An overlapping domain decomposition dual reciprocity method

Kamal Shanazari* and Ke Chen†
Department of Mathematical Sciences,
The University of Liverpool, Liverpool L69 7ZL, UK.

Abstract

We present a domain decomposition boundary integral equation method for two dimensional partial differential equations. The overlapping Schwarz method is employed to improve the dual reciprocity boundary element method. The resulting algorithm turns out to be more accurate than a non-overlapping approach previously proposed. Some numerical results showing the improved accuracy and efficiency of the methods are given.

Key words. Domain decomposition, boundary integral equation, dual reciprocity method, overlapping Schwarz method.
AMS subject classifications. 65D05, 31B10, 5N38.

1 Introduction

The boundary element method (BEM) is a powerful method for solving a class of partial differential equations (PDEs) [1]. The main idea in this method is to convert the original differential equation into an equivalent boundary integral equation by using Green's theorem and a fundamental solution. The boundary integral equation is then solved numerically on the boundary of the whole domain. Consequently the advantage in this method over the classical domain methods such as finite elements and finite differences is that only boundary discretization is required due to dimension reduction.

The method becomes less attractive when dealing with nonhomogeneous problems and more general linear PDE's that do not have an explicit fundamental solution. One way to generalise BEMs to these cases is to apply a fundamental solution corresponding to the homogeneous part of the equation to express the remaining ('inhomogeneous') term as domain integrals. Further one considers various ways to tackle these domain integrals. Of the methods proposed, the particular solution and multiple reciprocity methods are most commonly used [5, 9]. One of the most efficient methods to eliminate the domain integrals is the dual reciprocity method (DRM) due to Nardini and Brebbia [9] as this method applies to general linear PDEs. In this method the inhomogeneous

*E-mail: kamals@liverpool.ac.uk. Support of a research studentship by the Ministry of Education (Kurdistan University, Iran) is gratefully acknowledged.
†For correspondence, E-mail: k.chen@liverpool.ac.uk, URL: http//www.liv.ac.uk/∼cmchenke

term of the equation is approximated by interpolation in terms of some well-known functions $\phi(r)$, called radial basis functions (RBFs), where r is the distance between a source point and the field point. These RBFs are useful because the equation $\nabla^2 \hat{u} = \phi(r)$ can be solved analytically. There exist a large class of interpolating RBFs [7, 13] that can be used by DRM. These include the linear $1 + r$, the polynomial $P_k(r)$, the thin plate spline (TPS) $r^2 \log r$, the Gaussian $\exp(-r^2/\beta^2)$ and the multiquadrics $\sqrt{\beta^2 + r^2}$ (with β a constant parameter). Note that RBFs can be used alone to solve PDEs in the so-called meshless approach [6, 8]. We shall mainly use the polynomial RBF here.

Our motivation to use domain decomposition method (DDM) is to overcome the ill-conditioning of the RBF interpolation matrix whose inverse is explicitly used by DRM. Better representation of the inverse of such a matrix subsequently improves the DRM accuracy. We note that DDM techniques are widely used in solving PDEs. The popularity is largely due to its inherent parallelism in the DDM idea. Previous attempts were applying a specialised non-iterative and non-overlapping DDM idea to DRM by Popov and Power [10] in one dimensions and others [14, 2, 4, 3, 11, 12] in two dimensions. There the conditioning of the RBF interplation matrix is improved but the overall method is not tuned to parallel computation. Similar idea using DDM to improve the ill-conditioning was indicated earlier in Kensa and Carlson [6] suggesting that the best accuracy was achieved when the original domain was split into matching subdomains.

Motivated by general recommendation of overlapping DDM for PDEs, this paper presents an overlapping DDM approach for the DRM method. With it, we have achieved better accuracy than using single domain DDM and non-overlapping DDM. Not surprisingly the number of arithmetic operations is considerably decreased due to dealing with small size matrices. An attractive feature of the overlapping DDM is that there is no need to satisfy the physical matching conditions (involving the potential and flux depending on the type of the underlying PDE).

The rest of this paper will be organized as follows. We first describe the DRM in Section 2. Then the alternating Schwarz method will be reviewed and applied to DRM in Section 3. Finally, some numerical results will be presented in Section 4.

