An analysis of Seydel's test function methods for nonlinear power flow equations

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Abstract

One of the important planning and operations problems in electrical power systems is to determine the distance of a loading parameter from the voltage collapse point — the so-called performance index. Such an index indicates the severity of a power network with regard to voltage collapse.

This paper investigates a class of test function methods due to Seydel (1979) that may be adapted to produce a performance index. Experiments have shown that an arbitrary choice of the parameters in the method leads to unreliable indexes. We first present a result that gives an equivalent formula, involving determinants, for the Seydel test function. We then analyse the behaviour of such determinants in order to determine what parameters are suitable to produce reliable indexes. For typical power systems with reactive power changes, it turns out that only a restricted set of parameters can ensure a reliable index by the Seydel method; for other cases, a general modification is proposed. Promising numerical results using several standard IEEE test examples are presented.

Keywords: Nonlinear systems, Test functions, Bifurcation, Turning points, Performance index, Electrical power systems.

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1 Introduction

An electrical power system can be modelled in nodal analysis by a system of ordinary differential equations. The power network is usually runing at an equilibrium and a planned or an unexpected change in loading demand can set the system to a new equilibrium. Voltage collapse in a power network is a real world problem that can occur in association with bifurcation of the underlying nonlinear power equations, particularly saddle-node bifurcations or fold bifurcations [2, 3, 6, 10, 24, 9]. We concentrate on identifying the fold bifurcation point (or turning point) in the nonlinear system [12, 11, 8, 21, 25].

Denote a nonlinear system of equations by

$$\mathbf{f}(\mathbf{x}, \lambda) = \mathbf{0} \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^n$ is the state vector, $\lambda \in \mathbf{R}$ is the bifurcation parameter and $\mathbf{f} : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ with $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \ \ldots \ f_n(\mathbf{x})]^\top$. Assume that the current state variable is $\mathbf{x} = \mathbf{x}_0$ corresponding to the parameter λ_0. Away from the fold point, to find the state variable $\mathbf{x} = \mathbf{x}_1$ for any $\lambda = \lambda_1$, we can use the Newton method for $k = 1, 2, 3 \ldots$

$$\begin{cases} J(\mathbf{x}_1^{(k-1)})\Delta\mathbf{x}^{(k)} = -\mathbf{f}(\mathbf{x}_1^{(k-1)}), \\ \mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \Delta\mathbf{x}^{(k)} \end{cases} \tag{2}$$

to achieve $\lim_{k\to\infty} \mathbf{x}_1^{(k)} = \mathbf{x}_1$, provided that $det(J) \neq 0$.

For power systems, $\mathbf{x}$ denotes the vector of bus voltage magnitudes and angles, λ refers to the power demand and $\mathbf{f}$ the power balance equation from Kirchhoff laws; see also [12, 26, 1]. More specifically,

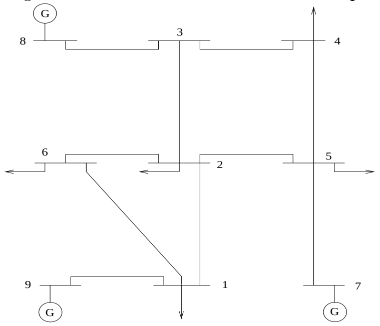
$$\mathbf{x} = [x_1 \ \cdots \ x_n]^\top = [\delta_1 \ \delta_2 \ \ldots \ \delta_m \ v_1 \ v_1 \ \cdots v_{m_1}]^\top,$$

and $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ is defined as

$$\mathbf{f}(\mathbf{x}, \lambda) = \begin{bmatrix} f_1(\mathbf{x}, \lambda) \\ \vdots \\ f_m(\mathbf{x}, \lambda) \\ f_{m+1}(\mathbf{x}, \lambda) \\ \vdots \\ f_n(\mathbf{x}, \lambda) \end{bmatrix} = \begin{bmatrix} \Delta P_1(\mathbf{x}, \lambda) \\ \vdots \\ \Delta P_m(\mathbf{x}, \lambda) \\ \Delta Q_1(\mathbf{x}, \lambda) \\ \vdots \\ \Delta Q_{m_1}(\mathbf{x}, \lambda) \end{bmatrix} = 0, \tag{3}$$

where we assume that an electrical power network has $m+1$ nodes (called buses): $j = 1, 2, \ldots, m, m+1$

Figure 1: The standard IEEE 9-bus example



with the last one, $m+1$, used as the reference bus. Here

$$\begin{cases} \Delta P_j = P_{Gj} - P_{Lj} - v_j \sum_{k=1}^{m+1} y_{jk} v_k \cos(\delta_j - \delta_k - \theta_{jk}) - \lambda \alpha_j, \\ \Delta Q_j = Q_{Gj} - Q_{Lj} - v_j \sum_{k=1}^{m+1} y_{jk} v_k \sin(\delta_j - \delta_k - \theta_{jk}) - \lambda \beta_j, \end{cases} \tag{4}$$

where P_{L_j} and Q_{L_j} are the real and reactive loads, $P_{Gj} + iQ_{Gj}$ denotes the total generation power at bus j and α_j and β_j describe the load pattern increase at bus j for the real and reactive loads respectively. At bus j, write the voltage into polar form $V_j = v_j \exp(i\delta_j)$ with v_j and δ_j denoting the voltage magnitude and phrase angle. In equation (3), the precise number of equations to be solved was decided upon the bus type assumption:

Type 0: the reference $j = m+1$ where $v_j = 1$ and $\delta_1 = 0$ and no equation is needed;

Type 1: a load bus at $j = 1, \ldots, m_1$ where two power equations for v_j and δ_j are needed;

Type 2: a voltage control bus at $j = m_1 + 1, \ldots, m$ where v_j is given and only one power equation is needed for δ_j.

