \tag{13}$$

where we only allow fixed power changes to distribute the total system load change represented by λ. Here the constant vector $\mathbf{b} \in \mathbf{R}^n$ denotes the system load pattern i.e. $\mathbf{b} = [\alpha_1 \ \ldots \alpha_m \ \beta_1 \ldots \beta_{m_1}]^\top$ and is such that $\sum_{k=1}^{n} b_k = 1$. In this paper we also consider the special case

$$\mathbf{b} = \sum_{k=1}^{s} w_{l_k}\mathbf{e}_{l_k} \qquad \text{with } w_{l_k} = 1/s \tag{14}$$

where $\mathbf{e}_{l_k}$ is the l_kth column of $I_{n\times n}$ with $m+1 \le l_k \le n$ i.e. we only consider variations of reactive load for a single bus or a selection of any s load buses.

Finally, to get familiarised with the unified notation, we expand the Jacobian equation in terms of the power quantities

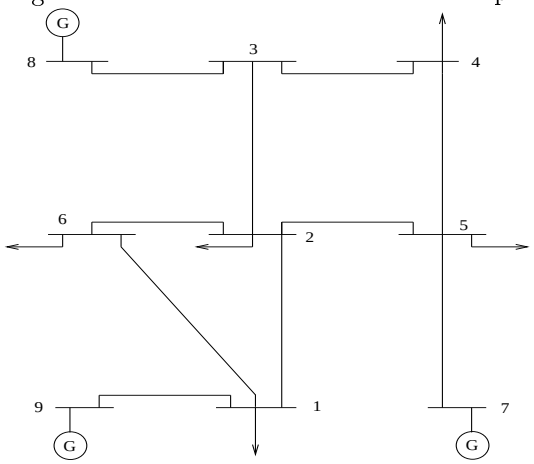
$$\begin{bmatrix} \frac{\partial\Delta P_1}{\partial\delta_1} & \cdots & \frac{\partial\Delta P_1}{\partial\delta_m} & \frac{\partial\Delta P_1}{\partial v_1} & \cdots & \frac{\partial\Delta P_1}{\partial v_{m1}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial\Delta P_m}{\partial\delta_1} & \cdots & \frac{\partial\Delta P_m}{\partial\delta_m} & \frac{\partial\Delta P_m}{\partial v_1} & \cdots & \frac{\partial\Delta P_m}{\partial v_{m1}} \\ \frac{\partial\Delta Q_1}{\partial\delta_1} & \cdots & \frac{\partial\Delta Q_1}{\partial\delta_m} & \frac{\partial\Delta Q_1}{\partial v_1} & \cdots & \frac{\partial\Delta Q_1}{\partial v_{m1}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial\Delta Q_{m_1}}{\partial\delta_1} & \cdots & \frac{\partial\Delta Q_{m_1}}{\partial\delta_m} & \frac{\partial\Delta Q_{m_1}}{\partial v_1} & \cdots & \frac{\partial\Delta Q_{m_1}}{\partial v_{m1}} \end{bmatrix} \begin{bmatrix} \Delta\delta_1(\mathbf{x}) \\ \vdots \\ \Delta\delta_m(\mathbf{x}) \\ \Delta v_1(\mathbf{x}) \\ \vdots \\ \Delta v_{m_1}(\mathbf{x}) \end{bmatrix} = - \begin{bmatrix} \Delta P_1(\mathbf{x}) \\ \vdots \\ \Delta P_m(\mathbf{x}) \\ \Delta Q_1(\mathbf{x}) \\ \vdots \\ \Delta Q_{m_1}(\mathbf{x}) \end{bmatrix}. \tag{15}$$

To show the performance of various methods, we use the following IEEE 9-bus system which is in IEEE Common Data Format; see [31]. This system as shown in Figure 1 has 9 buses: $m_1 = 6$ load buses (Type 1: bus 1, 2,... ,6), $m = 8 - m_1 = 2$ generators (Type 2: bus 7, 8) and 1 reference generator bus (Type 2: bus 9).

3 Bifurcation methods and test functions

To locate a fold point the usual bifurcation technique, based on the well-known ABCD lemma [10, 20, 28], extends the nonlinear system by including an eigensystem for an zero eigenvalue or other suitable equations to ensure solvability. Suppose $(\mathbf{x}^*, \lambda^*)$ is a fold point of (13). While the solvability is properly addressed at $(\mathbf{x}^*, \lambda^*)$, the domain of convergence is practically small i.e. the current parameter λ_0 must be close to the unknown parameter λ^* in order for the Newton method to converge.

Figure 1: The standard IEEE 9-bus example



In this section we shall first discuss the bifurcation methods and continuation methods, and then survey several test functions methods in order to highlight the relative advantages and disadvantages of each approach.

3.1 Direct bifurcation methods

To compute the fold point we set up the extended $(2n+1)$ dimensional system [22, 24]

$$\mathbf{G}(\mathbf{Y}) = \begin{bmatrix} \mathbf{f}(\mathbf{x}, \lambda) \\ \mathbf{f}_{\mathbf{x}}(\mathbf{x}, \lambda)\mathbf{h} \\ h_k - 1 \end{bmatrix} = 0, \tag{16}$$

where

$$\mathbf{Y} = \begin{bmatrix} \mathbf{x} \\ \mathbf{h} \\ \lambda \end{bmatrix} \in \mathbf{R}^{2n+1}.$$

Here $\mathbf{h} \in \mathbf{R}^n$ is some suitable vector with the normalising condition $h_k = 1$ for some $1 \leq k \leq n$. Other normalising conditions for $\mathbf{h}$ such as $\|\mathbf{h}\|_2 = 1$ and $\|\mathbf{h}\|_\infty = 1$ can also be defined [24]. Notice that at the fold point $(\mathbf{x}^*, \lambda^*)$, $\mathbf{h}^*$ is the right eigenvector of the Jacobian $\mathbf{f}_{\mathbf{x}}^*$. We use Newton's method to solve (16). Linearising, we iterate for $\ell \geq 0$

$$\underbrace{\begin{bmatrix} \mathbf{f}_{\mathbf{x}}^{(\ell)} & \mathbf{0} & \mathbf{f}_{\lambda}^{(\ell)} \\ \mathbf{f}_{\mathbf{xx}}^{(\ell)}\mathbf{h} & \mathbf{f}_{\mathbf{x}}^{(\ell)} & \mathbf{f}_{\mathbf{x}\lambda}^{(\ell)}\mathbf{h} \\ \mathbf{0} & \mathbf{e}_k^\top & 0 \end{bmatrix}}_{\mathbf{G}_{\mathbf{Y}}^{(\ell)}} \underbrace{\begin{bmatrix} \Delta\mathbf{x}^{(\ell)} \\ \Delta\mathbf{h}^{(\ell)} \\ \Delta\lambda^{(\ell)} \end{bmatrix}}_{\Delta\mathbf{Y}^{(\ell)}} = -\underbrace{\begin{bmatrix} \mathbf{f}(\mathbf{x}^{(\ell)}, \lambda^{(\ell)}) \\ \mathbf{f}_{\mathbf{x}}(\mathbf{x}^{(\ell)}, \lambda^{(\ell)})\mathbf{h}^{(\ell)} \\ h_k - 1 \end{bmatrix}}_{\mathbf{G}^{(\ell)}} \tag{17}$$