2 The dual reciprocity method

We describe the method for the following Poisson equation,

$$\bigtriangledown^2(u) = b, \qquad \text{in } \Omega, \tag{1}$$

where u is the dependent variable, Ω is the domain and b is the non- homogeneous part, which can generally be in the form,

$$b = f(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}).$$

The boundary condition can be of the Dirichlet type ($u = \bar{u}$ on Γ_D) or Neumann type ($\frac{\partial u}{\partial n} = \bar{q}$ on Γ_N) or Robin type (a combination of both), where $\Gamma_D + \Gamma_N = \Gamma$, is the boundary enclosing Ω. Applying the weighted residual technique and using Green's theorem, the integral formulation for equation (1) is concluded by,

$$c_i u_i = \int_\Gamma (u\frac{\partial u^*}{\partial n} - u^*\frac{\partial u}{\partial n}) d\Gamma + \int_\Omega u^* b d\Omega, \tag{2}$$

where c_i are coefficients depending on the location of the point i and weighting function u^*, called fundamental solution is given by,

$$u^* = -\frac{1}{2\pi} \ln r.$$

The above integral equation contains a domain integral corresponding to the non-homogeneous term, b. The main idea in DRM is converting this domain integral to an equivalent boundary integral by approximating the function b in terms of interpolating functions at a certain number of boundary N and internal L nodes, that is,

$$b = \sum_{k=1}^{N+L} \alpha_k \phi_k, \tag{3}$$

where ϕ_k are the RBF and α_k are the corresponding coefficients. As remarked earlier, $\hat{u}_k$ can be analytically obtained from

$$\bigtriangledown^2 \hat{u}_k = \phi_k.$$

With this approximation for b, the domain integral in equation (2) is converted to an equivalent boundary integral by applying the same idea as for the main equation, that is,

$$\int_\Omega b u^* d\Omega = \sum_{k=1}^{N+L} \alpha_k \int_\Omega (\bigtriangledown^2 \hat{u}_k) u^* d\Omega = \sum_{k=1}^{N+L} \alpha_k \{(u^* \frac{\partial \hat{u}_k}{\partial n} - \hat{u}_k \frac{\partial u^*}{\partial n}) d\Gamma - \hat{u}_{ik}\} \tag{4}$$

where $\hat{u}_{ik}$ is the known value of $\hat{u}_k$ at the source point i. Substituting (4) in (2), leads to the system of equations

$$Hu - Gq = (H\hat{U} - G\hat{Q})F^{-1}b, \tag{5}$$

where H and G are the BEM matrices, $\hat{U}$ and $\hat{Q}$, are matrices relating to $\hat{u}_k$ and their normal derivatives, and F is the interpolation matrix relevant to the approximation in (3). Also u and q are the vectors containing the potentials and the normal derivatives and b is a vector of the values of the right side of the equation (1). To find more details see [9].

There are at least three motivations behind the idea of decomposing the problem into smaller subproblems in DRM. Firstly, calculating the inverse and multiplications in (5) is expensive, especially when dealing with a more general form of PDEs for which the formulation in (5) becomes more costly [9]. Secondly, the matrices H, G, $\hat{U}$, $\hat{Q}$ and F are dependent on the geometry only. This is one desirable feature of DDM when applied to DRM since the original domain can be partitioned into a number of subdomains with the same shape and size consequently the above matrices can be evaluated only once for all subproblems. Thirdly, the accuracy of the interpolation in (3) depends upon the condition number of the interpolation matrix F. The condition number is sensitive to the size of matrix. The effect of reducing the size of matrix F by using domain partitioning on the condition number is shown in Figure 1. Furthermore, a cost analysis for DRM, containing the number of arithmetic operations to construct the linear system of equations in (5) and solving it by a direct method, was accomplished. Although apparently some extra points arising from the overlap regions are involved in calculations, the overall operations indicate a considerable reduction in comparison with the case of using a single domain, as depicted in Figure 2.

3 An overlapping Schwarz method

We first describe an overlapping Schwarz method for PDEs using two subdomains. We then generalize it to the DRM and finally describe multi-domain DDM-DRM algorithms.

Figure 1: The condition number of the interpolation matrix is plotted for the case of using different number of overlapping subdomains for a rectangular domain.