Here m_1 is the total number of load buses (which are assumed to be the *first* m_1 buses out of the network). Figure 1 shows the standard IEEE 9-bus system with $m_1 = 6$ load buses (Type 1), $m = 8 - m_1 = 2$ generator buses (Type 2) and 1 reference generator bus (Type 0).

At the fold point, the Jacobian matrix $J = \mathbf{f}_{\mathbf{x}}$ is singular and one of its real eigenvalues becomes zero. In particular, we consider the case of a normal turning point, that is, assume that the rank of the $n \times n$ Jacobian matrix at a fold point is $(n-1)$; see [16]. The obvious way of finding such a λ associated

with the fold point is to solve $det(J) = 0$. Unfortunately this is not feasible numerically, because (i) it is not easy to find an analytical formula for $det(J)$; (ii) it is not easy to decide when $\det(J) = 0$ and a well-conditioned matrix such as $J = diag(0.1, \ldots, 0.1)$ may have a zero determinant in the numerical sense; see [13]. The popular and reliable method of locating the fold bifurcation is the so-called arc-length (or pseudo-arc length) continuation method [8, 17, 21, 25].

Essentially a continuation method solves equation (3) or (1) together with an extra equation so that the augmented Jacobian matrix is non-singular near the points where $\det(J) = 0$; see [8, 25]. However, a continuation method is usually expensive because many path-following steps must be carried out to reach a bifurcation point (x^*, λ^*). Test functions are needed for a continuation method to know if a bifurcation point is reached; e.g. by computing at each step the minimal singular value or eigenvalue of J. This testing and monitoring process can also be expensive.

Moreover, the exact bifurcation point (x^*, λ^*) is not always needed and only an approximate location of λ^* is practically required in power system analysis. Thus it has been realized that test functions (usually denoted by $\tau(\lambda)$) may be used directly to provide approximations to λ^* [15, 6, 3]. Several test functions reported in the literature, achieving somewhat limited success, are based on the work of [20]; see also [24]. Therefore it is of importance to study the Seydel method [20] in details.

Section 2 of this paper reviews the Seydel method [20] briefly and tests its suitability for producing a performance index that approximates λ^*. Section 3 presents an alternative formulation for the Seydel method. Coupled with a recent analysis [5] of Jacobian matrices and their submatrices, Section 4 discusses the suitable choice of the two integer parameters in the Seydel method and proposes a modified Seydel method based on the analysis in [5]. Finally Section 5 presents some experimental results using widely available and standard IEEE test systems to illustrate the effectiveness of our proposed parameter choice of the Seydel method. Promising numerical results have been achieved.

2 Test function methods

A test function $\tau(\lambda)$ is a function of $\mathbf{x}$ and λ taking zero at $(\mathbf{x}^*, \lambda^*)$ and nonzero elsewhere. All test functions may be used by a continuation method as monitoring functions to detect if a bifurcation point has been reached. A simple choice is $\tau(\lambda) = det(J)$. However such a test function is not well-conditioned. The challenge is therefore to find an easily computable and well-conditioned function $\tau(\lambda)$ which is directly or indirectly linked to $det(J)$, that is,

$$\tau(\lambda) = \frac{det(J)}{K(\lambda)}, \tag{5}$$

where the scalar function K is used to scale $det(J)$; e.g. refer to [21, 18].

Once a test function τ is proposed, the usual philosophy is to test the behaviour of τ; if λ is approximately a quadratic or quartic function in terms of τ (not other way around), then the following Newton-like formula is used to produce a performance index $\hat{\lambda}$ (or sometimes called the load margin) — an approximation to λ^*:

$$\hat{\lambda} = \lambda_1 - \frac{1}{c}\frac{\tau(\lambda_1)}{d\tau/d\lambda(\lambda_1)}, \tag{6}$$

where $c = 2$ (quadratic case) and $c = 4$ (quartic case).

Our work in this paper will help determine more precisely what this c is.

2.1 The Seydel's test function

Seydel [20] proposed a test function based on a method that solves approximately the right zero eigenvector problem by a reduced linear system solver. It does not seem to be related to test function type (5) but we shall establish that it is of this type in the next section.

This method aims to provide a good initial approximation vector $\mathbf{h}_0$ for the eigenvalue problem $J\mathbf{h} = 0$ by converting it to a solvable linear system in the following way. After fixing the kth component $h_k = 1$, the homogeneous system

$$J\mathbf{h} = \mathbf{f}_{\mathbf{x}}(\mathbf{x}, \lambda)\mathbf{h} = 0 \tag{7}$$

becomes solvable but is overdetermined; there are n equations and $n-1$ unknowns. It is therefore still solvable at the fold point $(\mathbf{x}^*, \lambda^*)$ where $rank(\mathbf{f}^*_{\mathbf{x}}) = n-1$. In order to construct a solvable square system, Seydel [20] suggests that one of the n equations (say ℓth) is removed; equivalently replace the removed equation by a normalising condition, in this case with $h_k = 1$. Seydel then defines the removed (residue) equation to be the required test function τ.

