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An efficient method for evaluating the integral of a class of highly oscillatory functions

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ABSTRACT

Highly oscillatory integrals require special techniques for their effective evaluation. Various studies have been conducted to find computational methods for evaluating such integrals. In this paper we present an efficient numerical method to evaluate a class of generalised Fourier integrals (on a line or a square) with integrands of the form $f(x)e^{ikg(x)}$, under the assumption that in the domain of integration, both f and g are sufficiently smooth and that g does not have any stationary/critical points. Numerical analysis and results are given to illustrate the effectiveness of our method for computing generalised Fourier integrals.
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1. Introduction

Integrating an oscillatory kernel function of the following type for large $k \in \mathbb{R}$ over a closed manifold Γ of an open, connected domain $\Omega \subset \mathbb{R}^d$

$$I(f) = \int_\Gamma f(\mathbf{x}) e^{ikg(\mathbf{x})} d\Gamma$$

is of fundamental importance in many applications e.g. the boundary integral formulation of the exterior Helmholtz problem [1–4] where k is the acoustic wavenumber. In this work we present an efficient method for evaluating the above integral under the assumption that f and g are sufficiently smooth and that g does not have any stationary points. In addition, we assume that g does not depend on k, whilst if f does depend on k, we assume that it will only slowly vary with k. The simplest method for integrating the above integral is a high-order quadrature rule, such as a compound Gauss-type rule [5]. However, for very high wavenumbers k, the computational cost of such methods prohibits their use in practice, as the cost required to maintain accuracy grows at least in direct proportion to k as k increases. Instead, this paper will introduce a new quadrature method which can be used to quickly evaluate such integrals for virtually any wavenumber using relatively few quadrature points.

This paper is concerned with the problem of evaluating an integral of the form (one dimensional $d = 2$)

$$I(f) = \int_0^h f(x) e^{ikg(x)} dx \tag{1}$$

or (two dimensional $d = 3$)

$$I(f) = \int_0^h \int_0^h f(x, y) e^{ikg(x,y)} dx dy. \tag{2}$$

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Here h refers to the size of integration domain (in the context of the boundary element method). In the one-dimensional case, if f is the derivative of g then the integral can be trivially evaluated using the substitution $u = g(x)$, but often in more general cases it is necessary to resort to numerical methods. However, conventional quadrature rules are not well suited to evaluating integrals such as (1) or (2) due to the rapid oscillations in the integrand for large values of k. Such quadrature rules are based on interpolating the whole integrand using polynomials, or piecewise polynomials. Special quadrature rules based on product-integration type methods can only be used when it is possible to evaluate integrals such as (in the one-dimensional case)

$$\int_0^h x^n e^{ikg(x)} dx \tag{3}$$

exactly. Since this is not always possible, such rules cannot be used for general functions $g(x)$. In the two dimensional case, these problems become even more difficult to overcome. Hence in either the one or two dimensional case, we need to develop special quadrature methods for evaluating the integrals. Refer to [6] for a related but different application.

Many related studies are concerned with infinite series methods for (1) and (2), where one derives series solutions by assuming f and g are sufficiently smooth, and then obtaining a numerical method by truncating the series. In particular, Iserles et al. [7] have worked out the exact series solutions for (1) and (2) over a regular simplex, where all partial derivatives of f and g are needed up to order infinity, if the series is to be precisely evaluated. Huybrechs et al. [8] proposed a method to search for a special integrating path related to g so that the integration can be done easily, while Averbuch et al. [9] and Bruno et al. [10] proposed Fourier series related ideas to compute (1) and (2).

In this paper we shall first develop a quadrature rule for evaluating the one dimensional integral (1) and then show how the method can be generalised for evaluating the integral (2) in two dimensions. Our preferred idea is a mixed method of analytical product integration combined with numerical approximation, which is easy to apply to practical problems.

The rest of the paper is organised as follows. Section 2 will present a numerical method for solving the univariable case (1) and show a simple analysis of its properties. Section 3 will consider how to use this new quadrature method for the case of high wavenumbers for computing (1) and (2). Finally, Section 4 will present the results of employing the new quadrature method for typical integrals that arise from a boundary integral method, and will show that by using our quadrature method it is possible to get accurate solutions for cases of high wavenumbers without making an excessive number of quadrature points.

2. A method for evaluating oscillatory integrals

The methods proposed here are based on using piecewise polynomial approximations to both f and g.

1D Case. Suppose the interval $[0, h]$ is divided in n sub-intervals, not necessarily of equal length. Let x_{j-1} and x_j denote the ends of the jth sub-interval. In this sub-interval it is possible to approximate both $f(x)$ and $g(x)$ by linear functions of the form

$$\begin{aligned} f(x) &\approx f(x_{j-1})\left(\frac{x-x_j}{x_{j-1}-x_j}\right) + f(x_j)\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right) \\ g(x) &\approx g(x_{j-1})\left(\frac{x-x_j}{x_{j-1}-x_j}\right) + g(x_j)\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right). \end{aligned} \tag{4}$$

Hence the approximation to (1) can be expressed as

$$\begin{aligned} I_n(f) = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} & \left[f(x_{j-1})\left(\frac{x-x_j}{x_{j-1}-x_j}\right) + f(x_j)\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right)\right] \\ & \times \exp\left[ik\left[g(x_{j-1})\left(\frac{x-x_j}{x_{j-1}-x_j}\right) + g(x_j)\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right)\right]\right] dx. \end{aligned} \tag{5}$$

The integrals appearing in (5) can all be evaluated exactly, and so it is possible to construct a quadrature rule which uses the values of f and g at just the $n + 1$ points $x_0, \dots, x_n$. This rule will be referred to as the zeroth order one-dimensional high-frequency (0-1DHF) quadrature rule.