with updates

$$\underbrace{\begin{bmatrix} \mathbf{x}^{(\ell+1)} \\ \mathbf{h}^{(\ell+1)} \\ \lambda^{(\ell+1)} \end{bmatrix}}_{\mathbf{Y}^{(\ell+1)}} = \underbrace{\begin{bmatrix} \mathbf{x}^{(\ell)} \\ \mathbf{h}^{(\ell)} \\ \lambda^{(\ell)} \end{bmatrix}}_{\mathbf{Y}^{(\ell)}} + \underbrace{\begin{bmatrix} \Delta\mathbf{x}^{(\ell)} \\ \Delta\mathbf{h}^{(\ell)} \\ \Delta\lambda^{(\ell)} \end{bmatrix}}_{\Delta\mathbf{Y}^{(\ell)}}. \tag{18}$$

The main problem with direct methods is their stringent requirement of an accurate approximation to $(\mathbf{x}^*, \mathbf{h}^*, \lambda^*)$; otherwise the formulation does not converge. While it is of interest to find the exact solution

Table 1: Performance of the direct bifurcation method

Varied bus	Known index	Starting λ_0	Newton steps	λ^* located
1	5.246842	0.0	-	*
		4.6	7	5.2579741845
2	2.447133	0.0	-	*
		2.0	9	2.4548231884
3	5.801273	0.0	10	5.8140201900
4	3.192469	0.0	-	*
		1.9	12	3.2019586939
5	5.235171	0.0	-	*
		4.9	8	5.2452563631
6	2.339682	0.0	-	*
		1.5	11	2.3426212061

$(\mathbf{x}^*, \mathbf{h}^*, \lambda^*)$, once a bifurcation point is reached, this exercise is not as important as finding an accurate approximation to λ^* which is useful for control purposes.

For illustration, Table 1 shows the results of the above bifurcation method for IEEE 9 example with single bus variations. There "Newton steps" refer to the number of iteration steps required to satisfy $\|\mathbf{f}(\mathbf{x}^{(\ell)}, \lambda^{(\ell)})\| \leq TOL = 10^{-5}$ and "*" means no convergence. Clearly one can see that a suitable starting value for λ_0 is essential. Again the point is that it is difficult to know if a λ_0 is good enough *apriori*.

3.2 Continuation methods

The convergence problem with direct bifurcation methods is well known (due to nonlinearity) and one usually uses these methods only for accurately locating the fold point $(\mathbf{x}^*, \lambda^*)$ given a good estimate. Ideally, we want a method that has no difficulties near or passing round a fold point. This isn't unreasonable since there is nothing wrong geometrically with the curve but the parameter λ is not the right parameter to describe the curve. With it we can numerically trace the solution path by generating a sequence of points satisfying a chosen tolerance criterion.

One such method that overcomes the convergence problem completely is the continuation method as extensively discussed in the literature [1, 10, 24, 28]. These kinds of methods together with suitable monitoring steps for locating the fold bifurcation (e.g. computing the minimum singular value) are very reliable in most cases. In this section we briefly discuss the popular pseudo-arclength continuation method which is due to Keller [19].

Let s denote the parameter describing the solution path of (13). Suppose we have a solution $(\mathbf{x}_0, \lambda_0)$ of (13), then Keller's method consists of solving the following equations for $(\mathbf{x}, \lambda) = (\mathbf{x}_1, \lambda_1)$ close to $(\mathbf{x}_0, \lambda_0)$

$$\mathbf{G}(\mathbf{Y}, s) = \begin{bmatrix} \mathbf{f}(\mathbf{x}, \lambda) \\ \hat{\mathbf{x}}_0^\top (\mathbf{x} - \mathbf{x}_0) + \hat{\lambda}_0 (\lambda - \lambda_0) - \Delta s \end{bmatrix} = \mathbf{0} \tag{19}$$

where $\mathbf{Y} = (\mathbf{x}^\top, \lambda)^\top \in \mathbf{R}^{n+1}$ and $\mathbf{G} : \mathbf{R}^{n+2} \to \mathbf{R}^{n+1}$ and $(\hat{\mathbf{x}}_0, \hat{\lambda}_0)$ is the normalised tangent vector at $(\mathbf{x}_0, \lambda_0)$.

This system is solvable from the ABCD lemma [28] and can be solved by using a predictor-corrector procedure as follows:

- **Predictor** (Euler's Method)

$$\begin{bmatrix} \mathbf{x}^{(1)} \\ \lambda^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \lambda_0 \end{bmatrix} + \Delta s \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \hat{\lambda}_0 \end{bmatrix}$$

- **Corrector** (Newton's Method)

  for $\ell \geq 1$ iterate

$$\begin{bmatrix} (\mathbf{f}_{\mathbf{x}})^{(\ell)} & (\mathbf{f}_{\lambda})^{(\ell)} \\ \hat{\mathbf{x}}_0^{\top} & \hat{\lambda}_0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x}^{(\ell)} \\ \Delta \lambda^{(\ell)} \end{bmatrix} = - \begin{bmatrix} \mathbf{f}(\mathbf{x}^{(\ell)}, \lambda^{(\ell)}) \\ \hat{\mathbf{x}}_0^{\top}(\mathbf{x}^{(\ell)} - \mathbf{x}_0) + \hat{\lambda}_0(\lambda^{(\ell)} - \lambda_0) - \Delta s \end{bmatrix} \tag{20}$$

  with updates

$$\begin{bmatrix} \mathbf{x}^{(\ell+1)} \\ \lambda^{(\ell+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^{(\ell)} \\ \lambda^{(\ell)} \end{bmatrix} + \begin{bmatrix} \Delta \mathbf{x}^{(\ell)} \\ \Delta \lambda^{(\ell)} \end{bmatrix}. \tag{21}$$

On convergence of the corrector step, we obtain $(\mathbf{x}_1, \lambda_1)$. Then starting at $(\mathbf{x}_1, \lambda_1)$ to find $(\mathbf{x}, \lambda) = (\mathbf{x}_2, \lambda_2)$, the next tangent vector is computed from (note $\mathbf{f}_{\mathbf{x}}^1 = \mathbf{f}_{\mathbf{x}}(\mathbf{x}_1, \lambda_1)$ and $\mathbf{f}_{\lambda}^1 = \mathbf{f}_{\lambda}(\mathbf{x}_1, \lambda_1)$)

$$\begin{bmatrix} \mathbf{f}_{\mathbf{x}}^1 & \mathbf{f}_{\lambda}^1 \\ \hat{\mathbf{x}}_0^{\top} & \hat{\lambda}_0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\lambda}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{22}$$

It is interesting to observe that the predictor step is simply the initial Newton step because $(\mathbf{x}_0, \lambda_0)$ is a point on the solution curve.