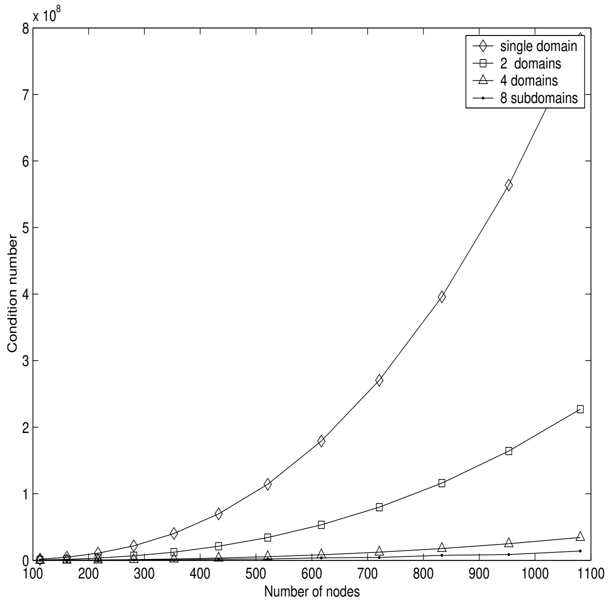


Figure 2: The number of operations in the case of using different number of subdomains is compared.

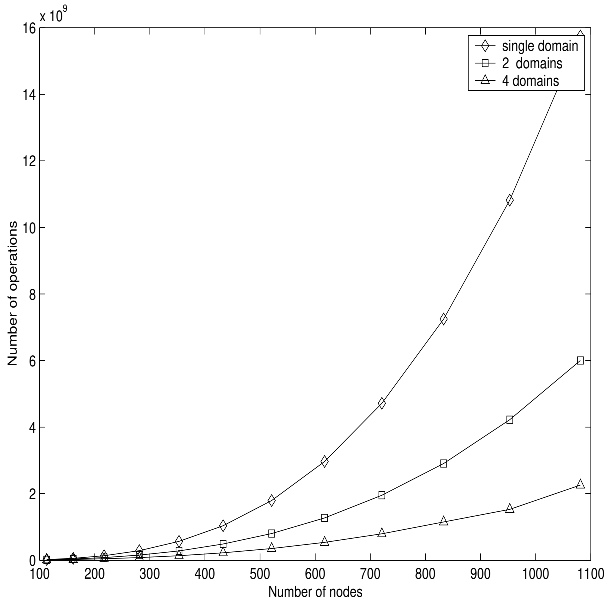
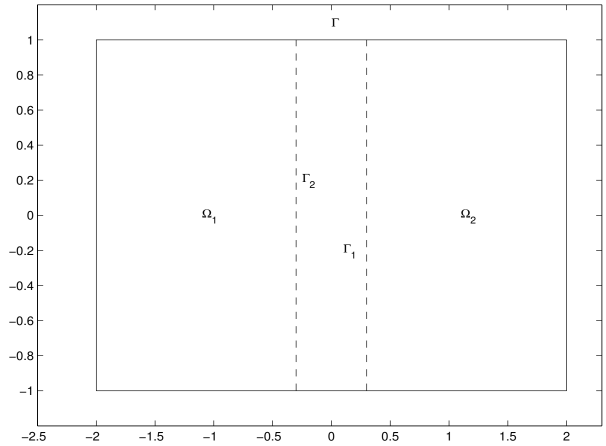


Figure 3: The domain Ω has been split into subdomains Ω_1 and Ω_2.



3.1 The Schwarz method

We explain the method for the following Poisson equation with a Dirichlet boundary condition,

$$\begin{array}{lll} \nabla^2(u) = b, & \text{in} & \Omega, \\ u = \bar{u}, & \text{on} & \Gamma. \end{array} \tag{6}$$

Of course other kinds of boundary conditions can be considered similarly. Let the domain Ω bounded by Γ be divided into two subdomains Ω_1 and Ω_2 bounded by $\partial\Omega_1$ and $\partial\Omega_2$ respectively (Fig 3). The artificial boundaries for these subdomains are denoted by Γ_1 and Γ_2 consequently $\partial\Omega_i\backslash\Gamma_i$ is the part of the boundary of the subdomain Ω_i which is not interior to Ω. Also let $\bar{\Omega} = \Omega \cup \partial\Omega$ denote the closure of the domain. In order to describe the method, we introduce some notation. Suppose that u_i^n stands for the approximate solution on $\bar{\Omega}_i$ after n iterations, $u_1^n|_{\Gamma_2}$ is the restriction of u_1^n to Γ_2 and $u_2^n|_{\Gamma_1}$ is the restriction of u_2^n to Γ_1. In this method an initial guess u_2^0, for the value in Ω_2, is required to start the iteration. In fact only the values along the Γ_1 is necessary. The Schwarz method is then proceeded with the following iterations to obtain the solution.