2D Case. We now generalise the above idea to integral (2). Let S denote the square which is the domain of integration in (2). We divide S into M triangles $T_1, T_2, \dots, T_M$ connected together at N node points. Consider the triangle T_j and let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the vertices of T_j. Further, let $g_1 = g(x_1, y_1)$, $g_2 = g(x_2, y_2)$ and $g_3 = g(x_3, y_3)$. Then by mapping T_j into a reference triangle in the (s, t) plane, and approximating g by a linearly interpolating polynomial we get

$$\int\int_{T_j} f(x, y) e^{ikg(x,y)} dx dy \approx \int_0^1 \int_0^{1-t} f(x(s, t), y(s, t)) e^{ik(g_1+(g_2-g_1)s+(g_3-g_1)t)} J_j ds dt \tag{6}$$

where

$$\begin{aligned} x(s,t) &= (1-s-t)x_1 + sx_2 + tx_3 \\ y(s,t) &= (1-s-t)y_1 + sy_2 + ty_3 \\ J_j &= |(x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1)|. \end{aligned} \tag{7}$$

We can now construct a quadrature rule for evaluating (6) of the form

$$\int_0^1 \int_0^{1-t} f(x(s,t), y(s,t)) e^{ik(g1+(g2-g1)s+(g3-g1)t)} J_j \mathrm{d}s\mathrm{d}t \approx \sum_{l=1}^{N} \tilde{w}_l(g) f(x(s_l,t_l), y(s_l,t_l)) \tag{8}$$

by choosing the weights $\tilde{w}_1(g), \ldots, \tilde{w}_N(g)$ which are local to the current triangle such that the rule is exact for some set of basis functions. If these basis functions are chosen to be the usual parametric basis functions (that is, they are equal to one at one node point and zero at all the other node points) then the quadrature rule for evaluating (2) can be expressed as

$$\int\int_S f(x,y) e^{ikg(x,y)} \mathrm{d}x\mathrm{d}y \approx \sum_{j=1}^{N} w_j(g) f(x_j, y_j) \tag{9}$$

where each $w_j(g)$ can be found by simply summing the appropriate local weights from the triangles where the corresponding basis function is non-zero, as described below.

In (6) we now also approximate f by a linear function to give

$$\int\int_{T_j} f(x,y) e^{ikg(x,y)} \mathrm{d}x\mathrm{d}y \approx \int_0^1 \int_0^{1-t} ((1-s-t)f_1 + sf_2 + tf_3)\, e^{ik((1-s-t)g_1+sg_2+tg_3)} J_j \mathrm{d}s\mathrm{d}t. \tag{10}$$

The three terms on the right-hand side of (10) can be evaluated separately to give an expression of the form

$$\tilde{w}_1(g)f_1 + \tilde{w}_2(g)f_2 + \tilde{w}_3(g)f_3 \tag{11}$$

where

$$\begin{aligned} \tilde{w}_1(g) &= J_j \int_0^1 \int_0^{1-t} (1-s-t) e^{ik[g_1+(g_2-g_1)s+(g_3-g_1)t]} \mathrm{d}s\mathrm{d}t \\ \tilde{w}_2(g) &= J_j \int_0^1 \int_0^{1-t} s e^{ik[g_1+(g_2-g_1)s+(g_3-g_1)t]} \mathrm{d}s\mathrm{d}t \\ \tilde{w}_3(g) &= J_j \int_0^1 \int_0^{1-t} t e^{ik[g_1+(g_2-g_1)s+(g_3-g_1)t]} \mathrm{d}s\mathrm{d}t, \end{aligned} \tag{12}$$

noting that J_j is a constant. Further details of evaluating the integrals in (12) can be found in the Appendix. We note that in order to avoid problems associated with dividing by zero, we have to consider the five cases: g_1, g_2 and g_3 all different; $g_1 = g_2$; $g_1 = g_3$; $g_2 = g_3$; and $g_1 = g_2 = g_3$. The above quadrature rule (9) obtained by summing up all the contributions of the M triangles will be denoted as the 0-2DHF rule.

Analysis. Our quadrature method is related to the Filon type idea [11] as described in [12]. In details, let a Gauss rule for integrating $f(u, v)$ be denoted by

$$T(f) = \int_0^h \int_0^h f(u,v) \mathrm{d}v\mathrm{d}u = \bar{T}(f) + O(h^{q+3}) \tag{13}$$

with

$$\bar{T}(f) = \sum_{\ell=1}^{L} \bar{w}_\ell f(u_\ell, v_\ell),$$

$$f(u,v) \approx \sum_{\ell=1}^{L} f(u_\ell, v_\ell) B_\ell(u,v), \quad \text{and} \quad \bar{w}_\ell = \int_0^h \int_0^h B_\ell(u,v) \mathrm{d}v\mathrm{d}u$$

where B_ℓ is a polynomial of degree up to $q = \frac{4\sqrt{L}-1}{2} \approx 2\sqrt{L} - 1$ for large L, and the total number of degrees of freedom is $2L$. Then a Filon type quadrature for $I(f)$ is the following

$$\bar{I}^F(f) = \sum_{\ell=1}^{L} w_\ell(g) f(u_\ell, v_\ell), \quad w_\ell(g) = \int_0^h \int_0^h B_\ell(u,v) e^{ikg(u,v)} \mathrm{d}v\mathrm{d}u. \tag{14}$$

For large $k \geq 1$ and $h \leq 1$, it is proved in [12,7] that $|I(f) - \bar{I}^F(f)| \leq O(h^2/k)$ and the error becomes $O(h^2/k^2)$ if a special choice of nodes is made.

However, the weights in (14) must be integrated analytically, which can be as challenging as the original integral (2). Our method (6) represents a practical computational algorithm to improve (14).