However there are at least *two* reasons why a continuation method is not the answer to the practical power control problem: (1) a continuation method (either arclength or pseudo-arclength based) is usually expensive because many path-following steps (including monitoring steps) must be carried out to reach a bifurcation point λ^*; (2) the exact bifurcation point $\mathbf{x}^*$ is not always needed and only the parameter location λ^* is practically required in power system analysis.

Table 2 shows the performance of the Keller's method for the IEEE 9 example. There Δs_I is the initial steplength used, Δs_L the steplength that must be reduced to in order for the continuation method to converge near the fold point and "C steps" refer to the total number of continuation steps required before an interval $(\lambda_a^*, \lambda_b^*)$ is detected that contains the fold point and the index value $\lambda^* = (\lambda_a^* + \lambda_b^*)/2$ is shown. Clearly the method is robust and reliable in locating the fold point but it is expensive as many Newton steps have been used. Although there exist improved versions of continuation methods, we shall concentrate on investigating an alternative and faster method — the test function method.

3.3 Test functions

Since both direct methods and continuation methods cannot provide a practical solution to the power control problem, there have been several attempts to develop a (scalar) test function $\tau(\lambda)$ which is only zero at $\lambda = \lambda^*$ and thus gives an accurate indication of the distance of an operational point $\mathbf{x}$ from the bifurcation point $\mathbf{x}^*$ (this implies that all test functions may be used as monitoring functions to detect if a bifurcation point has been reached). Obviously this test function must involve the information of the Jacobian matrix $J = \mathbf{f}_{\mathbf{x}}(\mathbf{x}, \lambda)$ and a simple choice is $\tau(\lambda) = det(J)$. However it is in general extremely difficult to get an analytical formula for $det(J)$ unless the underlying network is a trivial one (then J is extremely sparse). The challenge is therefore to find an easily computable function $\tau(\lambda)$ which is directly or indirectly linked to $det(J)$ [24]. We now discuss several test functions.

Table 2: Performance of the Keller's method

Varied bus	Known index	Δs_I size	Δs_L size	C steps	λ^* located
1	5.246842	0.05	0.05	291	5.255383
		0.10	0.10	169	5.250686
		1.00	0.125	95	5.246239
2	2.447133	0.05	0.05	160	2.450491
		0.10	0.10	84	2.430568
		1.00	0.0625	103	2.446636
3	5.801273	0.05	0.05	313	5.804349
		0.10	0.10	181	5.781905
		1.00	0.0625	120	5.799214
4	3.192469	0.05	0.05	201	3.193430
		0.10	0.05	128	3.193541
		1.00	0.0625	107	3.188414
5	5.235171	0.05	0.05	291	5.239677
		0.10	0.10	168	5.212326
		1.00	0.0625	117	5.237614
6	2.339682	0.05	0.05	150	2.335057
		0.10	0.10	100	2.335548
		1.00	0.0625	109	2.330990

Once a test function τ is proposed, the usual philosophy is to test the behaviour of τ; if λ is approximately a quadratic or quartic function in terms of τ (not other way around), then the following formula is used to produce a performance index $\hat{\lambda}$ (or sometimes called the load margin) — an approximation to λ^*:

$$\hat{\lambda} = \lambda_0 - \frac{1}{c}\frac{\tau(\lambda_0)}{d\tau/d\lambda(\lambda_0)}, \tag{23}$$

where $c = 2$ (quadratic case) and $c = 4$ (quartic case).

3.3.1 Method 1 — Seydel's test function

Seydel [23] proposed a test function based on a method that solves approximately the right zero eigenvector problem by a reduced linear system solver. The method finds a good approximation for the initial vector $\mathbf{h}$ of the extended system (16) in the following way. The homogeneous (zero eigenvector) system with a fixed $h_k = 1$

$$J\mathbf{h} = \mathbf{f}_{\mathbf{x}}(\mathbf{x}, \lambda)\mathbf{h} = 0 \tag{24}$$

is solvable but is overdetermined; there are n equations and $n-1$ unknowns. It is therefore also solvable at the fold point $(\mathbf{x}^*, \lambda^*)$ where $rank(\mathbf{f}^*_{\mathbf{x}}) = n-1$. In order to construct a square system regardless of where $(\mathbf{x}, \lambda)$ is, Seydel [23] suggests that one of the n equations is removed; equivalently replace the removed equation by a normalising condition: $h_k = 1$. Seydel then defines the removed (residue) equation to be the required test function τ.

Formally, let $\mathbf{A}_{lk}$ be the Jacobian J with row l replaced by $\mathbf{e}_k$. Then we can write

$$\mathbf{A}_{lk} = (I - \mathbf{e}_l\mathbf{e}_l^\top)J + \mathbf{e}_l\mathbf{e}_k^\top \tag{25}$$

where l denotes the index of the removed equation, $\mathbf{e}_l$ and $\mathbf{e}_k$ are the lth and kth unit vectors respectively. A family of test functions τ_{lk} are defined as follows

$$\tau_{lk} = \tau_{lk}(\lambda) = \mathbf{e}_l^\top J\mathbf{h} = \mathbf{e}_l^\top J\mathbf{A}_{lk}^{-1}\mathbf{e}_l \tag{26}$$

where $\mathbf{h}$ is the solution of the equation $\mathbf{A}_{lk}\mathbf{h} = \mathbf{e}_l$. The test function τ_{lk} described in (26) has the characteristic that (i) it is a function of $\mathbf{x}$ and λ and (ii) it is zero at $(\mathbf{x}^*, \lambda^*)$ and nonzero elsewhere. Therefore τ_{lk} is a well-defined test function. What is the optimal choice of indices l and k? Seydel [23, 27] suggests that l and k can be chosen such that $\mathbf{A}_{lk}$ is nonsingular in order to give a good test function and that there are many such functions. The general belief is that if the pair of indices l and k corresponds to two nonzero positions of the left eigenvector of J at the fold point, then τ_{lk} qualifies as a good test function.

However, our test results for the power system show that the choice of l and k is very restrictive with most choices yielding incorrect results. We will discuss suitable choices for l and k later on. Examples of using this test function can be found in Chiang et al [4, 17, 6].

3.3.2 Method 2 — the Lof test function

Simple variants of the Jacobian can be used as test functions. Lof et al [21] proposed to use

$$\tau_{Lof} = \frac{det(J)}{det(J_1)}$$

where J_1, sized $m \times m$, represents the active power mismatches and corresponds to the partition of the Jacobian matrix J for the voltage angles [21, 4]. We have tried to use this test function for predicting indices and found that it often gives under-estimates of the true indices (see §5).

3.3.3 Method 3 — Canizares test function

The Canizares test function [4, 5] is defined as

$$\tau_{caniz} = \frac{det(J)}{det(A)}$$

where A, sized $(n-2)\times(n-2)$, represents the active and reactive power mismatches excluding a particular load bus and corresponds to the partition of the Jacobian matrix J excluding two rows and two columns. We have tested this function for predicting indices and found that it gives reasonable indices (although sometimes under-estimates; see §5).