Formally, let $\mathbf{A}_{\ell k}$ be the Jacobian J with row ℓ replaced by $\mathbf{e}_k$. Then we can write

$$\mathbf{A}_{\ell k} = (I - \mathbf{e}_\ell \mathbf{e}_\ell^\top)J + \mathbf{e}_\ell \mathbf{e}_k^T \tag{8}$$

where ℓ denotes the index of the removed equation, $\mathbf{e}_\ell$ and $\mathbf{e}_k$ are the ℓth and kth unit vectors respectively. A family of test functions $\tau_{\ell k}$ are defined as follows

$$\tau_{\ell k} = \tau_{\ell k}(\lambda) = \mathbf{e}_\ell^\top J\mathbf{h} = \mathbf{e}_\ell^\top J\mathbf{A}_{\ell k}^{-1}\mathbf{e}_\ell \tag{9}$$

where $\mathbf{h}$ is the solution of the equation $\mathbf{A}_{\ell k}\mathbf{h} = \mathbf{e}_\ell$. It is not trivial to see that $\tau_{\ell k}(\mathbf{x}^*, \lambda^*) = 0$ but this was proved in [20]. What is a good choice for indices ℓ and k? Seydel [20, 24] suggests that ℓ and k can be chosen such that $\mathbf{A}_{\ell k}$ is nonsingular to give a good test function and that there are many such functions. The general belief is that if the pair of indices ℓ and k corresponds to two nonzero positions of the left eigenvector of J at the fold point, then $\tau_{\ell k}$ qualifies as a good test function.

Examples using this test function for several data sets can be found in Chiang et al [3, 15, 6]. However, our test results for several power systems show that the choice of ℓ and k is very restrictive with most choices yielding incorrect results. For the test system in Figure 1, consider the load variation of bus number 1. The true index is $\lambda^* = 5.2$. But with different parameters ℓ, k, very different results can be obtained using formula (6) with $c = 2$:

$\ell = 1$,	$k = 10$	$\hat{\lambda} = 13.9$
$\ell = 2$,	$k = 4$	$\hat{\lambda} = 8.4$
$\ell = 9$,	$k = 8$	$\hat{\lambda} = 1.9$
$\ell = 9$,	$k = 9$	$\hat{\lambda} = 5.3$
$\ell = 10$,	$k = 10$	$\hat{\lambda} = 9.9$
$\ell = 13$,	$k = 14$	$\hat{\lambda} = 15.1$

Note that only one pair of parameters lead to good approximation.

We will discuss suitable choices for ℓ and k later on in the paper.

2.2 Can formula (6) be exact for solving $\tau(\lambda) = 0$?

All test functions $\tau(\lambda)$, readily usable for monitoring purposes in a continuation method, may be adapted to yield an approximate performance index (or load margin) i.e. an approximation to a true fold bifurcation parameter λ^*. However the accuracy of such an approximation (hence the reliability of an index) relies on an assumption of the $\tau(\lambda)$ behaviour. A commonly used assumption [20, 6, 3] is the following

$$\tau(\lambda) = (b - d\lambda)^{1/c} \tag{10}$$

where b, c, d are some suitable (positive) constants; in particular the case of $c = 2$ (quadratic) or $c = 4$ (quartic) was practically used.

The first question is, given the behaviour of τ in (10), how we can solve $\tau(\lambda) = 0$ to find an approximate solution. At the present point (x_1, λ_1), the formula often recommended [20, 6, 3] is

$$\lambda^* = \lambda_1 - \frac{1}{c}\frac{\tau(\lambda_1)}{d\tau/d\lambda(\lambda_1)}. \tag{11}$$

Motivated by the discussions in [3, 4], we now prove that (11) gives the exact performance index (i.e. root of $\tau(\lambda)$) *if* the test function behaves as in (10). More precisely, because

$$W(\lambda) = \frac{1}{c}\frac{\tau(\lambda)}{d\tau/d\lambda(\lambda)} = -\frac{b}{d} + \lambda \tag{12}$$

is linear in λ, it has the same simple root as $\tau(\lambda)$, namely, $\lambda^* = \frac{b}{d}$ that can be worked out using

$$\frac{b}{d} = \lambda_1 - W(\lambda_1)$$

leading to the exact formula in (11).

As we see later, many test functions do not behave like (10) — this is why a seemingly reasonable test function cannot always provide a reliable index.

3 Reformulation of the Seydel function method

Seydel [20] proposed the above test function (§2.1) for a generic bifurcation system and was later adapted to produce performance indexes by several authors [3, 15, 6]. The behaviour of Seydel's test function (9) in terms of λ was not analysed before although many experiments have suggested that this τ can behave like (10). Part of the difficulty of analysing Seydel's test function lies in the simple fact that it is not a simple function (defined implicitly by a computable procedure). Seydel's original proof used the perturbation idea i.e. if $\lambda \to \lambda^*$, then $\tau_{\ell,k} \to 0$ i.e. $\tau_{\ell,k}$ is indeed a proper test function. However more needs to be known about this $\tau_{\ell,k}$ in terms of λ before we may use it properly for a performance index.

As a first step towards an analysis, we show that Seydel's test functions are of type (5). Then we analyse determinants next section.