To analyse (6), we may describe a general polynomial approximation instead of just linear functions. Suppose that we approximate the smooth function $g(u, v)$ by a polynomial of degree s. Let $P_s(u, v)$ be the interpolating polynomial of degree s for g:

$$g(u, v) = P_s(u, v) + E_s(u, v) \tag{15}$$

with $|E_s| \le O(h^{s+1})$. Then we can approximate (2) by

$$I^P(f) = \int_0^h \int_0^h f(u, v) \mathrm{e}^{\mathrm{i}kP_s(u,v)} \mathrm{d}v\mathrm{d}u \tag{16}$$

which is now computed by the Filon quadrature method, as with (14)

$$\bar{I}^P(f) = \sum_{\ell=1}^{L} w_\ell f(u_\ell, v_\ell), \quad w_\ell = \int_0^h \int_0^h B_\ell(u, v) \mathrm{e}^{\mathrm{i}kP_s(u,v)} \mathrm{d}v\mathrm{d}u \tag{17}$$

although the method breaks down if the integrals in (14) cannot be evaluated analytically. Practically $s = 1, 2$.

The accuracy of this method can be stated as follows.

Lemma 1. *Let $k \ge 1, 0 < h \le 1$. Assume that smooth functions $f \in C^1$, $g \in C^{s+1}$ for some $s \ge 1$, and P_s satisfies the nonresonance condition i.e. $\frac{\partial P_s}{\partial u} \ne 0$, $\frac{\partial P_s}{\partial v} \ne 0$, $\frac{\partial P_s}{\partial u} \ne \frac{\partial P_s}{\partial v}$ at the underlying reference triangle. The quadrature method (17) for (2) is bounded by*

$$|I(f) - \bar{I}^P(f)| \le O(h^2/k) + kO(h^{s+3}) \tag{18}$$

which implies that $|I(f) - \bar{I}^P(f)| \le O(h)$ if $h \le 1/k^{2/(s+2)}$ for large k.

Proof. Since $|I^P(f) - \bar{I}^P(f)| \le O(h^2/k)$ following (14), from the triangle inequality

$$|I(f) - \bar{I}^P(f)| \le |I(f) - I^P(f)| + |I^P(f) - \bar{I}^P(f)|$$

we only need to bound the first term. Recall that the real-valued function $g = P_s + E_s$ so $f\mathrm{e}^{\mathrm{i}kg} = f\mathrm{e}^{\mathrm{i}kP_s} + f\mathrm{e}^{\mathrm{i}kP_s}(\mathrm{e}^{\mathrm{i}kE_s} - 1)$. Using the boundedness of f and E_s and $|\mathrm{e}^{\mathrm{i}kE_s}| = 1$, we have

$$|f\mathrm{e}^{\mathrm{i}kg} - f\mathrm{e}^{\mathrm{i}kP_s}| \le |f|\,|\mathrm{e}^{\mathrm{i}kP_s}|\ \max k|E_s||\mathrm{e}^{\mathrm{i}kE_s}| \le CkO(h^{s+1})$$

and hence for the integrals

$$|I(f) - I^P(f)| \le h^2 \max |f\mathrm{e}^{\mathrm{i}kg} - f\mathrm{e}^{\mathrm{i}kP_s}| \le CkO(h^{s+3}).$$

This proves the lemma. $\square$

We remark that whenever $s \ge 1$ our algorithm will perform better than the standard quadrature method which requires $hk = O(1)$ i.e. $h \le 1/k$. However if $kO(h^{s+3}) = O(1)$ (e.g. when $s = 1, h = 0.1, k = 10^4$), our method will not have any accuracy, even if the error is independent of k. If the nonresonance condition is not met, refer to [13] for a different method.

In what follows, we shall consider two applications of (17) which make use of (18)

$$\frac{1}{k^\tau}|I(f) - \bar{I}^P(f)| \le O(h^2/k^{\tau+1}) + O(k^{1-\tau}h^{s+3}) \le O(h^2) \tag{19}$$

where $\tau = 1, 2$ and $s \ge 1$ — essentially we apply (16)-(17) to reformulated integrals.

3. Combination with product integration methods

For large k, we hope to reformulate (1) and (2) so that application of our quadrature method (17) will lead to error bounds that are independent of k, or even decaying as k increases.

3.1. Extension to one dimensional integrals

For the remaining presentation, it is necessary to assume that f is twice differentiable and that g is three-times differentiable, and that $g'(x) \ne 0$ for $x \in [0, h]$. Under these assumptions, it is possible to re-write (1) as

$$I(f) = \int_0^h \frac{f(x)}{g'(x)} g'(x) \mathrm{e}^{\mathrm{i}kg(x)} \mathrm{d}x \tag{20}$$

which can now be integrated by parts to give

$$I(f)=\frac{1}{\mathrm{i}k}\left[\frac{f(x)}{g'(x)}\mathrm{e}^{\mathrm{i}kg(x)}\right]_0^h-\frac{1}{\mathrm{i}k}\int_0^h\frac{g'(x)f'(x)-f(x)g''(x)}{(g'(x))^2}\mathrm{e}^{\mathrm{i}kg(x)}\mathrm{d}x. \tag{21}$$

The integral appearing in (21) contains a factor of $1/k$ and so will decrease in magnitude as k increases, thus helping to reduce the problems encountered at higher frequencies. The integral appearing in (21) can be evaluated using 0-1DHF quadrature rule described above. This second quadrature scheme will be referred to as the first order one-dimensional high-frequency (1-1DHF) quadrature rule.

If we repeat the above process and integrate by parts again, we obtain

$$\begin{aligned}I(f)=&\left[\left(\frac{f(x)}{\mathrm{i}kg'(x)}-\frac{g'(x)f'(x)-f(x)g''(x)}{\mathrm{i}^2k^2(g'(x))^3}\right)\mathrm{e}^{\mathrm{i}kg(x)}\right]_0^h\\&+\int_0^h\left(\frac{f''(x)(g'(x))^2-f(x)g'(x)g'''(x)-3f'(x)g''(x)g'(x)+3f(x)(g''(x))^2}{(\mathrm{i}k)^2(g'(x))^4}\right)\mathrm{e}^{\mathrm{i}kg(x)}\mathrm{d}x\end{aligned} \tag{22}$$

where the final integral appearing in (22) can be evaluated using the 0-1DHF quadrature rule. Here this final integral contains a $1/k^2$ factor, and so will decrease in magnitude more quickly than the first order 1-1DHF described above. The scheme will be referred to as the second order one-dimensional high-frequency (2-1DHF) quadrature rule.