3.3.4 Method 4 — Tangent vector function

The tangent vector computed at the current operational point $(\mathbf{x}, \lambda)$ is often used for 'early' detection of the system critical bus [5]. It has been independently discovered and used for predicting the fold bifurcation parameter λ^* (performance index) in [8, 25]

$$\tau = \frac{1}{\mathbf{g}_k}$$

where $J\mathbf{g}_k = \mathbf{b} = \mathbf{e}_k$. Here $\mathbf{g}$ is the tangent vector because differentiating the nonlinear system $\mathbf{f}(\mathbf{x}, \lambda) = \mathbf{f}(\mathbf{x}) - \lambda\mathbf{b} = 0$ with respect to λ yields

$$J\frac{d\mathbf{x}}{d\lambda} - \mathbf{b} = 0.$$

As proved in Seydel [25], this test function is simply a special case of the general Seydel method with $l = k$ for the case $s = 1$ in (14). However this equivalence is not true if $det(J) \neq 0$.

3.3.5 Method 5 — test functions for a transformed system

For the case of (14) with $s = 1$, Chiang et al [18, 7] proposed to transform the usual power system parameter λ into a new κ before employing test function techniques. This idea produces new power systems, similar to continuation systems, that have nonsingular Jacobian matrices even at λ^*. A new test function $\tau(\kappa)$ is used to detect if λ^* has been reached. If the original system is $\mathbf{f}(\mathbf{x}, \lambda) = \mathbf{f}(\mathbf{x}) - \lambda \mathbf{b} = 0$, the new system is

$$\begin{cases} \mathbf{g}(\mathbf{x}, \kappa) = \mathbf{f}(\mathbf{x}) - \kappa \mathbf{b}^\top \mathbf{x} \mathbf{b} = 0, \\ \mathbf{g}_\mathbf{x}(\mathbf{x}, \kappa)\mathbf{c} - \mathbf{b} = 0, \\ \tau(\kappa) = \kappa \mathbf{b}^\top \mathbf{c} + 1 = 0. \end{cases}$$

We remark that this method is closer to a continuation method than a index predicting method because even if κ^* is found, one needs information of $\mathbf{x}^*$ before finding λ^*. Therefore the method is beyond the scope of this paper.

We also remark that in the discussion of [7] it is proved that the above $\tau(\kappa)$ satisfies

$$\tau(\kappa) = \frac{det(\mathbf{f}_\mathbf{x}(\mathbf{x}, \lambda))}{det(\mathbf{g}_\mathbf{x}(\mathbf{x}, \kappa))},$$

where $det(\mathbf{g}_\mathbf{x}(\mathbf{x}, \kappa)) \neq 0$. It is also interesting to point out that for a continuation method like (19), Govaerts [14, 15] proposed a similar test function

$$g(\mathbf{x}, \lambda) = \frac{det(\mathbf{f}_\mathbf{x}(\mathbf{x}, \lambda))}{det(\mathbf{M}(\mathbf{x}, \kappa))}, \tag{27}$$

where $\mathbf{M}$ is the coefficient matrix of (20). However as discussed below, all such functions (based on different *scaled* determinants of J) require a careful study of their behaviour in terms of λ before one may consider a use for performance indices i.e. approximations to λ^*.

3.4 Approximate performance indices

All test functions $\tau(\lambda)$, readily usable for monitoring purposes in a continuation method, may be adapted to yield an approximate performance index (or load margin) i.e. an approximation to a true fold bifurcation parameter λ^*. However the accuracy of such an approximation (hence the reliability of an index) relies on an assumption of the $\tau(\lambda)$ behaviour. A commonly used assumption [23, 6, 4] is the following

$$\tau(\lambda) = (b - d\lambda)^{1/c} \tag{28}$$

where b, c, d are some suitable (positive) constants; in particular the choice of $c = 2$ (quadratic) or $c = 4$ (quartic) was practically used. We remark that as known from [15], assuming the fold point is in the neighbourhood i.e. if $\lambda^* \approx \lambda_0$, all test functions that have a regular zero satisfies $c = 2$. However, outside a neighbourhood (i.e. globally), only some test functions may satisfy $c = 2$ and our work will offer one way to construct such test functions of type (27) for certain $\mathbf{M}$.

The first question is, given the behaviour of τ in (28), how to solve $\tau(\lambda) = 0$ to get an approximate solution. The formula often recommended [23, 6, 4] is (given the present point $(\mathbf{x}_0, \lambda_0)$)

$$\lambda^* = \lambda_0 - \frac{1}{c} \frac{\tau(\lambda_0)}{d\tau/d\lambda(\lambda_0)}. \tag{29}$$

We find it surprising that this Newton method like formula was not formally explained in the literature. A use of the formula in another context can be found in [4, 5]. We can prove that (29) gives the exact performance index *if* the test function behaves as in (28). More precisely, because

$$W(\lambda) = \frac{1}{c}\frac{\tau(\lambda)}{d\tau/d\lambda(\lambda)} = -\frac{b}{d} + \lambda = 0 \tag{30}$$

has the root $\lambda^* = \frac{b}{d}$, then using

$$W(\lambda_0) = -\frac{b}{d} + \lambda_0,$$

we can verify that (29) is the *exact* solution of $\tau(\lambda) = 0$. As we see later, many test functions do not behave like (28) — this is why a seemingly reasonable test function cannot always provide a reliable index.

The second and related question is how to verify that a test function behaves like (28) for some c. This motivates the work of next section.

4 A new study of the Jacobian matrix and its submatrices

As the determinant of the Jacobian matrix J is often used in design of indices, we wish to know more of its behaviour before analysing existing and proposing new indices. In this section we first present some analytical results on J and its submatrices and then propose two new indices that are based upon these results.

4.1 Behaviour of the voltage angles and magnitudes

Clearly $det(J)$ is a nonlinear function of voltage angles δ_j and magnitudes v_j. To study $det(J)$ and its submatrices, we need some reasonable assumptions. For the case of reactive power injections changes (i.e. vary Q_{L_j}), experiments have shown that the differences of voltage angles $\delta_j - \delta_k$ do not vary a great deal. Also as noted in the literature, the bifurcation parameter λ has a quadratic behaviour in voltage magnitudes (this can be deduced from the main equation (4) or (5))

$$\lambda = a_0 v_j^2 + a_1 v_j + a_2 \tag{31}$$

where the coefficients a_0, a_1, a_2 depend on j. This suggests that it is reasonable to assume that

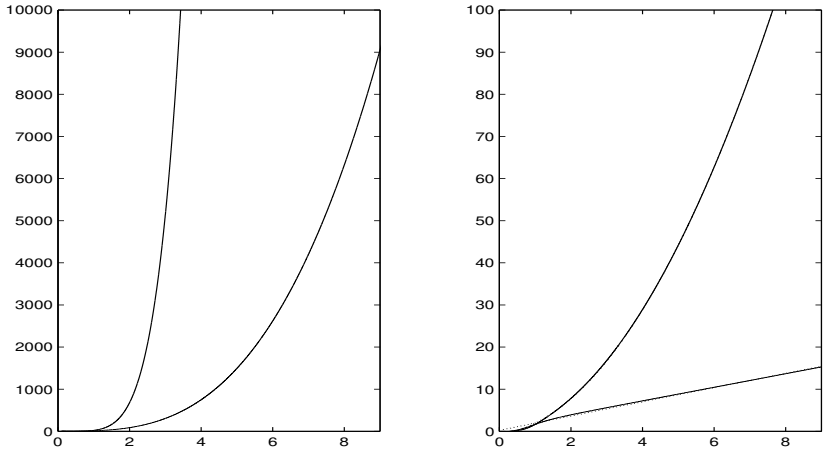
$$v_j \approx v_j - \gamma_j = (b_j - d_j\lambda)^{1/2} \tag{32}$$

because the constant $\gamma_j = a_1/(2a_0)$ is often small in our application.