The Cramer's rule is used extensively in the literature of bifurcation studies; this is also used here. Define $J(\ell_1, \ldots, \ell_s; k_1, \ldots, k_s)$ as the submatrix of J, sized $(n-s) \times (n-s)$, after taking out rows $\ell_1, \ldots, \ell_s$ and columns $k_1, \ldots, k_s$. To present our analysis, note that matrix $\mathbf{A}_{\ell,k}$, of size $n \times n$ and used in the test function of (9), is related to the $(n-1) \times (n-1)$ submatrix $J(\ell; k)$; in fact we can verify that $det(\mathbf{A}_{\ell,k}) = det(J(\ell; k))$.

Theorem 1 *The Seydel's test function described in (9) can be represented by*

$$\tau_{\ell k} = (-1)^{\ell+k}\frac{det(J)}{det(J_{\ell k})} = (-1)^{\ell+k}\frac{det(J)}{det(J_{\ell;k})}. \tag{13}$$

Proof. First note that there exists at least one submatrix $J_{\ell;k}$ that is nonsingular (due to fold bifurcation) for some $1 \le \ell, k \le n$. We consider the system

$$J(\mathbf{x}_0, \lambda_0)\mathbf{h} = J\mathbf{h} = \mathbf{0}, \qquad h_k = 1.$$

Re-write this system as an overdetermined one:

$$\left\{\begin{array}{ccccccccccc}
J_{11}h_1 & + \ldots + & J_{1,k-1}h_{k-1} & + & J_{1,k+1}h_{k+1} & + \ldots + & J_{1n}h_n & = & -J_{1,k} \\
\vdots & & \vdots & & \vdots & \vdots & & & \vdots \\
J_{\ell-1,1}h_1 & + \ldots + & J_{\ell-1,k-1}h_{k-1} & + & J_{\ell-1,k+1}h_{k+1} & + \ldots + & J_{\ell-1,n}h_n & = & -J_{\ell-1,k} \\
J_{\ell 1}h_1 & + \ldots + & J_{\ell,k-1}h_{k-1} & + & J_{\ell,k+1}h_{k+1} & + \ldots + & J_{\ell n}h_n & = & -J_{\ell,k} \\
J_{\ell+11}h_1 & + \ldots + & J_{\ell+1,k-1}h_{k-1} & + & J_{\ell+1,k+1}h_{k+1} & + \ldots + & J_{\ell+1n}h_n & = & -J_{\ell+1,k} \\
\vdots & & \vdots & & \vdots & \vdots & & & \vdots \\
J_{n1}h_1 & + \ldots + & J_{n,k-1}h_{k-1} & + & J_{n,k+1}h_{k+1} & + \ldots + & J_{nn}h_n & = & -J_{n,k}
\end{array}\right.$$

Using Seydel's method, we introduce a solvable system by removing equation ℓ:

$$\left\{\begin{array}{ccccccccccc}
J_{11}h_1 & + \ldots + & J_{1,k-1}h_{k-1} & + & J_{1,k+1}h_{k+1} & + \ldots + & J_{1n}h_n & = & -J_{1,k} \\
\vdots & & \vdots & & \vdots & \vdots & & & \vdots \\
J_{\ell-1,1}h_1 & + \ldots + & J_{\ell-1,k-1}h_{k-1} & + & J_{\ell-1,k+1}h_{k+1} & + \ldots + & J_{\ell-1,n}h_n & = & -J_{\ell-1,k} \\
J_{\ell+11}h_1 & + \ldots + & J_{\ell+1,k-1}h_{k-1} & + & J_{\ell+1,k+1}h_{k+1} & + \ldots + & J_{\ell+1n}h_n & = & -J_{\ell+1,k} \\
\vdots & & \vdots & & \vdots & \vdots & & & \vdots \\
J_{n1}h_1 & + \ldots + & J_{n,k-1}h_{k-1} & + & J_{n,k+1}h_{k+1} & + \ldots + & J_{nn}h_n & = & -J_{n,k}
\end{array}\right. \tag{14}$$

This gives rise to the same solution as Seydel i.e. $\mathbf{h} = \mathbf{A}_{\ell,k}^{-1}\mathbf{e}_\ell$ and

$$\tau_{\ell,k} = J_{\ell 1}h_1 + \ldots + J_{\ell,k-1}h_{k-1} + J_{\ell,k}h_k + J_{\ell,k+1}h_{k+1} + \ldots + J_{\ell n}h_n = e_\ell^\top J\mathbf{h} = e_\ell^\top J\mathbf{A}_{\ell,k}^{-1}\mathbf{e}_\ell. \tag{15}$$

This seemingly messy procedure has a simple alternative expression using the Cramer's rule. Let $\mathbf{a}_k$ be the kth column of matrix J and define $\mathbf{B}_j$ as the $(n-1) \times (n-1)$ submatrix of J formed by removing row l and replacing column j, for any $j = 1, \ldots, n$, by $-\mathbf{a}_k$

$$\mathbf{B}_j = \begin{bmatrix}
J_{11} & \ldots & -J_{1k} & \ldots & J_{1,k-1} & J_{1,k+1} & \ldots & J_{1n} \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots \\
J_{\ell-1,1} & \ldots & -J_{\ell-1,k} & \ldots & J_{\ell-1,k-1} & J_{\ell-1,k+1} & \ldots & J_{\ell-1,n} \\
J_{\ell+1,1} & \ldots & -J_{\ell+1,k} & \ldots & J_{\ell+1,k-1} & J_{\ell+1,k+1} & \ldots & J_{\ell+1,n} \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots \\
J_{n1} & \ldots & -J_{nk} & \ldots & J_{n,k-1} & J_{n,k+1} & \ldots & J_{nn}
\end{bmatrix}.$$