It is worth noting at this point that it is possible to integrate by parts again, which will introduce a more pleasing $1/k^3$ factor in the final integral which needs to be evaluated. However, it will also require both f and g to be more times differentiable. In addition, as the numerical results below show, the accuracy of the rules described here is sufficient for most purposes, and so in a practical situation there is little to be gained by developing such rules, unless one is interested in theoretical analysis which has of course been done in [12,7]. We also note that although the integrands in (22) are a lot more complicated than those in the original integral (1), the results of applying a quadrature rule to (22) are a lot more accurate than applying the rule directly to (1), as shown by our numerical results given below.

3.2. Extension to two dimensional integrals.

Assume that all the partial derivatives of order 2 or less of f and all the partial derivatives of order 3 or less of g exist and are continuous, and further assume that neither of the first order partial derivatives of g are zero in the domain of integration. Re-write (2) as

$$I(f)=\int_0^h\int_0^h\frac{f}{\mathrm{i}kg_y}ikg_y\mathrm{e}^{\mathrm{i}kg}\mathrm{d}y\mathrm{d}x. \tag{23}$$

Integrating (23) with respect to y using the parts rule gives

$$I(f)=\int_0^h\left[\frac{f}{\mathrm{i}kg_y}\mathrm{e}^{\mathrm{i}kg}\right]_{y=0}^{y=h}\mathrm{d}x-\int_0^h\int_0^h\left(\frac{1}{\mathrm{i}k}\frac{f_yg_y-fg_{yy}}{g_y^2}\right)\mathrm{e}^{\mathrm{i}kg}\mathrm{d}y\mathrm{d}x. \tag{24}$$

Rewrite the first term in (24) as

$$\int_0^h\left[\frac{\mathrm{i}kg_xf}{(ik)^2g_xg_y}\mathrm{e}^{\mathrm{i}kg}\right]_{y=0}^{y=h}\mathrm{d}x \tag{25}$$

and integrate by parts to get

$$\begin{aligned}&\left.\left(\frac{f\mathrm{e}^{\mathrm{i}kg}}{(ik)^2g_xg_y}\right)\right|_{x=h,y=h}-\left.\left(\frac{f\mathrm{e}^{\mathrm{i}kg}}{(ik)^2g_xg_y}\right)\right|_{x=0,y=h}-\left.\left(\frac{f\mathrm{e}^{\mathrm{i}kg}}{(ik)^2g_xg_y}\right)\right|_{x=h,y=0}+\left.\left(\frac{f\mathrm{e}^{\mathrm{i}kg}}{(ik)^2g_xg_y}\right)\right|_{x=0,y=0}\\&\quad-\int_0^h\frac{1}{(ik)^2}\left[\left(\frac{g_{xy}g_xf+g_{xx}g_yf-g_yg_xf_x}{g_x^2g_y^2}\right)\mathrm{e}^{\mathrm{i}kg}\right]_{y=0}^{y=h}\mathrm{d}x.\end{aligned} \tag{26}$$

Substituting (26) into (24) yields

$$\begin{aligned}I(f)=&\left.\left(\frac{f\mathrm{e}^{\mathrm{i}kg}}{(ik)^2g_xg_y}\right)\right|_{x=h,y=h}-\left.\left(\frac{f\mathrm{e}^{\mathrm{i}kg}}{(ik)^2g_xg_y}\right)\right|_{x=0,y=h}-\left.\left(\frac{f\mathrm{e}^{\mathrm{i}kg}}{(ik)^2g_xg_y}\right)\right|_{x=h,y=0}+\left.\left(\frac{f\mathrm{e}^{\mathrm{i}kg}}{(ik)^2g_xg_y}\right)\right|_{x=0,y=0}\\&-\int_0^h\frac{1}{(ik)^2}\left[\left(\frac{g_{xy}g_xf+g_{xx}g_yf-g_yg_xf_x}{g_x^2g_y^2}\right)\mathrm{e}^{\mathrm{i}kg}\right]_{y=0}^{y=h}\mathrm{d}x-\int_0^h\int_0^h\frac{1}{\mathrm{i}k}\frac{f_yg_y-fg_{yy}}{g_y^2}\mathrm{d}y\mathrm{d}x\end{aligned} \tag{27}$$

where the integrals appearing in (27) can be evaluated using the 0-1DHF rule or 0-2DHF rule as appropriate. This rule will be referred to as the first order (1-2DHF) quadrature rule. It is noted that all the remaining integrals appearing in (27) contain factors of $1/k$ or higher.

As in the one-dimensional case, it is possible to get a factor of at least $1/k^2$ by integrating by parts twice. This yields

$$\begin{aligned} I(f) = & \left(\frac{f}{(\mathrm{i}k)^2 g_x^2 g_y^2} - \frac{f_y g_y - g_{yy} f}{(\mathrm{i}k)^3 g_y^3 g_x}\right) \mathrm{e}^{\mathrm{i}kg}\Bigg|_{x=h,y=h} - \left(\frac{f}{(\mathrm{i}k)^2 g_x^2 g_y^2} - \frac{f_y g_y - g_{yy} f}{(\mathrm{i}k)^3 g_y^3 g_x}\right) \mathrm{e}^{\mathrm{i}kg}\Bigg|_{x=h,y=0} \\ & - \left(\frac{f}{(\mathrm{i}k)^2 g_x^2 g_y^2} - \frac{f_y g_y - g_{yy} f}{(\mathrm{i}k)^3 g_y^3 g_x}\right) \mathrm{e}^{\mathrm{i}kg}\Bigg|_{x=0,y=h} + \left(\frac{f}{(\mathrm{i}k)^2 g_x^2 g_y^2} - \frac{f_y g_y - g_{yy} f}{(\mathrm{i}k)^3 g_y^3 g_x}\right) \mathrm{e}^{\mathrm{i}kg}\Bigg|_{x=0,y=0} \\ & - \int_0^h \left[\left(\frac{g_x g_y f_x - g_{xx} g_y f - g_{xy} g_x f}{(\mathrm{i}k)^2 g_x^2 g_y^2} - \frac{w_1}{(\mathrm{i}k)^3 g_x^2 g_y^4}\right) \mathrm{e}^{\mathrm{i}kg}\right]_{y=0}^{y=h} \mathrm{d}x + \int_0^h \int_0^h \left(\frac{w_2}{(\mathrm{i}k)^2 g_y^4}\right) \mathrm{e}^{\mathrm{i}kg} \mathrm{d}x\mathrm{d}y \end{aligned} \tag{28}$$