To proceed with our analysis on determinants, our main assumptions are

$$\begin{cases} \delta_{j_1}(\lambda) - \delta_{j_2}(\lambda) \approx 1, & \text{for any } j_1,\ j_2 \\ \sum\limits_{\substack{j=1 \\ j\neq i_1}}^{m} w_j v_j \approx v_{i_1}, & \text{for } i_1 = 1, 2, \ldots, m_1 \quad \text{(rows } i_1\text{)}, \\ \sum\limits_{\substack{j=1 \\ j\neq i_2}}^{m} v_j \approx v_{i_2}, & \text{for } i_2 = 1, 2, \ldots, m - m_1 \quad \text{(rows } i_2 + m_1\text{)}, \end{cases} \tag{33}$$

Figure 2: Illustration of rational functions of polynomials



where w_j's are generic constants independent of λ. By "$\approx$", we mean that two quantities are at most differing from each other by factors that do not depend on λ; e.g. if $f_4 = 6\lambda^4 + 5$, $f_3 = 1.5\lambda^3$, then $f_4 \approx \lambda f_3$ for $\lambda \geq 0.8$. The idea of considering only leading terms of polynomials is reasonable as illustrated in Figure 2, where $P_5(t) = 21t^5 + t^4 - 2t^3 + t^2 + 3t - 1$ (the steep curve) and $P_3(t) = 13t^3 - 5t^2 - 3t + 9$ (the other curve) are plotted on the top graph. We can observe from the right graph in Figure 2 that the rational function of two polynomials $P_5(t)/P_3(t)$ behaves like a quadratic and $P_5(t)/t/P_3(t)$ behaves linearly (for $t \geq 0$) so we may write $P_5(t)/P_3(t) \approx t^2$ and $P_5(t)/t/P_3(t) \approx t$.

Here the second assumption in (33) is reasonable because at least one neighbouring voltage magnitude behaves similarly to v_{i_1} of load bus i_1 (regardless of constants w_j). The third one is because $v_{i_2+m_1} \approx 1$ remains fixed during power load changes so the left sum is dominated by v_{i_1} with $1 \leq i_1 \leq m_1$.

These facts enable us to investigate and simplify the relations between various determinants (hence test functions) and voltage magnitudes, and consequently between test functions and the bifurcation parameter λ.

4.2 Determinants of the Jacobian matrix and its submatrices

With assumption (33) we can prove the following:

Theorem 1 *If (33) holds, the determinant of the Jacobian matrix J from (15) can be approximated by a high order multi-variate polynomial* $\mathbf{P}(v_1, \ldots, v_m)$ *which has the highest and lowest order terms respectively*

$$v_1^4 \cdots v_{m-m_1}^4 v_{m-m_1+1}^3 \cdots v_{m_1}^3 v_{m_1+1}^2 \cdots v_m^2,$$
$$v_1^3 \cdots v_{m-m_1}^3 v_{m-m_1+1}^3 \cdots v_{m_1}^3 v_{m_1+1}^2 \cdots v_m^2.$$

Proof. Using assumption (33), we may simplify the Jacobian matrix in (15) as

$$J = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \approx \left[\begin{array}{ccccccc|ccc} v_1v_1 & \cdots & v_1v_{m_1} & v_1v_{m_1+1} & \cdots & v_1v_m & & v_1 & \cdots & v_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ v_{m_1}v_1 & \cdots & v_{m_1}v_{m_1} & v_{m_1}v_{m_1+1} & \cdots & v_{m_1}v_m & & v_{m_1} & \cdots & v_{m_1} \\ v_{m_1+1}v_1 & \cdots & v_{m_1+1}v_{m_1} & v_{m_1+1}v_1 & \cdots & v_{m_1}v_m & & v_{m_1} & \cdots & v_{m_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ v_mv_1 & \cdots & v_mv_{m_1} & v_mv_{m_1+1} & \cdots & v_mv_{m-m_1} & & v_m & \cdots & v_m \\ \hline v_1v_1 & \cdots & v_1v_{m_1} & v_1v_{m_1+1} & \cdots & v_1v_m & & v_1 & \cdots & v_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ v_{m_1}v_1 & \cdots & v_{m_1}v_{m_1} & v_{m_1}v_{m_1+1} & \cdots & v_{m_1}v_m & & v_{m_1} & \cdots & v_{m_1} \end{array}\right].$$

The relevant quantities $\delta_j(\lambda_0), y_{jk}, \theta_{jk}$ will make this approximate Jacobian matrix differ from the true matrix by constant factors that are not needed in our context. Using the properties of determinants, constant factors can come out rows or columns. Also noting that v_j $(m_1 + 1 \le j \le m)$ does not depend on λ, we obtain that

$$det(J) \approx v_1^3 \cdots v_{m-m_1}^3 v_{m-m_1+1}^3 \cdots v_{m_1}^3 v_{m_1+1}^2 \cdots v_m^2 \left|\begin{array}{ccccccc|ccc} 1 & \cdots & 1 & 1 & \cdots & 1 & & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & 1 & & 1 & \cdots & 1 \\ 1 & \cdots & 1 & v_1 & \cdots & 1 & & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & v_{m-m_1} & & 1 & \cdots & 1 \\ \hline 1 & \cdots & 1 & 1 & \cdots & 1 & & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & 1 & & 1 & \cdots & 1 \end{array}\right|.$$

With a further simplification of the determinant, the theorem is proved. ■

This idea of simplifying the Jacobian matrix (15) carries over to study of its submatrices. Define $J(\ell_1, \ldots, \ell_s; k_1, \ldots, k_s)$ as the submatrix of J, sized $(n-s) \times (n-s)$, after taking out rows $\ell_1, \ldots, \ell_s$ and columns $k_1, \ldots, k_s$. It is feasible to study such a general submatrix; however for illustration, we only consider some special cases. We give the following results.