1. Solving equation (6) for the subdomain Ω_1 as follows,

$$\begin{array}{lll} \nabla^2 u = b & \text{in} & \Omega_1, \\ u_1^n = \bar{u} & \text{on} & \partial\Omega_1 \setminus \Gamma_1, \\ u_1^n = u_2^{n-1}|_{\Gamma_1} & \text{on} & \Gamma_1. \end{array} \tag{7}$$

2. Solving equation (6) for the subdomain Ω_2 as follows,

$$\begin{array}{lll} \nabla^2 u = b & \text{in} & \Omega_2, \\ u_2^n = \bar{u} & \text{on} & \partial\Omega_2 \setminus \Gamma_2, \\ u_2^n = u_1^n|_{\Gamma_2} & \text{on} & \Gamma_2. \end{array} \tag{8}$$

The above iteration is performed until a desirable accuracy is achieved. More precisely for a given positive value, ϵ the inequalities $|u_1^n - u_1^{n-1}| < \epsilon$ and $|u_2^n - u_2^{n-1}| < \epsilon$ are satisfied. It should be noted that the values $u_2^n|_{\Gamma_1}$ and $u_1^n|_{\Gamma_2}$ are approximated by interpolating in Ω_2 and Ω_1 respectively. To find more details on alternative approaches see [15].

3.2 Formulation of the DRM in two subdomains

We explain the Schwarz formulation when applied to DRM with a Dirichlet boundary condition. First, a discrete operator is introduced, used in updating the interior boundary by interpolation. Let $\tau_{\Omega_j \to \Gamma_i}$, $i \neq j$, represent an operator which interpolates values from the interior points of Ω_j to the interior boundary Γ_i. From the DRM formulation in (5), for subproblem i we have

$$H_i u_i - G_i q_i = f_i \tag{9}$$

where

$$f_i = (H_i \hat{U}_i - G_i \hat{Q}_i) F_i^{-1} b_i.$$

In order to use the boundary condition, the matrices H_i are G_i are partitioned as

$$H_i = \begin{pmatrix} H_{\partial\Omega_i \setminus \Gamma_i} & H_{\Gamma_i} & H_{\Omega_i} \end{pmatrix}, \qquad G_i = \begin{pmatrix} G_{\partial\Omega_i} & 0 \end{pmatrix},$$

so are the vectors u_i and q_i. The portions of G_i associated with the interior points take zero due to the fact that the normal derivatives corresponding to interior points vanish. Employing the above partitions and using the interpolation operator τ, the system of equations (9) for subproblems (7) and (8) take the form

$$\begin{array}{lr} \begin{pmatrix} H_{\partial\Omega_1 \setminus \Gamma_1} & H_{\Gamma_1} & H_{\Omega_1} \end{pmatrix} \begin{pmatrix} u^n_{\partial\Omega_1 \setminus \Gamma_1} \\ u^n_{\Gamma_1} \\ u^n_{\Omega_1} \end{pmatrix} + \begin{pmatrix} G_{\partial\Omega_1} & 0 \end{pmatrix} \begin{pmatrix} q^n_{\partial\Omega_1} \\ 0 \end{pmatrix} = f_1 & \text{in} \quad \Omega_1 \\ u^n_{\partial\Omega_1 \setminus \Gamma_1} = \bar{u} & \text{on} \quad \partial\Omega_1 \setminus \Gamma_1 \\ u^n_{\Gamma_1} = \tau_{\Omega_2 \to \Gamma_1} u^{n-1}_{\Omega_2} & \text{on} \quad \Gamma_1. \end{array} \tag{10}$$

$$\begin{array}{lr} \begin{pmatrix} H_{\partial\Omega_2 \setminus \Gamma_2} & H_{\Gamma_2} & H_{\Omega_2} \end{pmatrix} \begin{pmatrix} u^n_{\partial\Omega_2 \setminus \Gamma_2} \\ u^n_{\Gamma_2} \\ u^n_{\Omega_2} \end{pmatrix} + \begin{pmatrix} G_{\partial\Omega_2} & 0 \end{pmatrix} \begin{pmatrix} q^n_{\partial\Omega_2} \\ 0 \end{pmatrix} = f_2 & \text{in} \quad \Omega_1 \\ u^n_{\partial\Omega_2 \setminus \Gamma_2} = \bar{u} & \text{on} \quad \partial\Omega_2 \setminus \Gamma_2 \\ u^n_{\Gamma_1} = \tau_{\Omega_1 \to \Gamma_2} u^{n}_{\Omega_1} & \text{on} \quad \Gamma_2, \end{array} \tag{11}$$