where

$$w_1 = -2g_{xy}g_xg_yf_y + g_xg_y^3f_{xy} + 3g_{yy}g_{xy}g_xf - g_{yy}g_xg_yf_x - g_{yyx}g_xg_yf - g_{xx}g_y^2f_y + g_{xx}g_{yy}g_yf$$

$$w_2 = -3g_{yy}g_yf_y + g_y^2f_{yy} + 3g_{yy}^2f - g_{yyy}g_yf$$

and the remaining integrals can be evaluated using the 0-1DHF or 0-2DHF rules as appropriate. Clearly all the remaining integrals contain factors of $1/k^2$.

As remarked in the one-dimensional case, it is possible to apply this procedure again and get additional powers of $1/k$. However, this would require more restrictions on the continuity of the derivatives of f and g, and lead to even more complicated expressions for the various terms in the quadrature rules.

Moreover, we can use Lemma 1 to demonstrate that further integration by parts is not necessary. Indeed, as our reformulated integrals have factors $1/k$ (1-2DHF, $\tau = 1$) and $1/k^2$ (2-2DHF, $\tau = 2$), the new errors with $s = 1$ will be bounded as in (19) i.e.

$$\frac{1}{k^\tau}|I(f) - \bar{I}^p(f)| \le O(h^2/k^{\tau+1}) + O(k^{1-\tau}h^4) \le O(h^2) \tag{29}$$

which, for a fixed h, actually becomes more accurate as k increases.

4. Numerical results

Below we shall present numerical results for two purposes: firstly to show that our proposed quadrature method (16) and (17) applied to the reformulated integrals is much more efficient than the naive method of Gaussian quadrature with many nodes, and secondly to show that the above simple error bound (29) is reliable. Thus the new results here represent major improvements over our preliminary results reported in [2] where the 0-2DHF rule was used to directly evaluate (2).

The test problems considered here are (one-dimensional)

$$I(f) = \int_0^1 x\sqrt{1+x}\,\mathrm{e}^{\mathrm{i}k\sqrt{2+2x+x^2}}\mathrm{d}x \tag{30}$$

$$I(f) = \int_0^1 \left(x^4 + x^2 + 1\right) \mathrm{e}^{\mathrm{i}k(4x^4+5x^3+6x^2+7x+8)}\mathrm{d}x \tag{31}$$

$$I(f) = \int_0^1 \frac{1}{x+1}\mathrm{e}^{\mathrm{i}k/(x^2+2x+2)}\mathrm{d}x \tag{32}$$

and (two-dimensional)

$$I(f) = \int_0^1\int_0^1 \sqrt{1+x^2+y^2}\,\mathrm{e}^{\mathrm{i}k\sqrt{2+2x+2y+x^2+y^2}}\mathrm{d}y\mathrm{d}x \tag{33}$$

$$I(f) = \int_0^1\int_0^1 (x+y+1)^4\,\mathrm{e}^{\mathrm{i}k(x+y+2)^3/128}\mathrm{d}x\mathrm{d}y \tag{34}$$

$$I(f) = \int_0^1\int_0^1 \frac{1}{x+y+1}\mathrm{e}^{\mathrm{i}k/(x+y+2)}\mathrm{d}x\mathrm{d}y. \tag{35}$$

Each of the rules developed in Section 2 will be used to evaluate (30), along with a compound Gauss rule with the same number of nodes, to give a comparison with a naive use of standard quadrature rule for evaluating such integrals. The exact answer was obtained by using a very high order compound Gauss rule to evaluate the integral, where the relative error is less than 2×10^{-10}%.

Table 1
The relative error (%) in evaluating (30) using various quadrature rules.

k	0-1DHF	1-1DHF	2-1DHF	GAUSS
100	4.42E−01	2.43E−02	2.44E−04	2.40E+03
200	5.25E+00	5.46E−02	2.70E−05	8.59E+02
300	6.19E−01	4.00E−02	5.59E−05	2.35E+03
400	2.99E+00	1.98E−02	2.12E−05	5.51E+03
500	1.42E+00	1.76E−02	9.10E−06	3.65E+03
600	3.32E+00	2.19E−02	7.51E−06	8.01E+03
700	1.03E+00	1.14E−02	9.04E−06	1.59E+04
800	1.54E+00	1.01E−02	6.47E−06	1.38E+04
900	1.79E+00	4.78E−03	8.63E−07	1.00E+04
1000	2.81E+00	7.63E−03	4.00E−06	1.89E+04

Table 2
The relative error (%) in evaluating each of the one-dimensional test integrals using the 2-1DHF quadrature rule.

k	Example 1 (30)	Example 2 (31)	Example 3 (32)
100	2.44E−03	2.02E−04	9.95E−03
200	2.71E−05	4.22E−05	6.33E−03
300	5.59E−05	9.45E−06	7.78E−03
400	2.12E−05	1.06E−05	6.04E−03
500	9.11E−06	7.01E−06	1.64E−03
600	7.51E−06	1.37E−05	1.85E−03
700	9.04E−06	4.59E−06	6.01E−04
800	6.47E−06	2.38E−06	6.00E−04
900	8.63E−07	1.66E−06	1.97E−03
1000	4.00E−06	2.05E−06	1.54E−04

The relative errors (expressed as a percentage) in using the various rules to evaluate (30), where a 10 point 0-1DHF rule is used to work out the various final integrals, for different values of k, are given in Table 1. The column headed "Gauss" is where a 10 point compound Gauss rule has been used. The table clearly shows that using a simple compound Gauss rule of the same order yields totally unacceptable results. In the course of our numerical experiments we found that we would need to use at least 1000 times as many quadrature points as used here, in order to get a similar level of accuracy with a compound Gauss rule.