Here by Cramer's rule for (14)

$$h_j = \frac{\det(\mathbf{B}_j)}{\det(\mathbf{A}_{\ell k})}. \tag{16}$$

Relating $\det(\mathbf{B}_j)$ to $\det(\mathbf{A}_{\ell j})$, we can verify that (using elementary properties of determinants)

$$h_1 = \frac{1}{\det(\mathbf{A}_{\ell k})}\begin{vmatrix} -J_{1k} & \ldots & J_{1,k-1} & J_{1,k+1} & \ldots & J_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -J_{\ell-1,k} & \ldots & J_{\ell-1,k-1} & J_{\ell-1,k+1} & \ldots & J_{\ell-1,n} \\ -J_{\ell+1,k} & \ldots & J_{\ell+1,k-1} & J_{\ell+1,k+1} & \ldots & J_{\ell+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -J_{nk} & \ldots & J_{n,k-1} & J_{n,k+1} & \ldots & J_{nn} \end{vmatrix} = (-1)^{1+k}\frac{\det\mathbf{A}_{\ell 1}}{\det\mathbf{A}_{\ell k}},$$

$$h_2 = \frac{1}{\det(\mathbf{A}_{\ell k})}\begin{vmatrix} J_{11} & -J_{1k} & \ldots & J_{1,k-1} & J_{1,k+1} & \ldots & J_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ J_{\ell-1,1} & -J_{\ell-1,k} & \ldots & J_{\ell-1,k-1} & J_{\ell-1,k+1} & \ldots & J_{\ell-1,n} \\ J_{\ell+1,1} & -J_{\ell+1,k} & \ldots & J_{\ell+1,k-1} & J_{\ell+1,k+1} & \ldots & J_{\ell+1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ J_{n1} & -J_{nk} & \ldots & J_{n,k-1} & J_{n,k+1} & \ldots & J_{nn} \end{vmatrix} = (-1)^{2+k}\frac{\det\mathbf{A}_{\ell 2}}{\det\mathbf{A}_{\ell k}},$$

$$\ldots$$

$$h_{k-1} = \frac{\det(\mathbf{B}_{k-1})}{\det(\mathbf{A}_{\ell k})} = (-1)^1\frac{\det(\mathbf{A}_{\ell,k-1})}{\det(\mathbf{A}_{\ell k})} = (-1)^{(k-1)+k}\frac{\det(\mathbf{A}_{\ell,k-1})}{\det(\mathbf{A}_{\ell k})},$$

$$h_k = 1 = (-1)^{k+k}\frac{\det(\mathbf{A}_{\ell k})}{\det(\mathbf{A}_{\ell k})},$$

$$h_{k+1} = \frac{\det(\mathbf{B}_{k+1})}{\det(\mathbf{A}_{\ell k})} = (-1)^1\frac{\det(\mathbf{A}_{\ell,k+1})}{\det(\mathbf{A}_{\ell k})} = (-1)^{(k+1)+k}\frac{\det(\mathbf{A}_{\ell,k+1})}{\det(\mathbf{A}_{\ell k})},$$

$$\ldots$$

$$h_n = \frac{\det(\mathbf{B}_n)}{\det(\mathbf{A}_{\ell k})} = (-1)^{n-k}\frac{\det(\mathbf{A}_{\ell n})}{\det(\mathbf{A}_{\ell k})} = (-1)^{n+k}\frac{\det(\mathbf{A}_{\ell n})}{\det(\mathbf{A}_{\ell k})}.$$

That is, for any $j = 1, \ldots, n$,

$$h_j = \frac{\det(\mathbf{B}_j)}{\det(\mathbf{A}_{\ell k})} = (-1)^{j+k}\frac{\det(\mathbf{A}_{\ell j})}{\det(\mathbf{A}_{\ell k})} \quad \text{and} \quad \det(\mathbf{B}_j) = (-1)^{j+k}\det(\mathbf{A}_{\ell j}). \tag{17}$$

Substituting the above into (15), we obtain

$$\begin{aligned}\tau_{\ell k} &= J_{\ell 1}h_1 + J_{\ell 1}h_2 + \ldots + J_{\ell n}h_n \\ &= \frac{1}{\det(\mathbf{A}_{\ell k})}\left[(-1)^{1+k}J_{\ell 1}\det\mathbf{A}_{\ell 1} + (-1)^{2+k}J_{\ell 2}\det\mathbf{A}_{\ell 2} + \ldots + (-1)^{n+k}J_{\ell n}\det\mathbf{A}_{\ell n}\right] \\ &= (-1)^{\ell+k}\frac{1}{\det(\mathbf{A}_{\ell k})}\left[(-1)^{1+l}J_{\ell 1}\det\mathbf{A}_{\ell 1} + (-1)^{2+l}J_{\ell 2}\det\mathbf{A}_{\ell 2} + \ldots + (-1)^{n+l}J_{\ell n}\det\mathbf{A}_{\ell n}\right] \\ &= (-1)^{\ell+k}\frac{\det(J)}{\det(\mathbf{A}_{\ell k})}.\end{aligned}$$

Thus the proof is complete. □

Remark. The theorem establishes that the Seydel's test function for any parameters ℓ, k is of type (5) — a scaled determinant. Note that this result assumes $det(J_{\ell k}) \neq 0$ and it holds for both the case $det(J) = 0$ (at the fold point) and the case $det(J) \neq 0$ (away from the fold point). However when assuming $det(J) \neq 0$ only, [22, 23] have derived an equivalent formula for τ:

$$\begin{cases} J\mathbf{h} = \tau e_\ell, \\ e_k^\top \mathbf{h} = 1, \end{cases} \qquad \text{that is,} \qquad \begin{bmatrix} J & -e_\ell \\ e_k^\top & 0 \end{bmatrix}\begin{bmatrix} \mathbf{h} \\ \tau \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{18}$$

This formulation fits the framework of bordered systems as described by [14]. Again using the Cramer's rule, we can obtain a similar result to the above theorem (although this has not been done before). However this is only feasible if $det(J) \neq 0$.

