New wavelet preconditioner for solving boundary integral equations over nonsmooth boundaries

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Abstract

Matrices obtained by wavelet discretisation of partial differential equations or boundary integral equations (BIEs) are typically sparse with a finger-like sparsity pattern, in contrast to matrices obtained by traditional single scale discretisation, which are dense. In some cases diagonal preconditioning is sufficient, but there are effective preconditioners that can be used when this is not the case. Here we consider the particular case of BIEs whose boundary has a geometrical singularity, and propose a new preconditioning strategy based on permutations of the unknowns. The strategy's performance is analysed and compared with related techniques for the double layer equation for Laplace's equation.

Keywords: Preconditioning, continuous wavelet discretisation, boundary integral equation, permutation, geometric singularity.

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1 Introduction

Many classes of important partial differential equations defined over the interior or exterior of a particular region can be reformulated into a boundary integral equation (BIE) defined over the boundary of the region. Examples include the Helmholtz equation from wave scattering and Laplace's equation from potential theory [8]. While the traditional boundary element method using a single scale basis leads to dense matrices, using a wavelet basis will lead to nearly sparse matrices [3, 7, 10, 14]. Although the traditional boundary element method may be combined with the effective (matrix-free) fast multipole method [13, 18], we concentrate here on the wavelet approach, particularly on the iterative solution of the resulting linear system and on the necessary preconditioning.

Solution of a BIE using a traditional single scale boundary element method produces a linear system with a dense matrix having n^2 nonzero entries, where n is the number of unknowns. In contrast, a wavelet method produces a sparse matrix with only $O(n \log^k n)$ nonzeros, with k a generic constant depending on the operator and wavelets used. Wavelet discretisation schemes and their convergence have been heavily studied. Beylkin, Coifman and Rokhlin [3] showed that the use of a wavelet basis, in a non-standard form, produces elegantly a sequence of matrices that have many small entries and whose entries decay away from the diagonal. Under a suitable threshold these matrices are sparse. Developments by Dahmen, Kleemann, Prössdorf and Schneider [10], Rathsfeld [16, 17], and Harbrecht and Schneider [14] yield sophisticated wavelet discretisation and compression methods with the key properties: (1) the matrix has $O(n \log^k n)$ nonzero entries, for some small integer k; (2) the asymptotic order of convergence of the method is not changed by the compression. Property (1) reduces the cost of matrix vector multiplication when solving the system iteratively. Also, the pattern for the nonzero entries is known in advance so the cost of generating the matrix can be reduced to $O(n \log^k n)$, compared with $O(n^2)$ for traditional single scale boundary element methods (without combining with a fast multipole method). Rathsfeld [17], in particular, presents families of biorthogonal trial and test wavelets for use in a Petrov-Galerkin scheme for the double layer BIE formulation for Laplace's equation on a polygonal boundary.

In this paper we are concerned with preconditioning the linear system obtained with a wavelet discretisation. Norm equivalence results for the wavelet basis mean that there exists a diagonal preconditioner for the linear system that yields a uniformly bounded condition number. This preconditioner would be sufficient in many cases but in practice the condition number can still be large, or irrelevant in the case of highly non-normal matrices because then fast convergence may not be related to good conditioning.

In previous work, involving discrete wavelet transforms, attention was given to preconditioning and various sparse preconditioners were proposed [7, 11, 12]. In particular, Ford and Chen [11] considered a matrix having a region of nonsmoothness. In this context we say, loosely speaking, that a matrix is smooth when nearby entries take similar values. That problem is similar to the one considered here but we propose a different approach and use continuous wavelets. We remark that Cohen and Masson [9] show that the inverse of many wavelet matrices also has a finger pattern, although computing the inverse is not trivial.

When the boundary has points at which it is not smooth, for example a corner, a strong singularity is produced in the kernel of the integral operator and it becomes necessary for the preconditioner to incorporate those entries *outside* the finger pattern that are produced by the presence of the corner. The theoretical basis comes from the operator splitting as shown in [5, 6]. We focus on the construction of a sparse preconditioner that incorporates the off diagonal fingers and the entries due to the corner by manipulating the sparsity pattern of the matrix using permutations.

In Section 2 we state the double layer BIE formulation for Laplace's equation and discuss a Petrov-Galerkin solution technique. In Section 3 we briefly describe the biorthogonal wavelet bases used in the Petrov-Galerkin method and then the compression strategy applied to the matrix. In Section 4 we describe the preconditioner, whose effectiveness for boundaries with a corner is demonstrated in Section 5.