To show that the rule can be applied to a wide class of integrands, the 2-1DHF was also used to evaluate (31) and (32), and the results, expressed as a percentage relative error, are given in Table 2. For (30) and (31) the calculated errors are of the same magnitude, but in the case of (32) they are larger but still within acceptable limits, bearing in mind that there are only 10 quadrature points. These results show that the 2-1DHF rule is capable of accurately evaluating integrals of the form (1) for a wide range of functions f and g.

The results for evaluating (33) using each of the two-dimensional quadrature rules are given in Table 3. Similar to the one-dimensional cases above, the exact values have been found by using a very high-order compound Gauss rule, although in this case there is no comparison with a product Gauss rule of the same order. Here the square was divided into a 11×11 grid of points (with $h = 10^{-1}$) giving a total of 121 quadrature points. The results in Table 4 are for using a 101×101 grid of points (with $h = 10^{-2}$) giving a total of 10 201 quadrature points.

Table 5 gives a comparison of relative error when using the 2-2DHF rule to evaluate each of the two dimensional test integrals with $h = 10^{-2}$. As in the one-dimensional case discussed above, these results show that the rule gives similar levels of accuracy for a range of different integrands.

As expected, Tables 1, 3 and 4 confirm that the second order rules give the best accuracy, followed by the first order rules and then the zero order rules. Further in the case of the first and second order rules, increasing the value of k does not lead to a significant increase in the relative error.

To explain the above performance, we now illustrate the results of Table 1 in Fig. 1a and b, where we compare the scaled errors of the 0-1DHF against the errors 1-1DHF (by dividing the 0-1DHF errors by k) and 2-1DHF (by dividing the 0-1DHF errors by k^2). Clearly the curves match for a range of wavenumbers, indicating that our error bound (29) can predict the error behavior.

Remarks. In separate tests, we have found that the Levin type integration method [14–16] is comparable to our method in 1D; however in 2D, we believe our method is more efficient since we do not have to re-compute the weight for each element. A full comparison remains to be done.

5. Conclusions

A hybrid quadrature method of product integration combined with numerical approximation is proposed for evaluating a class of highly oscillatory integrals that arise from a number of practical applications. Analysis of the basic zeroth order rule can be used to predict the error behaviour of the first order and the second order rules. For the same mesh size, increasing

Table 3
The relative error (%) in evaluating (33) using our quadrature rules with $h = 10^{-1}$.

k	0-2DHF	1-2DHF	2-2DHF
100	3.77E+01	3.86E−02	4.54E−02
200	5.03E+01	2.94E−02	1.20E−02
300	3.23E+01	6.63E−02	8.52E−01
400	6.65E+01	5.78E−02	1.01E−02
500	3.85E+01	6.21E−02	1.09E−02
600	2.62E+01	4.61E−02	8.72E−04
700	2.02E+02	1.05E−01	7.69E−03
800	6.70E+02	5.63E−02	6.21E−03
900	2.12E+02	3.93E−02	2.85E−03
1000	4.06E+01	4.92E−02	1.35E−03

Table 4
The relative error (%) in evaluating (33) using our quadrature rules with $h = 10^{-2}$.

k	0-2DHF	1-2DHF	2-2DHF
100	4.22E−02	2.65E−04	1.21E−04
200	1.16E−01	5.97E−04	2.53E−04
300	1.67E−01	5.25E−04	1.50E−04
400	2.84E−01	4.87E−04	1.25E−04
500	4.69E−01	1.05E−03	2.07E−04
600	7.42E−01	6.94E−04	6.15E−05
700	9.22E+00	8.39E−03	8.17E−04
800	1.59E+01	9.24E−03	5.75E−04
900	2.42E+02	3.68E−02	9.99E−04
1000	1.24E+01	5.95E−04	3.72E−04

Table 5
The relative error (%) in evaluating each of the two-dimensional test integrals using the 2-2DHF quadrature rule with $h = 10^{-2}$.

k	Example 4 (33)	Example 5 (34)	Example 6 (35)
100	1.21E−04	3.86E−04	4.11E−04
200	2.53E−04	4.68E−04	4.03E−04
300	1.50E−04	2.36E−04	5.67E−05
400	1.25E−04	3.91E−04	4.21E−04
500	2.07E−03	1.42E−04	3.93E−04
600	6.15E−04	2.74E−04	1.43E−04
700	8.17E−04	5.22E−04	4.33E−04
800	5.75E−04	1.57E−04	3.83E−04
900	9.99E−04	3.28E−04	2.59E−04
1000	3.72E−04	1.42E−03	4.48E−04

the wavenumber leads to reduction of the quadrature errors, which is in agreement with analytical studies in the recent literature. Numerical results are presented to illustrate the effectiveness of the proposed method. Future work will include possible applications to boundary element methods for solving acoustic radiation and scattering problems.