where the restrictions $u_2^{n-1}|_{\Gamma_1}$ and $u_1^n|_{\Gamma_2}$ have been written in terms of the operation τ. Let $X_i = \begin{pmatrix} q_i \\ u_{\Omega_i} \end{pmatrix}$ be the vector containing of the unknown values and $A_i = \begin{pmatrix} G_{\partial\Omega_i} & H_{\Omega_i} \end{pmatrix}$ the coefficient matrix associated with X_i, then after enforcing the boundary condition and transferring the known values to the right side, the iterative equations (10) and (11) are reduced to

$$\begin{cases} A_1 X_1^n = f_1 - H_{\partial\Omega_1 \setminus \Gamma_1} \bar{u} - H_{\Gamma_1} \tau_{\Omega_2 \to \Gamma_1} u^{n-1}_{\Omega_2}, \\ A_2 X_2^n = f_2 - H_{\partial\Omega_2 \setminus \Gamma_2} \bar{u} - H_{\Gamma_2} \tau_{\Omega_1 \to \Gamma_2} u^{n}_{\Omega_1}. \end{cases} \tag{12}$$

On more simplification, let

$$\tilde{f}_1 = f_1 - H_{\partial\Omega_1 \setminus \Gamma_1} \bar{u},$$

$$\tilde{f}_2 = f_2 - H_{\partial\Omega_2 \setminus \Gamma_2} \bar{u},$$

then equations (12) may be written as

$$\left\{\begin{array}{l} A_1 X_1^n = \tilde{f}_1 - H_{\ \Gamma_1} \tau_{\Omega_2 \to \Gamma_1} u_{\Omega_2}^{n-1}, \\ \\ A_2 X_2^n = \tilde{f}_2 - H_{\ \Gamma_2} \tau_{\Omega_1 \to \Gamma_2} u_{\Omega_1}^{n}. \end{array}\right. \tag{13}$$

For the methods such as finite difference in which the normal derivatives are not involved in the discretized PDE, the corresponding iterative equations as in (13) is equivalent to the block Gauss-Seidel method (see [15]). The main difference here is existence of the q values which are not active in updating the u_{Γ_i} consequently the interpolation operator does not depend on $q_{\partial\Omega_i}$. However, by introducing a new operator on X_i, which acts the same as τ but not active on $q_{\partial\Omega_i}$, we show that (13) is equivalent to a block Gauss-Seidel method. The suitable operator is $T_{\Omega_i \to \Gamma_j} = \begin{pmatrix} 0 & \tau_{\Omega_i \to \Gamma_j} \end{pmatrix}$ for which

$$T_{\Omega_i \to \Gamma_j} X_i^n = \begin{pmatrix} 0 & \tau_{\Omega_i \to \Gamma_j} \end{pmatrix} \begin{pmatrix} q_{\partial\Omega_i} \\ u_{\Omega_i}^n \end{pmatrix} = \tau_{\Omega_i \to \Gamma_j} u_{\Omega_i}^n.$$

Equations (13) in terms of the new operator become

$$\left\{\begin{array}{l} A_1 X_1^n = \tilde{f}_1 - H_{\ \Gamma_1} T_{\Omega_2 \to \Gamma_1} X_{\Omega_2}^{n-1}, \\ \\ A_2 X_2^n = \tilde{f}_2 - H_{\ \Gamma_2} T_{\Omega_1 \to \Gamma_2} X_{\Omega_1}^{n}. \end{array}\right. \tag{14}$$

which is a block Gauss-Seidel method for the linear system

$$\begin{pmatrix} A_1 & H_{\ \Gamma_1} T_{\Omega_2 \to \Gamma_1} \\ \\ H_{\ \Gamma_2} T_{\Omega_1 \to \Gamma_2} & A_2 \end{pmatrix} \begin{pmatrix} X_1 \\ \\ X_2 \end{pmatrix} = \begin{pmatrix} \tilde{f}_1 \\ \\ \tilde{f}_2 \end{pmatrix}.$$