4 Parameter choice and modified Seydel methods

To determine the behaviour of Seydel's test functions $\tau_{\ell k}(\lambda)$, we now investigate the behaviour of determinants: $det(J)$ and $det(J_{\ell k})$. Clearly from equation (3), it is not easy to relate τ to λ directly. Numerical evidence has confirmed that all voltage magnitudes exhibit a quadratic behaviour i.e.

$$\lambda = a_j v_j^2 + b_j v_j + c_j \tag{19}$$

and the differences between voltage angles $\delta_i - \delta_j$ do not vary much with λ. Theoretically it can be shown that near the fold bifurcation, a quadratic behaviour such (19) holds for other nonlinear systems as well; see [19].

From (19), we see that

$$v_j + k_j = (r_j - s_j\lambda)^{1/2} \tag{20}$$

where $k_j = b_j/(2a_j)$ and so on. Therefore $v_j + k_j$ (and v_j if k_j is small) will satisfy the assumption in (10). This motivates us to relate determinants to v_j's first before we estimate behaviour of Seydel's test functions.

4.1 Properties of the Jacobian matrix and its submatrices

Given that the Seydel's test functions are determined by two determinants $det(J)$ and $det(J_{\ell k})$, we now consider the analytical behaviour these two determinants. This was studied in [5] and an extraction of the results is shown as follows:

Theorem 2 ([5]) *The determinant of the Jacobian matrix J from (3) can be approximated by a high order multi-variate polynomial $\mathbf{P}(v_1, \ldots, v_m)$ which has the highest and lowest order terms respectively*

$$v_1^4 \cdots v_{m-m_1}^4 v_{m-m_1+1}^3 \cdots v_{m_1}^3 v_{m_1+1}^2 \cdots v_m^2,$$
$$v_1^3 \cdots v_{m-m_1}^3 v_{m-m_1+1}^3 \cdots v_{m_1}^3 v_{m_1+1}^2 \cdots v_m^2.$$

Consequently, the ratio of two determinants has the following behaviour:

1. $det(J)/det(J(1;1)) \approx v_1^2$;
2. $det(J)/det(J(m+\ell; m+k)) \approx v_\ell$;
3. $det(J)/det(J(m+1, \ldots, n; m+1, \ldots, n)) \approx \prod_{j=1}^{m} v_j$,

where $\approx$ means two quantities differ by a constant multiplier that is independent of λ.

As the ratios of determinants are simple monomials of v_j's, it remains to study the behaviour of v_j's before the behaviour of test functions can be determined.

4.2 Parameter choice and behaviour of the scalar $v_j = v_j(\lambda)$

An analytical study of $v_j = v_j(\lambda)$ (or $x_j = x_j(\lambda)$) is difficult to conduct so we have carried out a numerical investigation. Following (20), we tried to find the constant k_j by fitting a quadratic function in (19). It turns out that if we consider the load variation for bus p, for $j = 1, 2, \ldots, m_1$,

$$|k_p| = \min |k_j|,$$

although most are small. Therefore we may ignore k_p and assume that (compare to (10))

$$v_p = (r_p - s_p\lambda)^{1/2},$$

which is the desirable property for (11) to be exact with $c = 2$ for solving $\tau(\lambda) = 0$. Thus we make the following recommendation, when varying bus p, in order for the Seydel's test function $\tau_{\ell k}$ to provide a useful performance index:

$$\begin{cases} \ell = p, & \text{(because } k_p \approx 0) \\ k = m+1, \cdots, m+m_1, & \text{(because of Theorem 2).} \end{cases}$$

4.3 Modified Seydel methods

In general, if $\ell \neq p$ (but $\ell > m$), our Theorems 1 and 2 state that for $j = m+1, \ldots, m+m_1$,

$$\tau_{\ell j} = (-1)^{\ell+j} \frac{\det(J)}{\det(J_{\ell j})} \approx v_\ell$$

and so the Seydel's test function cannot provide a good index because formula (11) cannot be used with any fixed c.

In this case, from equation (20), we can propose the following modified Seydel's test function:

$$\tau_{\ell j}^{\mathbf{m}} = \frac{\det(J)}{\det(J_{\ell j})} \frac{v_\ell + k_\ell}{v_\ell} \tag{21}$$

that will behave like the inverse quadratic function (i.e. the right hand side of (20)). Thus for a suitable constant k_ℓ, $\tau_{\ell j}^{\mathbf{m}}$ can be used to give a good performance index; e.g. for the example in Figure 1 when $p = 2$ and $\ell = 1$: $k_\ell = k_1 = -0.8211$ and the modified method gives the approximate index $\hat{\lambda} = 2.1$ (exact index $\lambda^* = 2.4$) while the standard Seydel's method gives the poor approximation $\hat{\lambda} = 7.3$.