Appendix. Evaluation of the integrals in (12)

We now give some details of computing the integrals in (12), which are used to find the approximate value of (2). For simplicity, the explicit dependence of the weights on the function g has been omitted. Let $\tilde{w}_1 = J_j\hat{w}_1$, $\tilde{w}_2 = J_j\hat{w}_2$ and $\tilde{w}_3 = J_j\hat{w}_3$ where

$$\hat{w}_1 = \int_0^1 \int_0^{1-t} (1-s-t)\mathrm{e}^{\mathrm{i}k[g_1+(g_2-g_1)s+(g_3-g_1)t]}\mathrm{d}s\mathrm{d}t$$
$$\hat{w}_2 = \int_0^1 \int_0^{1-t} s\mathrm{e}^{\mathrm{i}k[g_1+(g_2-g_1)s+(g_3-g_1)t]}\mathrm{d}s\mathrm{d}t$$
$$\hat{w}_3 = \int_0^1 \int_0^{1-t} t\mathrm{e}^{\mathrm{i}k[g_1+(g_2-g_1)s+(g_3-g_1)t]}\mathrm{d}s\mathrm{d}t.$$

The formulae for evaluating these integrals in each of the five cases, depending on the relative values of g_1, g_2 and g_3, are given below.

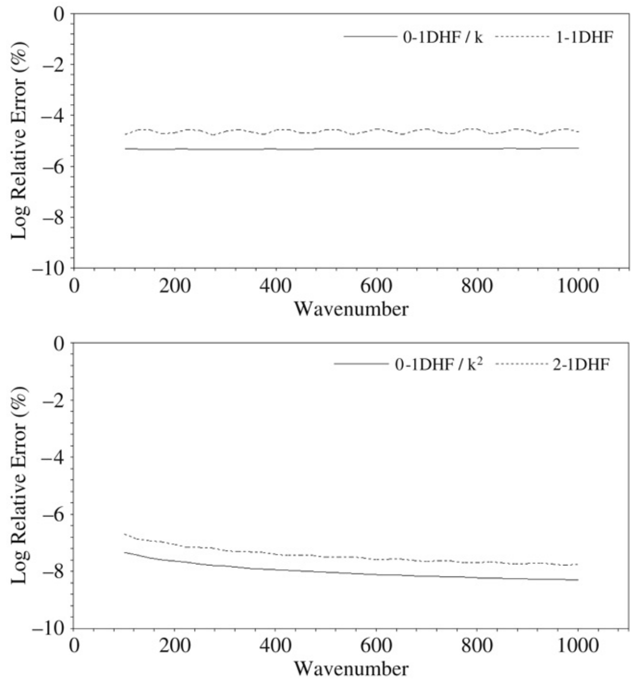

Fig. 1. Comparison of error behaviour of the new quadrature methods. Top (a): errors 0-1DHF/k versus 1-1DHF and Bottom (b): 0-1DHF/k^2 versus 2-1DHF.

If $g_1 = g_2 = g_3$ then

$$\hat{w}_1 = \hat{w}_2 = \hat{w}_3 = \frac{\mathrm{e}^{\mathrm{i}kg_1}}{6}.$$

If $g_1 = g_2$ and g_3 is different then

$$\hat{w}_1 = \frac{-\mathrm{i}\left(2\,\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_1\,k - 2\,\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_3\,k + \mathrm{e}^{\mathrm{i}kg_1}k^2{g_1}^2 - 2\,\mathrm{e}^{\mathrm{i}kg_1}k^2 g_1\,g_3 + \mathrm{e}^{\mathrm{i}kg_1}k^2{g_3}^2 - 2\,\mathrm{e}^{\mathrm{i}kg_1} + 2\,\mathrm{e}^{\mathrm{i}kg_3}\right)}{2\,(g_1 - g_3)^3\,k^3}$$

$$\hat{w}_2 = \frac{-\mathrm{i}\left(2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_1\,k - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_3\,k + \mathrm{e}^{\mathrm{i}kg_1}k^2{g_1}^2 - 2\,\mathrm{e}^{\mathrm{i}kg_1}k^2 g_1\,g_3 + \mathrm{e}^{\mathrm{i}kg_1}k^2{g_3}^2 - 2\,\mathrm{e}^{\mathrm{i}kg_1} + 2\,\mathrm{e}^{\mathrm{i}kg_3}\right)}{2\,(g_1 - g_3)^3\,k^3}$$

$$\hat{w}_3 = \frac{-\mathrm{e}^{\mathrm{i}kg_1}kg_1 + \mathrm{e}^{\mathrm{i}kg_1}kg_3 - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1} + 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_3} - \mathrm{e}^{\mathrm{i}kg_3}kg_1 + \mathrm{e}^{\mathrm{i}kg_3}kg_3}{(g_1 - g_3)^3\,k^3}.$$

If $g_1 = g_3$ and g_2 is different then

$$\hat{w}_1 = \frac{-\mathrm{i}\left(-2\,\mathrm{e}^{\mathrm{i}kg_2} - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_1\,k + 2\,\mathrm{e}^{\mathrm{i}kg_1} + 2\,\mathrm{e}^{\mathrm{i}kg_1}g_1\,g_2\,k^2 - \mathrm{e}^{\mathrm{i}kg_1}{g_2}^2k^2 - \mathrm{e}^{\mathrm{i}kg_1}{g_1}^2k^2 + 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_2\,k\right)}{2\,(-g_1 + g_2)^3\,k^3}$$

$$\hat{w}_2 = -\frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_2} + kg_2\,\mathrm{e}^{\mathrm{i}kg_2} - \mathrm{e}^{\mathrm{i}kg_2}kg_1 - \mathrm{e}^{\mathrm{i}kg_1}kg_1 - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1} + \mathrm{e}^{\mathrm{i}kg_1}kg_2}{(-g_1 + g_2)^3\,k^3}$$

$$\hat{w}_3 = \frac{\mathrm{i}\left(2\,\mathrm{e}^{\mathrm{i}kg_2} + 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_1\,k - 2\,\mathrm{e}^{\mathrm{i}kg_1} - 2\,\mathrm{e}^{\mathrm{i}kg_1}g_1\,g_2\,k^2 + \mathrm{e}^{\mathrm{i}kg_1}{g_2}^2k^2 + \mathrm{e}^{\mathrm{i}kg_1}{g_1}^2k^2 - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_2\,k\right)}{2k^3\,(-g_1 + g_2)^3}.$$