2 A Boundary Integral Formulation for Laplace's Equation

We briefly state the double layer equation, which is our model problem and arises as a BIE formulation for Laplace's equation [2, 15]. Our method will be applicable to other BIE formulations and equations with a geometric singularity, and generalisations to three dimensional problems are in progress.

Let $D \subseteq \mathbb{R}^2$ be a bounded domain with boundary curve ∂D. Consider Laplace's equation

$$\nabla^2 u(p) = 0, \quad p \in D \tag{1}$$

with Dirichlet boundary condition

$$u(p) = -\frac{1}{2} g(p), \quad p \in \partial D.$$

Then (1) can be reformulated [15]

$$\rho(p) - \int_{\partial D} K(p,q)\rho(q)dq = g(p), \quad p \in \partial D, \tag{2}$$

where

$$u(p) = \int_{\partial D} K(p,q)\rho(q)dq$$

with $K(p,q) = \frac{1}{\pi}\frac{\partial}{\partial n_q} G(p,q)$ where $G(p,q)$ is the Green's function for (1), and n_q is the unit outward normal to ∂D at q.

Let $\gamma : \mathbb{R} \to \partial D$ be a periodic mapping with period 1. Then, abusing our previous notation, we can write (2) in the form

$$\rho(t) - \int_{[0,1]} K(t,s)\rho(s)ds = g(t), \quad t \in [0,1]. \tag{3}$$

For brevity denote by A the operator on the left hand side of (3). We seek a solution of the weak form of (3), that is, we seek $\rho \in L_2([0,1])$ so that

$$\langle A\rho, v\rangle = \langle g, v\rangle, \quad \forall v \in L_2([0,1]), \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product $\langle u, v\rangle = \int_{[0,1]} u(s)\overline{v(s)}ds$.

The Petrov-Galerkin solution is obtained by replacing the trial space $L_2([0,1])$ by some ${}^{A}V_h \subseteq L_2([0,1])$ and the test space $L_2([0,1])$ by some ${}^{T}V_h \subseteq L_2([0,1])$ in (4) to seek $\rho \in {}^{A}V_h$ so that

$$\langle A\rho, v\rangle = \langle g, v\rangle, \quad \forall v \in {}^{T}V_h. \tag{5}$$

Writing $A_h = \langle A\,{}^{A}\mathbf{V}, {}^{T}\mathbf{V}\rangle$, $g_h = \langle g, {}^{T}\mathbf{V}\rangle$ where ${}^{T}\mathbf{V}$ and ${}^{A}\mathbf{V}$ are bases for ${}^{T}V_h$ and ${}^{A}V_h$ respectively, yields the linear system

$$A_h \rho_h = g_h.$$

In general the matrix A_h will not be symmetric (or Hermitian in the complex case) because it is obtained using a Petrov-Galerkin method.

3 Wavelet Bases, Matrix Assembly and Compression

In this section we first describe the two wavelet bases used for the Petrov-Galerkin solution of our model equation (3) before discussing how to determine a priori the location of the nonzero entries. The compression strategies that we apply are hybrids of those of Rathsfeld [17] and Petersdorff and Schwab [22]. The trial wavelets have two vanishing moments, that is, they are orthogonal to all polynomials of degree less than 2. The test wavelets have four vanishing moments, that is, they are orthogonal to all polynomials of degree less than 4. First wavelets on $\mathbb{R}$ are constructed. These wavelets are then made periodic so that they are suitable for application to (3). The construction is described in detail in [1, 17].

Define the trial mother scaling function and wavelet respectively by

$$\begin{aligned} {}^A\phi(t) &= \max\{0, 1-|t|\} \\ {}^A\psi(t) &= \frac{1}{2}{}^A\phi(2t) - {}^A\phi(2t-1) + \frac{1}{2}{}^A\phi(2t-2). \end{aligned}$$

Then for level $j \geq 0$ and integer k we have

$${}^A\psi_k^j(t) = \begin{cases} 2^{j/2}\ {}^A\psi(2^j t - k), & j \geq 0, \\ {}^A\phi(t), & j = -1, \end{cases}$$

which defines the family of wavelets on $\mathbb{R}$. Periodic wavelets are derived by

$$\mathrm{Per}\ {}^A\psi_k^j(t) = \sum_{n\in\mathbb{Z}} {}^A\psi_k^j(t-n).$$

Denote by $\delta_0(t)$ the Dirac distribution, which has the property $\int f(t)\delta_0(t)dt = f(0)$. Define test mother scaling function and wavelet respectively by

$$\begin{aligned} {}^T\phi(t) &= \delta_0(t), \\ {}^T\psi(t) &= -\frac{1}{16}{}^T\phi(2t+1) + \frac{1}{4}{}^T\phi(2t) - \frac{3}{8}{}^T\phi(2t-1) \\ &\qquad + \frac{1}{4}{}^T\phi(2t-2) - \frac{1}{16}{}^T\phi(2t-3). \end{aligned}$$