Corollary 1 *Under assumption (33), the determinants of three particular submatrices J admit the following approximations ($1 \le p \le m_1$):*

1. $det(J(1;1)) \approx det(J)/v_1^2$;
2. $det(J(m+p; m+p)) \approx det(J)/v_p$;
3. $det(J(m+1, \ldots, n; m+1, \ldots, n)) \approx det(J)/\prod_{j=1}^{m} v_j$.

Proof. It suffices to consider case 1. Using the properties of determinants, take out factor v_1 from row 1 and v_1 from column 1 from $det(J)$. Then what is left is approximately $det(J(1;1))$ and so the results are proved. ■

4.3 New Jacobian matrix based test functions and indices

We first remark that any function of λ can be a valid test function $\tau(\lambda)$ if it is zero at λ^* and nonzero elsewhere. This makes it relatively easy to propose a new test function e.g. by simple manipulation of

$det(J)$. However it is a harder problem to make use of a test function to yield a computable, reliable and accurate performance index that approximates λ^*. In the literature, the index formula (29) is frequently used but fewer work have verified that an underlying test function satisfies (28); without (28), (29) cannot provide a good performance index even if a test function is a valid one for monitoring purposes.

Our idea of a test function is simple: we want it to relate to $det(J)$ directly. The new idea is to be able to predict the behaviour of a test function in terms of λ on construction — ideally linking (28). Then we can be justified to use (29) to provide a good performance index.

To this end, using Theorem 1 and Corollary 1, we can construct test functions that possess known behaviour in terms of v_j or λ. To use formula (29), a test function to be constructed should be approximately equivalent to v_j that in turn behaves like (32).

Therefore, we propose the following two new test functions if we consider the variation at bus p $(1 \le p \le m_1)$:

1. $\tau_1(\lambda) = det(J)/det(J(m+p; m+p))$;
2. $\tau_2(\lambda) = det(J)/det(J(m+1, \ldots, n; m+1, \ldots, n))/\prod_{\substack{j=1 \\ j \neq p}} v_j$.

Corollary 2 *The two new test functions as defined above will behave like v_p or (32) i.e.*

$$\tau_1(\lambda) \approx (b_1 - d_1\lambda)^{1/2} \quad \textit{and} \quad \tau_2(\lambda) \approx (b_2 - d_2\lambda)^{1/2}.$$

Thus for both functions, one may use formula (29) with $c = 2$ to produce a reliable performance index.

Proof. Using Corollary 1, we see that both behave ($\approx$) like v_p and for our case of reactive power changes v_p (as a load bus) behaves as in (32) with $j = p$. The proof is complete. ■

Remark. It is possible to generalise τ_1 as

$$\tau_1(\lambda) = det(J)/det(J(\ell; k));$$

however care must be taken to assess the behaviour of τ_1 for specific ℓ, k before any use. For example, with $\ell = k = 1$ and using Corollary 1, we see that $\tau_1 \approx v_1^2$ so one cannot use (29) with either $c = 2$ or $c = 4$. We found that when considering load variations for bus p, γ_p is the smallest of all γ_j's so we recommend the choice $\ell = m + p$ and $k \ge m + 1$.

5 Numerical results

We now present some numerical results of our proposed index methods. To emphasize the robustness of our methods, we shall use standard test examples rather than specially tuned ones. In particular we shall consider the standard and widely available test systems

1. IEEE 9:
   the number of load buses $m_1 = 6$ and $m = 8$, and the Jacobian is of the size 14×14;
2. IEEE 14:
   the number of load buses $m_1 = 9$ and $m = 13$, and the Jacobian is of the size 22×22;
3. IEEE 30:
   the number of load buses $m_1 = 24$ and $m = 29$, and the Jacobian is of the size 53×53;

4. IEEE 57:
   the number of load buses $m_1 = 50$ and $m = 56$, and the Jacobian is of the size 106×106;
5. IEEE 118:
   the number of load buses $m_1 = 64$ and $m = 117$, and the Jacobian is of the size 181×181;
6. IEEE 300:
   the number of load buses $m_1 = 231$ and $m = 299$, and the Jacobian is of the size 530×530.

For each system, we shall test the efficiency and robustness of our two new methods in getting an 'on-line' index list for all load buses that can be used to identify the weakest bus. We shall compare the performance of

1. M1 — New test function τ_1
2. M2 — New test function τ_2
3. M3 — Seydel test function with $\tau = det(J)/det(J_{1;1})$ [23]
4. M4 — Canizares test function $\tau = det(J)/det(J_{p,m+p;p,m+p})$ [4]
5. M5 — Lof's test function $\tau = det(J)/det(J1)$ [21].

The computational cost required for each case depends on the total number of conventional Newton steps for power flow equations.

To implement (29), the calculation of the derivative $d\tau/d\lambda$ is done by finite differences as in [17, 6]. Therefore two successive Newton solves are applied at $\lambda_0 = 0$ and $\lambda_1 = 10^{-4}$; for these IEEE systems with tolerance $TOL = 10^{-5}$ for stopping Newton iterations, the number of such iterations is typically small (about 5) using the simple starting values $\delta_j = 0$ and $v_j = 1$. For other systems, one may have to use a continuous power flow program to find $\mathbf{x}_0$ and $\mathbf{x}_1$ (hence $\tau(\lambda_0)$ and $\tau(\lambda_1)$) if convergence is a problem. An alternative method for computing $d\tau/d\lambda$ may be based on semi-analytical formulas as in Seydel [23].

In Figures $3-8$, we display the predicted indices for each test system by the five methods, compared with the exact indices obtained by running a continuous power flow program. As some indices (mainly from $M3$) are over-estimates, we present each plot in two graphs with the top one showing all predicted indices and the bottom one excluding the outlier cases so that a better comparison is visible for those close cases. For some larger networks, we rotate the graphs so that bus labels are readable. For IEEE 300 bus, we only show the indices for the first 5 buses for clarity. A legend list can be stated as follows:

1. Symbol $\bigcirc$ — for M1
2. Symbol $+$ — for M2
3. Symbol $\bigtriangledown$ — for M3
4. Symbol $*$ — for M4
5. Symbol $\bigtriangleup$ — for M5
6. Symbol $\times$ — for exact indices.

Clearly we observe that two new test functions M1 and M2 provide very reliable performance indices for all cases considered. M4 gives a reasonable index for several cases but is less accurate and can fail in identifying the weakest bus. M3 shows results of a typical example of adapting generic bifurcation methods to power flow equations — it may or may not work for arbitrary choice of parameters of l, k. Note that Seydel [23, 27] did not recommend any choice of the parameters l, k; in fact [27, p.6] seems

to suggest that for a test system of $n = 4$ there are at most $16 = 4^2$ test functions! Our analytical and numerical results have confirmed only some choices lead to useful indices (e.g. $l = k = m + p$ if p corresponds to the load bus under variations). Method M5 can also provide useful profiles reflecting the true indices but they are usually overestimates and thus not reliable. The trouble with M5 is that it is 'too close' to $det(J)$ and it behaves like very high order polynomials that cannot provide a useful index.