If $g_2 = g_3$ and g_1 is different then

$$\hat{w}_1 = -\frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1} - \mathrm{e}^{\mathrm{i}kg_1}g_2\,k + kg_1\,\mathrm{e}^{\mathrm{i}kg_1} - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_2} + \mathrm{e}^{\mathrm{i}kg_2}kg_1 - \mathrm{e}^{\mathrm{i}kg_2}kg_2}{k^3\,(g_1 - g_2)^3}$$

$$\hat{w}_2 = \frac{\mathrm{i}\left(2\,\mathrm{e}^{\mathrm{i}kg_1} - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}kg_1 + \mathrm{e}^{\mathrm{i}kg_2}k^2{g_1}^2 + \mathrm{e}^{\mathrm{i}kg_2}k^2{g_2}^2 - 2\,\mathrm{e}^{\mathrm{i}kg_2} + 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}kg_2 - 2\,\mathrm{e}^{\mathrm{i}kg_2}k^2 g_1\,g_2\right)}{2k^3\,(g_1 - g_2)^3}$$

$$\hat{w}_3 = \frac{\mathrm{i}\left(2\,\mathrm{e}^{\mathrm{i}kg_1} - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}kg_1 + \mathrm{e}^{\mathrm{i}kg_2}k^2{g_1}^2 + \mathrm{e}^{\mathrm{i}kg_2}k^2{g_2}^2 - 2\,\mathrm{e}^{\mathrm{i}kg_2} + 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}kg_2 - 2\,\mathrm{e}^{\mathrm{i}kg_2}k^2 g_1\,g_2\right)}{2k^3\,(g_1 - g_2)^3}.$$

Otherwise (g_1, g_2 and g_3 all different)

$$\begin{aligned}
\hat{w}_1 \;=\; & [-\mathrm{i}(-\mathrm{e}^{\mathrm{i}kg_2}{g_3}^2 - \mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_1k{g_3}^2 - \mathrm{i}k{g_1}^2\mathrm{e}^{\mathrm{i}kg_1}g_2 + \mathrm{e}^{\mathrm{i}kg_3}{g_2}^2 + \mathrm{i}k{g_1}^2\mathrm{e}^{\mathrm{i}kg_1}g_3 - \mathrm{e}^{\mathrm{i}kg_2}{g_1}^2 \\
& + \mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_2k{g_3}^2 + 2\,\mathrm{e}^{\mathrm{i}kg_1}g_1g_2 - 2\,\mathrm{e}^{\mathrm{i}kg_1}g_3g_1 + 2\,\mathrm{e}^{\mathrm{i}kg_2}g_3g_1 + \mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}{g_2}^2kg_1 \\
& - 2\,\mathrm{e}^{\mathrm{i}kg_3}g_1g_2 + \mathrm{e}^{\mathrm{i}kg_3}{g_1}^2 + \mathrm{e}^{\mathrm{i}kg_1}{g_3}^2 - \mathrm{e}^{\mathrm{i}kg_1}{g_2}^2 - \mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}{g_2}^2kg_3)]/[(-g_3+g_2)\,(-g_1+g_2)^2\,(-g_3+g_1)^2\,k^3] \\
\hat{w}_2 \;=\; & [\mathrm{i}(2\,g_2\mathrm{e}^{\mathrm{i}kg_2}g_1 - \mathrm{i}{g_2}^2\mathrm{e}^{\mathrm{i}kg_2}kg_1 - 2\,\mathrm{e}^{\mathrm{i}kg_3}g_1g_2 - \mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}kg_2{g_3}^2 + \mathrm{i}{g_2}^2\mathrm{e}^{\mathrm{i}kg_2}kg_3 + \mathrm{e}^{\mathrm{i}kg_3}{g_2}^2 \\
& - \mathrm{e}^{\mathrm{i}kg_1}{g_3}^2 - \mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}{g_1}^2kg_3 + \mathrm{e}^{\mathrm{i}kg_2}{g_3}^2 + \mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}{g_1}^2kg_2 \\
& + \mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}kg_1{g_3}^2 + \mathrm{e}^{\mathrm{i}kg_3}{g_1}^2 - 2\,g_2\mathrm{e}^{\mathrm{i}kg_2}g_3 + 2\,\mathrm{e}^{\mathrm{i}kg_1}g_3g_2 - \mathrm{e}^{\mathrm{i}kg_2}{g_1}^2 - \mathrm{e}^{\mathrm{i}kg_1}{g_2}^2)]/[(g_3-g_1)(g_3-g_2)^2(g_1-g_2)^2k^3] \\
\hat{w}_3 \;=\; & [\mathrm{e}^{\mathrm{i}kg_3}{g_3}^2g_2k + \mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}{g_3}^2 + \mathrm{i}\mathrm{e}^{\mathrm{i}kg_3}{g_1}^2 - \mathrm{e}^{\mathrm{i}kg_3}g_1{g_3}^2k + 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_3}g_3g_2 - \mathrm{e}^{\mathrm{i}kg_3}g_3{g_2}^2k \\
& - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_3}g_3g_1 + \mathrm{e}^{\mathrm{i}kg_3}g_3{g_1}^2k - \mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}{g_1}^2 - \mathrm{i}\mathrm{e}^{\mathrm{i}kg_3}{g_2}^2 + \mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}{g_2}^2 - \mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}{g_3}^2 \\
& + 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_2}g_3g_1 - 2\mathrm{i}\mathrm{e}^{\mathrm{i}kg_1}g_3g_2 + \mathrm{e}^{\mathrm{i}kg_3}g_1{g_2}^2k - \mathrm{e}^{\mathrm{i}kg_3}g_2{g_1}^2k]/[(g_1-g_2)\,(-g_3+g_1)^2\,(-g_3+g_2)^2\,k^3].
\end{aligned}$$

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