Then for level $j \geq 0$ and integer k we have

$${}^T\psi_k^j(t) = \begin{cases} 2^{j/2}\ {}^T\psi(2^j t - k), & j \geq 0, \\ {}^T\phi(t), & j = -1. \end{cases}$$

Let $\Lambda_N = \{(j,k) \mid -1 \le j \le N, k = 0,\dots,\eta(j)\}$ for some $N > -1$, where $\eta(j) = 2^{\max\{0,j\}}$ is the number of basis functions at level j. The trial space and the test space respectively are defined by

$$^{A}V = \operatorname{span}\left\{\mathrm{Per}\ {}^{A}\psi_k^j \ : \ (j,k) \in \Lambda_N\right\}, \qquad {}^{T}V = \operatorname{span}\left\{{}^{T}\psi_k^j \ : \ (j,k) \in \Lambda_N\right\}.$$

We now consider practical aspects of constructing the Petrov-Galerkin matrix representing the BIE operator A with respect to the wavelet basis. In particular we describe how to efficiently compute the compressed matrix by computing a priori where the nonzero entries will be. Different compression strategies are used for the two boundaries considered.

Let A_Λ be the Petrov Galerkin matrix for A obtained from (5) with ${}^{T}V_h = {}^{T}V$ and ${}^{A}V_h = {}^{A}V$. This matrix is dense, though it has very many small entries that can be discarded. Consider the case where ∂D is an ellipse. We obtain the compressed matrix by applying

$$\left(\widetilde{A}_\Lambda\right)_{(j,k)(j',k')} = \begin{cases} 0, & \text{for } d_\gamma({}^{A}\psi_k^j, {}^{T}\psi_{k'}^{j'}) > B_{jj'}, \\ (A_\Lambda)_{(j,k)(j',k')}, & \text{otherwise}, \end{cases} \tag{6}$$

where

$$d_\gamma({}^{A}\psi_k^j, {}^{T}\psi_{k'}^{j'}) = \min_{\substack{p \in \gamma(\mathrm{supp}\,{}^{A}\psi_k^j) \\ q \in \gamma(\mathrm{supp}\,{}^{T}\psi_{k'}^{j'})}} \|p - q\|$$

and $B_{jj'} = c\max\{2^{-j}, 2^{-j'}\}$ where c is a constant depending on ∂D. This is a simplification of the strategy of Petersdofff and Schwab [21]. Since (6) can be used to compute a priori which entries in A_Λ are to be discarded in $\widetilde{A}_\Lambda$, those entries neeed not be computed at all. This facilitates fast computation of $\widetilde{A}_\Lambda$.

Consider the case where ∂D is a tear drop, whose corner is at $\gamma(0)$. The corner must be considered in the compression strategy. We obtain the compressed matrix by applying

$$\left(\widetilde{A}_\Lambda\right)_{(j,k)(j',k')} = \begin{cases} 0, & \text{for } d({}^{A}\psi_k^j, {}^{T}\psi_{k'}^{j'}) > B_{jj'}^1 \text{ and } {}^{A}d({}^{T}\psi_{k'}^{j'}) > B^2, \\ (A_\Lambda)_{(j,k)(j',k')}, & \text{otherwise}, \end{cases} \tag{7}$$

where

$$\begin{aligned} d({}^{A}\psi_k^j, {}^{T}\psi_{k'}^{j'}) &= \min_{\substack{s \in \mathrm{supp}\,{}^{A}\psi_k^j \\ t \in \mathrm{supp}\,{}^{T}\psi_{k'}^{j'}}} d_{[0,1]}(s,t), \\ {}^{A}d({}^{T}\psi_{k'}^{j'}) &= \min_{t \in \mathrm{supp}\,{}^{T}\psi_{k'}^{j'}} d_{[0,1]}(t,0), \end{aligned}$$

with

$$d_{[0,1]}(s,t) = \min\{|s-t|, |s-t+1|, |s-t-1|\}.$$

Also $B^1_{jj'} = c^1 \max\{2^{-j}, 2^{-j'}\}$ and $B^2 = c^2$ where c^1 and c^2 are constants depending on ∂D. This is a simplification of the strategy of Rathsfeld [17].