By adding and substructing X_i^{n-1} in (14) the following iterative formula in terms of the residual is obtained,

$$\begin{array}{l} X_1^n \leftarrow X_1^{n-1} + A_1^{-1}(\tilde{f}_1 - A_1 X_1^{n-1} - H_{\ \Gamma_1} T_{\Omega_2 \to \Gamma_1} X_{\Omega_2}^{n-1}), \\ X_2^n \leftarrow X_2^{n-1} + A_2^{-1}(\tilde{f}_2 - A_2 X_2^{n-1} - H_{\ \Gamma_2} T_{\Omega_1 \to \Gamma_2} X_{\Omega_1}^{n}). \end{array} \tag{15}$$

The alternating Schwarz method can be constructed based on the block Jacobi, that is, updating the boundary conditions for the subdomains together and solving each subproblem independently. This method, of course, reduces the rate of convergence but is suitable for parallel computing.

3.3 Multi-domain DRM algorithms

Extending the alternating Schwarz method to the case of many subdomains is straightforward. Here we present the algorithms based on both Gauss-Seidel and Jacobi. Let the domain Ω be partitioned into p overlapping subdomains $\Omega_k, \quad k = 1, 2, \cdots, p$, such that $\Omega = \cup\Omega_k$. Also let S_i be the union of all subdomains Ω_j which overlap the subdomain Ω_i. The above algorithms can be presented as follows,

Algorithm 1 (An alternating Schwarz algorithm based on Gauss-Seidel)

1. *For domain* $\Omega_i, \quad i = 1, 2, \cdots, p,$

2. *update the boundary conditions for subproblem i, using the most recent solution from the subdomain $\Omega_j \in S_i$.*

3. *Solve the subproblem i to obtain the new solution,*

4. *If the desirable accuracy has been achieved stop otherwise go to step 1.*

It should be noted in the step 2 for the first iteration, an initial guess for the boundary conditions of subproblem 1 will be required. For the other subproblems a combination of initial guess and most recent solution will be used until the first iteration be completed. In the next iterations the boundary conditions will be updated using only the most recent solution.

Algorithm 2 (An alternating Schwarz algorithm based on Jacobi)

1. *Update the boundary condition for all the subdomains,*

2. *Solve all the subproblems*

3. *If the desirable accuracy has been achieved stop otherwise go to step 1.*

In the first iteration, the boundary conditions are determined, using an initial guess for all the subdomains and for the next iterations the solution from the previous iteration is used to update the boundary conditions. One can see that in this method, not taking advantage of the most recent solution in updating the boundary conditions, the convergence becomes slower in comparison with the previous method. However, the method is suitable for parallel implementation as each subproblem can be solved independently from the others.

One useful feature of these alternating Schwarz algorithms is that subdomains can have their own grids independently of each other. If the mesh points over the artificial boundaries coincide, generally nonmatching grids can work together easily.

3.4 Multiplicative and additive variants

For many applications where matching grids are be used, one may exploit the fact that all overlapped values in (15) in the overlapped region can be updated easily (in addition to the artificial boundary points). The outcome of this idea has been two well known approaches, usually called multiplicative and additive methods [16], which are based on Gauss-Seidel and Jacobi methods respectively.

Here we only present the compact matrix form of these two methods which are a modification of the iterative equations (15). For more details about constructing the methods, see [15]. The performance of multiplicative method can be described in two steps. In the first half iteration, the part of u in the first subdomain is updated. That is,

$$u^{n+1/2} \leftarrow u^n + \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} (f - Au^n),$$

where A is the global matrix obtained from the discretizing the underlying PDE, A_1 and A_2 are the same matrices as in (13). Similarly in the second half iteration,

$$u^{n+1} \leftarrow u^{n+1/2} + \begin{pmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{pmatrix} (f - Au^{n+1/2}),$$

the other part of u including the overlap, updated in the first step, is updated. Note that in each half step the local part of the global residual $(f - Au)$ is involved. A similar iterative method based on a generalization of block Jacobi known as additive method, can be given as

$$u^{n+1} \leftarrow u^n + \left(\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{pmatrix} \right) (f - Au^n).$$

Here the only difference is that the most recent solution is not used in the second half step. One can see that using these versions of Schwarz method considerably reduces the number of iterations consequently it has been recommended for the methods such as FEM and FDM.

However, we do not recommend these DDM variants for DRM due to the fact that constructing the global matrix A is costly, taking into account that in DRM each source point couples to all the boundary points. Without constructing small matrices, a DDM approach cannot offer the advantage over our proposed methods in §3.3.