5 Numerical experiments

We now present some numerical results to demonstrate our analysis of and recommendations for Seydel's test function methods for index calculations. We shall use standard test examples rather than specially tuned ones. The following standard systems are tested:

1. IEEE 9:
   the number of load buses $m_1 = 6$ and $m = 8$, and the Jacobian is of the size 14×14;
2. IEEE 14:
   the number of load buses $m_1 = 9$ and $m = 13$, and the Jacobian is of the size 22×22;
3. IEEE 30:
   the number of load buses $m_1 = 24$ and $m = 29$, and the Jacobian is of the size 53×53;
4. IEEE 57:
   the number of load buses $m_1 = 50$ and $m = 56$, and the Jacobian is of the size 106×106.

For each example, we shall test the effectiveness of Seydel's test function methods with our recommended choice of parameters, in getting an 'on-line' index list for all load buses that can be used to identify the weakest bus. Assuming we consider the reactive power load variation for bus p, $p = 1, 2, \ldots, m_1$, we shall compare the performance of

1. S1 — The Seydel's test function with fixed $\ell = 1, k = 2$;

2. S2 — The Seydel's test function with fixed $\ell = 2, k = 2$;
3. S3 — The Seydel's test function with fixed $\ell = m+1, k = m+1$;
4. S4 — The Seydel's test function with variable $\ell = m+p, k = m+1$; (recommended)
5. S5 — The Seydel's test function with variable $\ell = m+p, k = m+p$; (recommended)
6. S6 — The Seydel's test function with variable $\ell = m+p, k = m+m_1$. (recommended)

To implement (11), the calculation of the derivative $d\tau/d\lambda$ is done by finite differences as in [15, 6]. Therefore two successive Newton solves are applied at $\lambda_0 = 0$ and $\lambda_1 = 10^{-4}$; for these IEEE systems with tolerance $TOL = 10^{-5}$ for stopping Newton iterations, the number of such iterations is typically small (about $3-5$) using the simple starting values $\delta_j = 0$ and $v_j = 1$. For other systems, one may have to use a continuous power flow program to find $\mathbf{x}_0$ and $\mathbf{x}_1$ (hence $\tau(\lambda_0)$ and $\tau(\lambda_1)$) if convergence is a problem. An alternative method for computing $d\tau/d\lambda$ may be based on semi-analytical formulas as in Seydel [20].

In Figures 1-4, we display the predicted indexes for each test example by the SIX methods, compared with the exact indexes obtained by running a continuous power flow program. As some predicted indexes are over-estimates, for clarity, we present each figure in two graphs with the top one showing $S1-S3$ and the bottom showing $S4-S6$. For IEEE 57, we rotate the graphs so that bus labels are readable. A legend list can be stated as follows:

Symbol $\times$	——	for S1
Symbol $\bullet$	——	for S2
Symbol $\bigtriangledown$	——	for S3
Symbol $\bigcirc$	——	for S4
Symbol $\triangle$	——	for S5
Symbol $+$	——	for S6
Symbol $\diamond$	——	for **exact** indexes.

Clearly we observe that the recommended choices $S4-S6$ provide very reliable performance indexes for all cases considered although $S5$ performs the best. Methods $S1-S3$ demonstrate results of a typical example of adapting generic bifurcation methods to power flow equations — they may or may not work for an arbitrary choice of parameters of ℓ, k. Notice that in Figure 4, method $S1$ has somewhat strange jumps because the missing index data are negative and so are not visible on the log scale. Recall that Seydel [20, 24] did not recommend any choice of the parameters l, k; in fact [24, p.6] seems to suggest that for a test system of $n=4$ there are at most $16 = 4^2$ test functions! Our analytical and numerical results have confirmed that only the theoretically justified and recommended m_1 choices lead to useful indexes (see §4.2).

Finally to demonstrate that it is possible to modify the Seydel's test function methods when the 'wrong' parameters are used, we show in Figure 6 the improved results for modified $S3$ as in §4.3 (i.e. fixed $\ell = k = m+1$) for IEEE 9 and IEEE 30 bus systems. Although this is an interesting way to improve the basic method, the modification does not yet lead to a practical method because it is not clear how to find constants k_j a priori. Future work should address this problem.

6 Conclusions

The Seydel's test function method is a useful tool for detecting fold bifurcations in an arc-length continuation method. When adapted to give a performance index for power system analysis, it often gives the incorrect results for an arbitrary choice of parameters ℓ, k. To analyse the method, with the aim of finding a suitable choice of ℓ, k, we present an alternative formulation for this method. Based on this formulation, we are able to estimate the analytical behaviour of the test function for different choices of ℓ, k that in turn leads to a useful performance index (the fold bifurcation parameter estimation). For reactive power load variations at a single bus, we found that there are at most m_1 choices of valid pairs of ℓ, k; for other choices we have shown, using our analytical results, examples of a modified Seydel's method that give rise to a useful index.