4 The Preconditioner

In this section we describe an effective sparse preconditioner for the sparse matrix $\widetilde{A}_\Lambda$ arising from our wavelet discretisation. Note that in our previous studies where discrete wavelet transforms are used, obtaining the sparse matrix $\widetilde{A}_\Lambda$ is not a cheap process because a dense matrix has to be formed first. Here, as shown, the matrix $\widetilde{A}_\Lambda$ is computed directly by integrating wavelet basis functions in a cheaper procedure.

Our idea is to reorder the rows and columns of $\widetilde{A}_\Lambda$ so that most of its nonzero entries lie close to the main diagonal or close to the bottom or right hand side of the matrix. Then incorporating only those entries into the preconditioner allows the preconditioner to be effectively inverted by an LU factorisation with minimal fill in. This improves on previous work [7, 11] not only by using continuous wavelets, but also by taking into account the location of any geometric singularities (corner induced non-smooth features), and we expect the new preconditioner to outperform that described in [7].

In contrast to previous work where explicit matrix permutation is carried out, we simply re-order the wavelet basis functions. Observe that the wavelets exhibit a tree structure and that each wavelet has two *children*. Representing the family of wavelets by a tree and each wavelet by a node in the tree we seek to reorder the wavelets by ordering the nodes in the tree.

Let ∂D have a corner located at $\gamma(z)$ for $z \in [0,1]$. Let w be a wavelet and denote by c_w the centre of the support of w. Define a metric on the wavelets by $s_z(w) = d_{[0,1]}(c_w, z)$. If w_i and w_j are wavelets with the same parent and $s_z(w_i) \leq s_z(w_j)$, then we say $w_i \preceq w_j$. Under $\preceq$ the tree is organised by proximity to the corner. Define an ordering $\trianglelefteq$ of the wavelets by

1. if $w_i \preceq w_j$ then the descendants of $w_i \trianglelefteq$ the descendants of w_j.

2. if w_i is an ascendant of w_j then $w_i \trianglelefteq w_j$.

This is a preorder ordering of the tree [4]. The purpose of Step 1 is to implement the so-called centering idea as in [7], while step 2 takes into account proximity to the singularity.

We summarise our method to compute the new preconditioner as follows:

Algorithm 1

1. *Obtain an ordering* $i_1, i_2, \ldots, i_n$ *such that* $w_{i_j} \trianglelefteq w_{i_{j+1}}$*;*

2. *Order the wavelets using the ordering from Step 1 and compute* $\widetilde{A}_\Lambda$*;*

3. *The new preconditioner with an arrow type sparsity pattern is obtained from* $\widetilde{A}_\Lambda$ *by retaining the k-diagonal part of* $\widetilde{A}_\Lambda$*, augmented with the bottom b rows and right b columns of* $\widetilde{A}_\Lambda$*.*

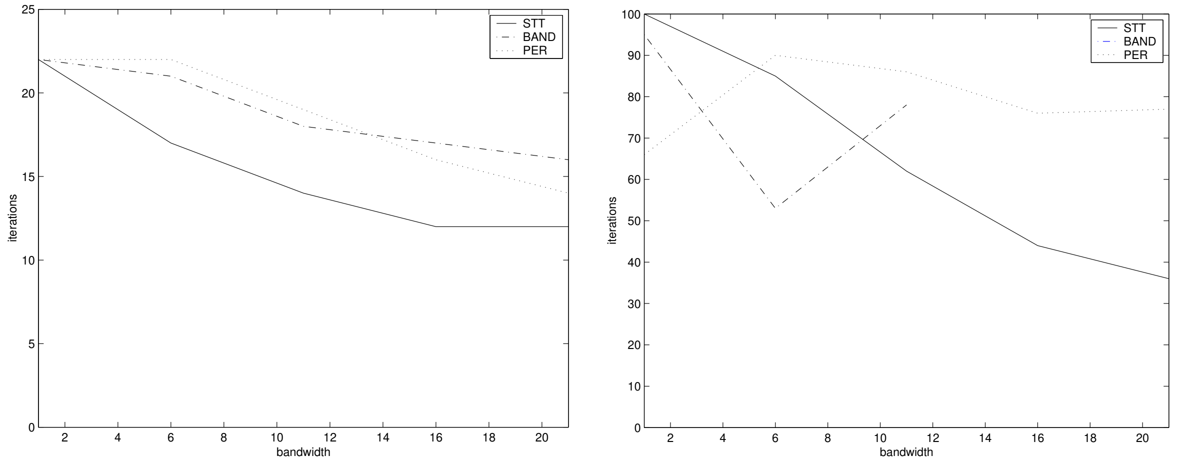
Remark 1 *The ordering is easily extended to the case where there are several corners. Replacing the corner* z *by a set of corners* Ω*, we define* $s_\Omega(w) = \min_{z\in\Omega} d_{[0,1]}(c_w, z)$*. The ordering follows on substituting* s_Ω *for* s_z *in the discussion above.*

5 Numerical Experiments

In this section we present results demonstrating the effectiveness of the new preconditioner for problems where the boundary possesses a corner. We compare the new preconditoner, which for convenience we refer to by STT, with banded preconditioners based on the Per ordering [7] and on no reordering. Two specific boundaries are tested here.