4 Numerical results

In this section some numerical experiments are presented to demonstrate the performance of the proposed over-lapping domain decomposition DRM. The simple polynomial RBF, $1 + r^3$, is employed for the interpolation part of DRM. The numerical solutions are obtained for three test examples when the elements are approximated by linear functions and the collocation points are selected inside the elements, that is, discontinuous elements. The solution error is measured at the collocation points as L^2 norm of the error values.

In the first example the convergence behaviour of the

- new method — Algorithm 1 of §3.3

with regard to the size of overlap is tested. In Examples 2 and 3, the results of the new method are compared with the

- old method — Non-iterative MD-DRM algorithm of [11]

in cases of single and multi-domains; in both cases, the total number of points at subdomains including the overlap is equal to the number of points in the single domain. The comparison with MD-DRM (old method) is only illustrative as the number of internal points in a subdomain will be $n_i \approx n/p$ with p the number of subdomains (processors); in practice, one may use MD-DRM [11] with much more subdomains to allow a much smaller n_i.

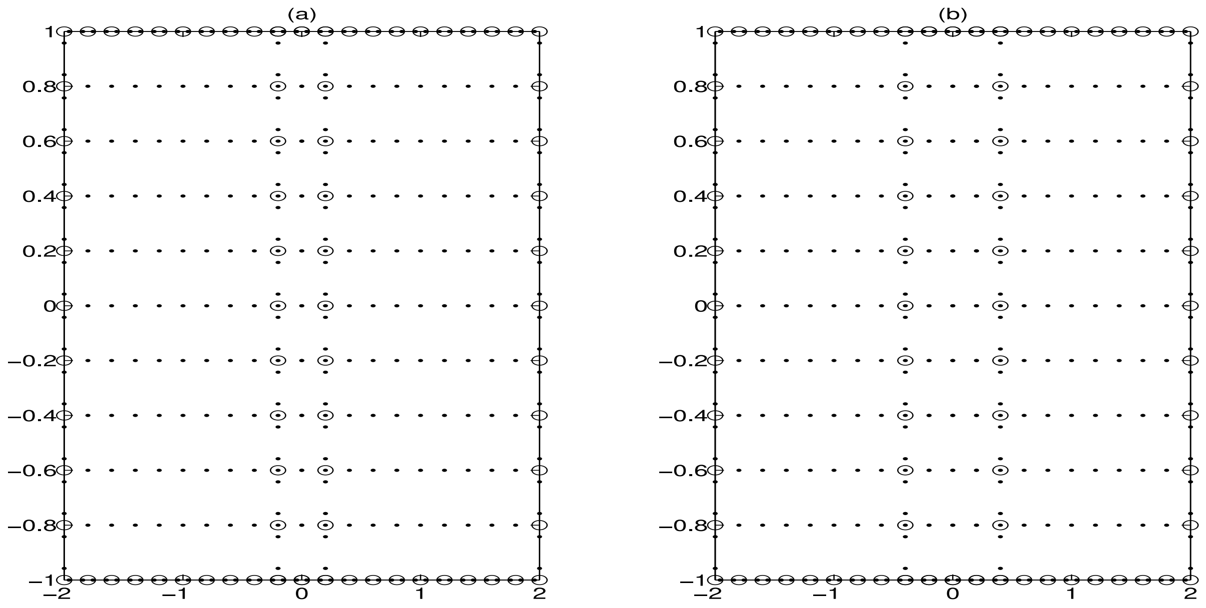
Example 1: In this example equation

$$\nabla^2 u + y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + xyu = d, \tag{16}$$

with a Dirichlet boundary condition is solved in the rectangle $\{(x, y), -2 \leq x \leq 2, -1 \leq y \leq 1\}$. In equation (16) d has been selected such that the exact solution is $u = \exp(-x) + \exp(-y)$. Two sort of splitting the domain with different amount of overlap is examined (Fig 4). Table 1 shows the numerical results for different number of nodes, using different partitioning.

As one can see, the number of iterations for convergence to the solution is only dependent on the amount of the overlap. It turns out that this is a general behaviour of the overlapping Schwarz method independent of the numerical method as it has been indicated for finite difference method as well (see [15]).

Figure 4: Two types of partitioning with different amount of overlap are displayed in Figures (a) and (b).