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Figure 2: Performance indexes by various methods for IEEE 9 bus

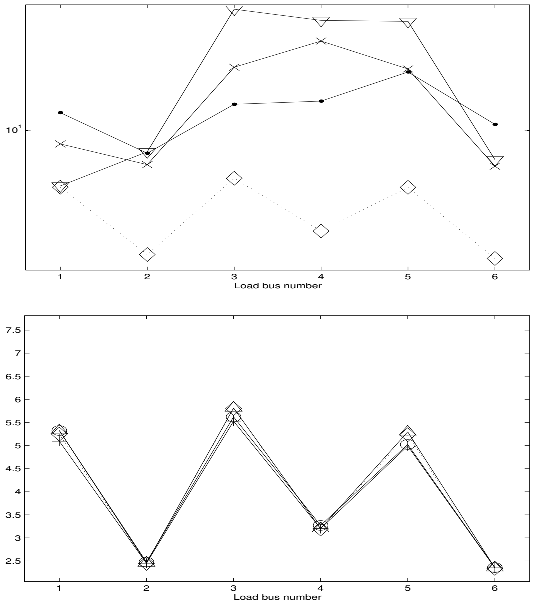


Figure 3: Performance indexes by various methods for IEEE 14 bus

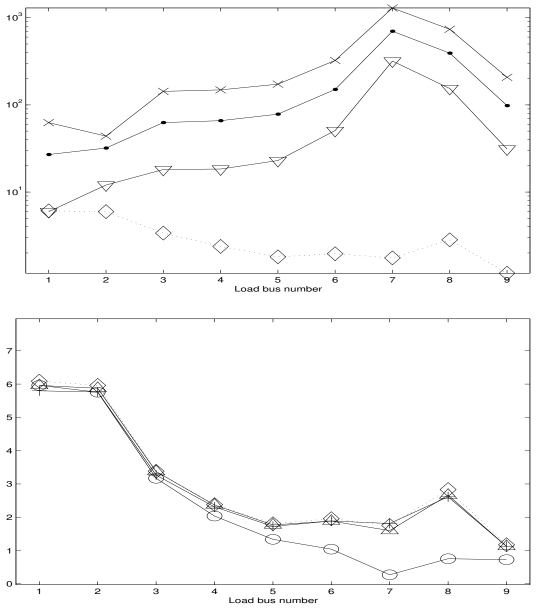


Figure 4: Performance indexes by various methods for IEEE 30 bus

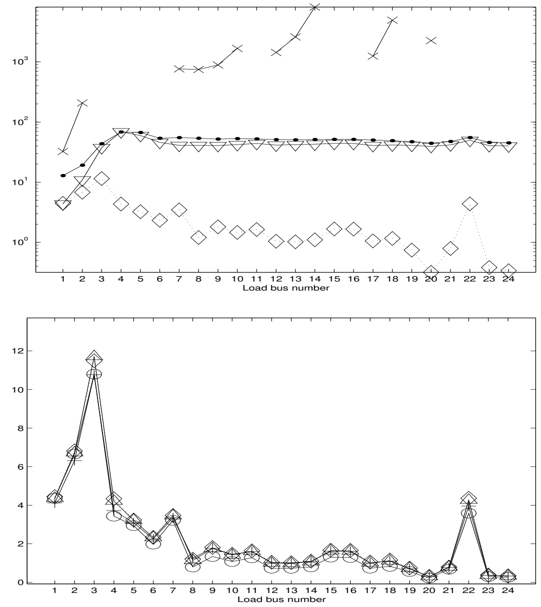


Figure 5: Performance indexes by various methods for IEEE 57 bus

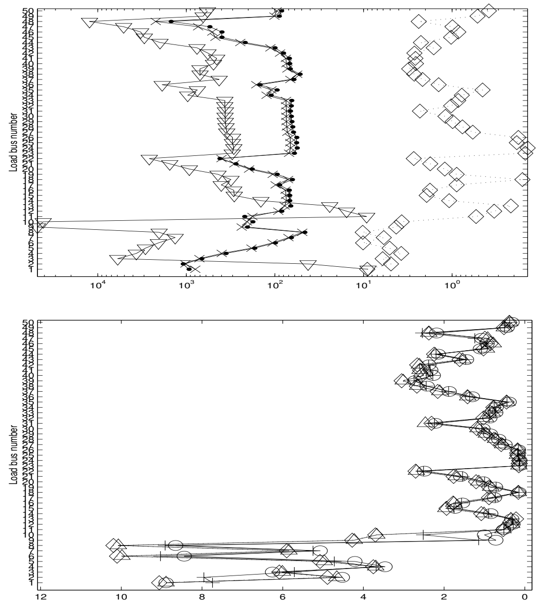
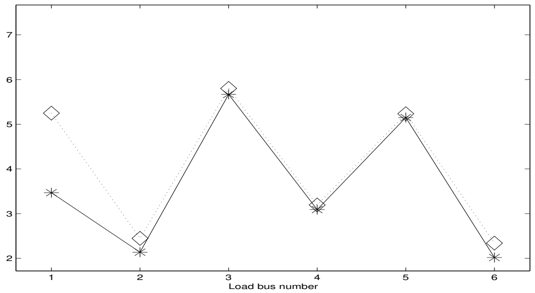
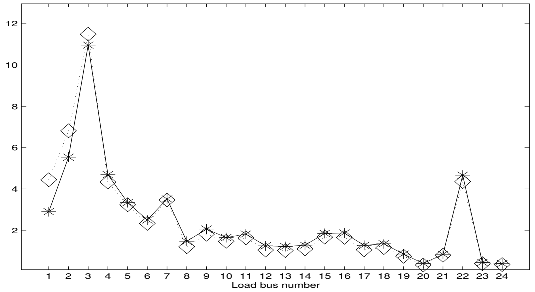


Figure 6: Performance indexes by modified Seydel methods for 9-bus (top) and 30-bus systems (bottom)





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