Firstly consider the case where ∂D is a tear drop shape with a corner at $\gamma(0)$ with internal angle $5\pi/9$. An adaptive wavelet discretisation [17], which uses high level wavelets only close to the corner is used.

Figure 1 (a) shows the number of preconditioned GMRES [19, 20] iterations required to reduce the residual norm by a factor of 10^{-8}. Figure 1 (b) shows the number of preconditioned CGNL [19] iterations required to reduce the residual norm by a factor of 10^{-8}. In both cases the number of iterations are presented as a function of bandwidth. The border width for the STT preconditioner is fixed at 45; smaller values were tested with slightly more iterations observed but the results presented are typical.



(a) (b)

Figure 1: Number of iterations plotted against bandwidth for the tear drop (a) GMRES (b) CGNL. The boundary for this problem possesses a corner.

For this problem, where the boundary possesses a corner, the STT preconditioner is more effective than the Per and band preconditioners.

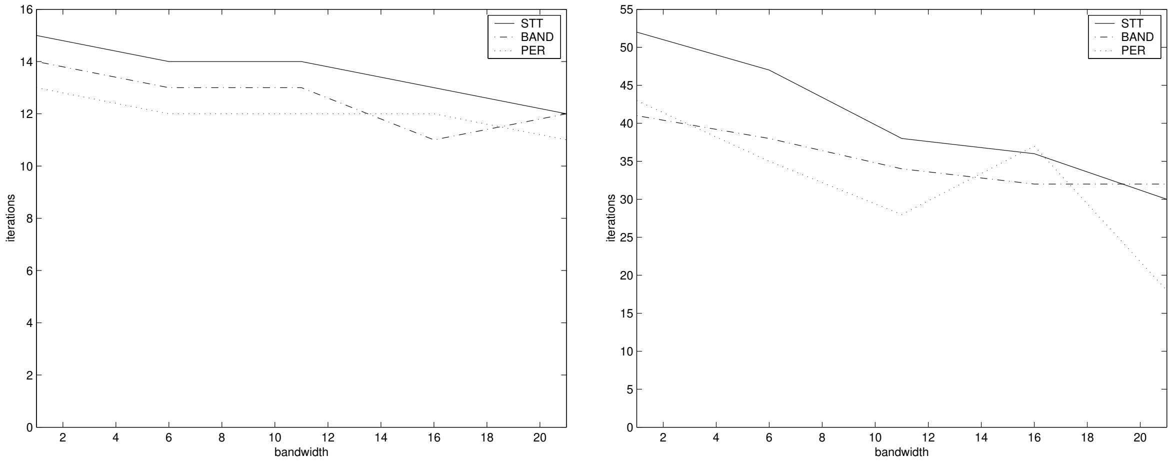
Secondly consider the case where ∂D is the ellipse $\{(x,y) \mid x^2/a^2+y^2/b^2=1\}$ where $a=1.1$ and $b=1$. This boundary is smooth everywhere.

Figure 2 (a) shows the number of preconditioned GMRES [19, 20] iterations required to reduce the residual norm by a factor of 10^{-8}. Figure 2 (b) shows the number of preconditioned CGNL [19] iterations required to reduce the residual norm by a factor of 10^{-8}. As before, in both cases the number of iterations are presented as a function of bandwidth, and the border width for the STT preconditioner is fixed at 45; smaller values were tested with slightly more iterations observed but the results presented are typical.

For this problem, where the boundary is smooth, the band and Per preconditioners are slightly more effective than the STT preconditioner but in all cases the performances are comparable.

6 Conclusions

We have introduced a new preconditioning strategy for finger pattern matrices arising from wavelet discretisation of BIEs with a geometric singularity caused by a corner in the bound-



(a) (b)

Figure 2: Number of iterations plotted against bandwidth for the ellipse (a) GMRES (b) CGNL. The boundary for this problem is smooth.

ary. The strategy is based on sparse approximation of the matrix after reordering the basis functions by proximity to the corner. The new preconditioner is shown numerically to be more effective than the Per [7, 11] and band preconditioners. For a problem where the boundary is smooth, the preconditioner is shown numerically to be comparable to the Per [7, 11] and band preconditioners.

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