6 Conclusions

We gave a brief review of the recent developments in voltage stability studies in computing fold bifurcations. The particular class of methods using test functions are investigated in details. We found that many test functions can provide useful indicators to determine if a fold point has been reached but are not useful in predicting the index (i.e. the distance away form the fold point).

This paper proposed a new way of analysing the behaviour of the determinants of the Jacobian matrix and its submatrices and consequently designing determinants-based test function methods. For a useful class of problems involving reactive power changes, we are able to propose new as well as analyse existing test functions. The central idea is the ability to determine whether a test function can lead to practical performance indices. Numerical results using widely available IEEE data (without any tuning) are promising and confirm our theoretical predictions.

Future work will involve applications to a wider class of test systems and generalise the new analytical and numerical methods to a wider class of problems.

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Figure 3: Performance indices by various methods for IEEE 9 bus

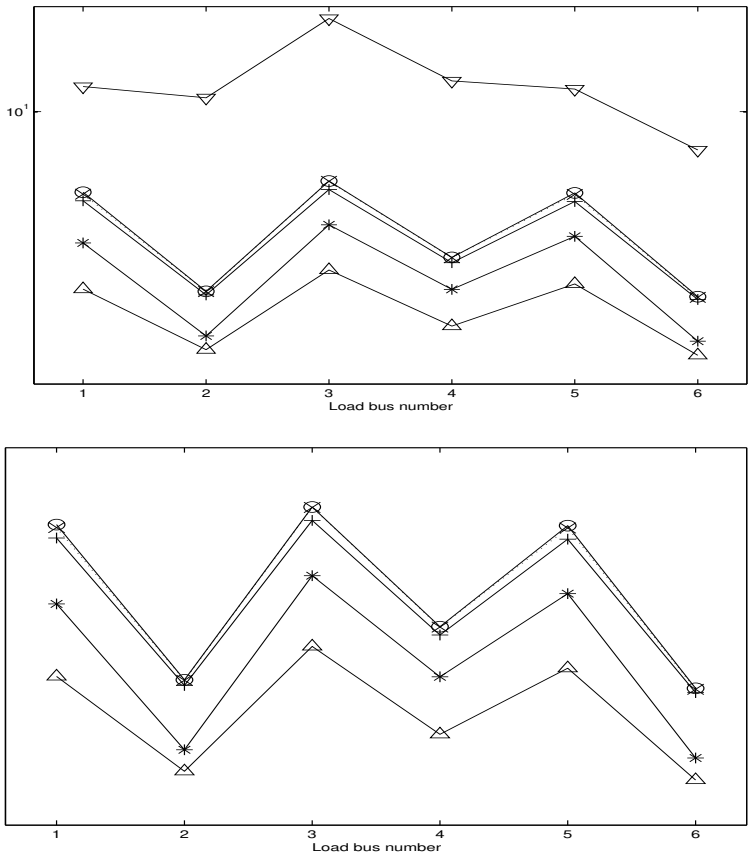


Figure 4: Performance indices by various methods for IEEE 14 bus

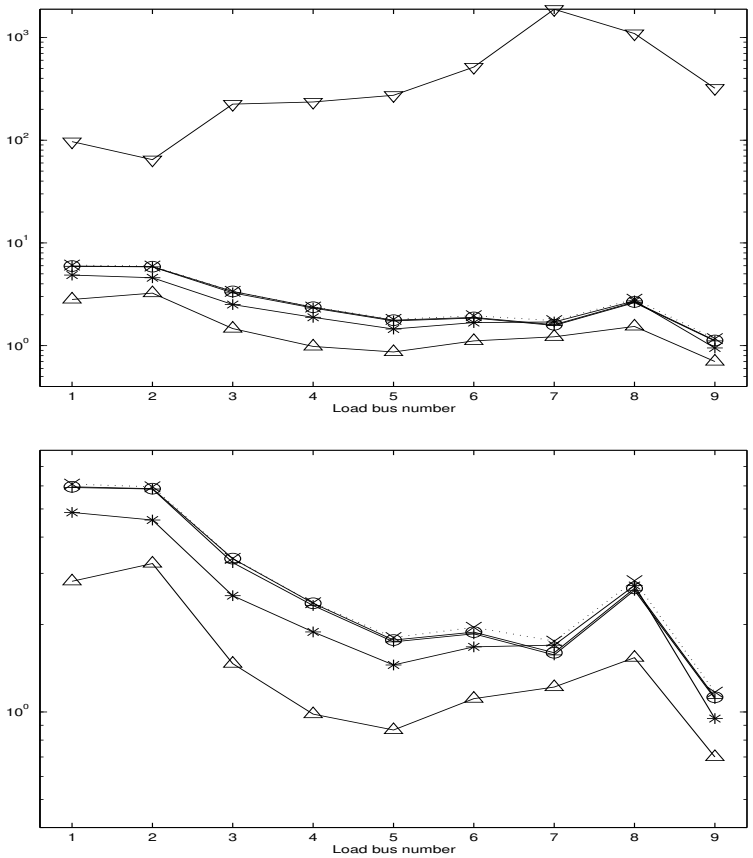


Figure 5: Performance indices by various methods for IEEE 30 bus

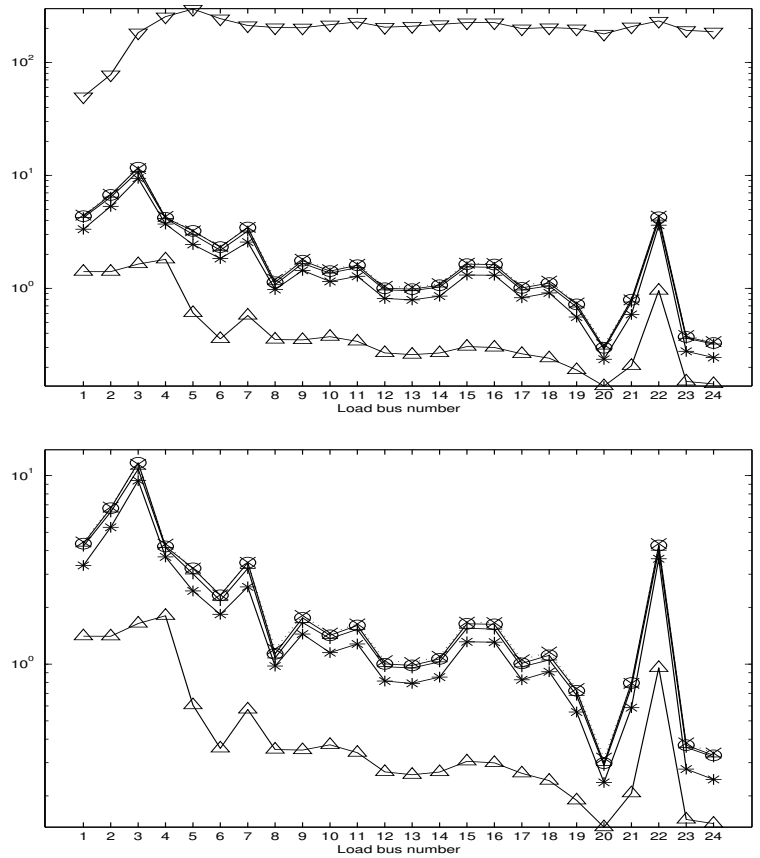


Figure 6: Performance indices by various methods for IEEE 57 bus

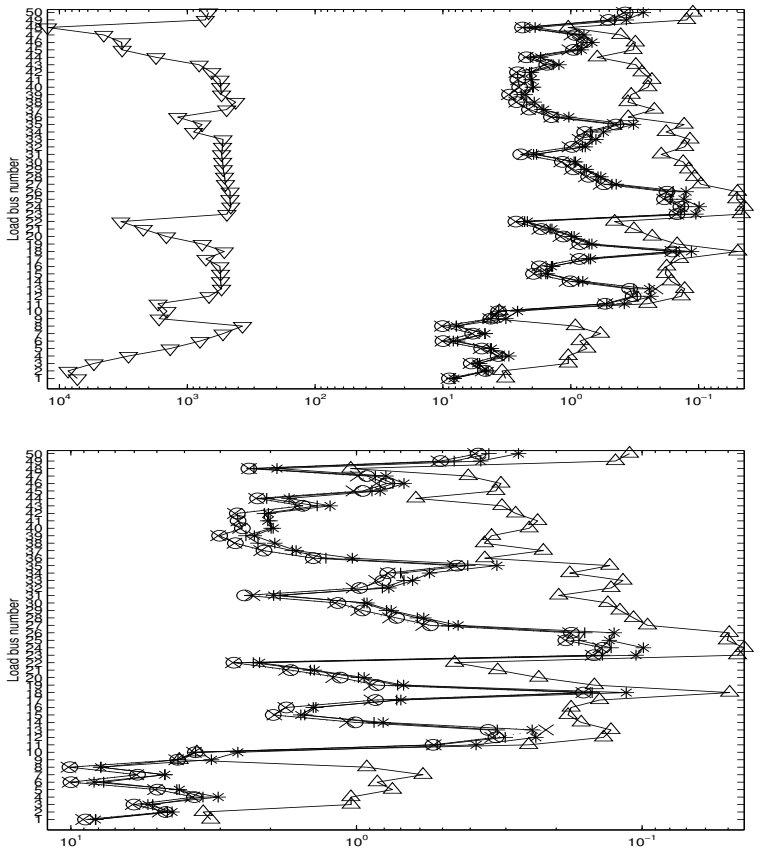


Figure 7: Performance indices by various methods for IEEE 118 bus

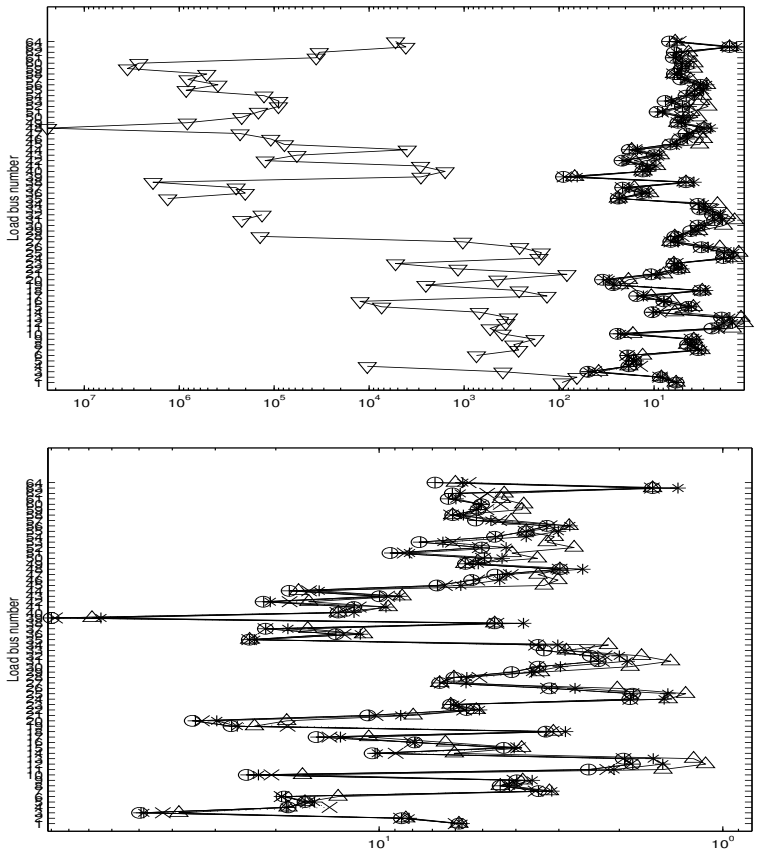
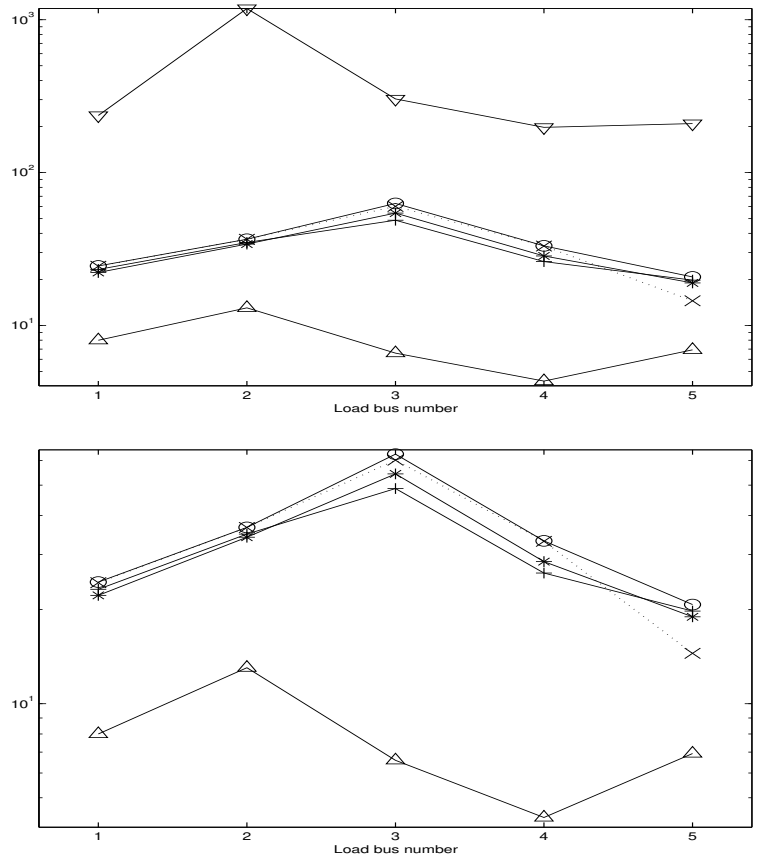


Figure 8: Performance indices by various methods for IEEE 300 bus (first 5 load buses)



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