Example 2: This example deals with the same equation as in (16) with solution function

$$u(x, y) = 0.5 - 0.5 \tanh(-4 + (4x - 2)^2 + (4y - 2)^2),$$

in the rectangle $\{(x, y), 0 \leq x \leq 1, 0 \leq y \leq 1\}$. The solution is harder than the one in the previous example and consequently a larger number of points is required to gain a desirable accuracy. The error values for the interior points are presented in three cases, single domain, the new overlapping method and the old non-overlapping method, in Table 2. A significant improvement in the accuracy is observed in the new method in comparison with the case of using single domain and the old non-overlapping method, especially in the case of using four and eight subdomains. We believe that this improvement is due to the nonsmooth nature of u which demands more interactions between subdomains. Our new method indeed interacts more between subdomains than the old method.

However, more the number of subdomains is, more the number of iterations is required to converge to the solution. This is a typical behaviour in the iterative DDM since the communication between the subproblems becomes slower when more number of subdomains are involved.

Example 3: This example involves a convection diffusion equation defined over a unit square

$$D \bigtriangledown^2 u - V_x \frac{\partial u}{\partial x} - ku = 0$$

where u (kgm^{-3}) is the concentration of the transport of a substance, V_x (ms^{-1}) is the velocity field in the x direction, depending on the y coordinate,

$$V_x(y) = \frac{\lambda^2}{c_2}(y - 0.5)^2$$

with $\lambda = k - c_2^2$ and $c_2 = \log[c(1,0)/c(0,0)]$ and k (s^{-1}) is the reaction constant. The boundary conditions are :

$$c(0, y) = 300 \exp\left[\frac{\lambda}{2}(y^2 - y)\right], \qquad c(1, y) = 10 \exp\left[\frac{\lambda}{2}(y^2 - y)\right],$$

and

$$\frac{\partial u}{\partial y}|_{y=0} = -150\lambda \exp(c_2 x), \qquad \frac{\partial u}{\partial y}|_{y=1} = 150\lambda \exp(c_2 x).$$

For the case $D = 1$ ($m^2 s^{-1}$) and $k = 50$, the exact solution for the above equation is given by

$$c = 300 \exp\left[\frac{\lambda}{2}(y^2 - y)\right] \exp(c_2 x).$$

Again the problem was solved by three methods as in Example 2. This equation has been already solved in [11] using relative error at the boundary points. In order to compare the performance of the new method with old one the L^2 norm of relative error for the boundary points are measured. The error values for different number of subdomains are presented in Table 3.

From Table 3, one can observe that the overall results using multi-domains are better than applying single domain. The two multi-domain algorithms performed similarly. This may be due the smooth nature of the solution u which does not require many interactions between subdomains. In particular more improvement is achieved when applying the new overlapping method for a larger number of subdomains. However, the accuracy in the other cases is reasonable taking into account the considerable saving in the cost, indicated in Figure 2.

5 Conclusions

A development of the overlapping Schwarz method was presented for the dual reciprocity method to improve its accuracy and efficiency. A considerable improvement in the accuracy was observed when the method was applied to Poisson and convection diffusion equations. In particular, more accuracy were achieved for the Poisson type equations when a larger number of subdomains was employed, although slightly more iterations were demanded as expected. Numerical experiments show that our new method is more robust than the old non-overlapping method, yielding at least the same accuracy in all tests but requiring less computational work.

References

[1] Brebbia C. A. and Dominguez J., *Boundary Elements — An Introductory Course*, Computational Mechanics Publications, McGraw-Hill, 1989.

[2] Florez W. F., Power H. and Chejne F., *Multi-domain dual reciprocity BEM approach for the Navier-Stokes system of Equations*, Commun. Numer. Meth. Engng., 16, pp. 671-681, 2000.

[3] Florez W. F., Power H. and Janna F. C., *Multi-domain dual reciprocity for solution of inelastic non-Newtonian flow problems at low Reynolds number*, Comput. Mech., 27, pp. 396-411, 2001.

[4] Florez W. F. and Power H., *DRM Multi-domain Mass Conservative Interpolation Approach for the BEM Solution of the Two-Dimensional Navier-Stokes Equations*, Computers and Mathematics with Applications, 43, pp. 457-472, 2002.

[5] Golberg M., *The method of fundamental solutions for Poissons equations*, Engineering Analysis with boundary Elements, 16, pp.205-213, 1995.

[6] Kansa E. J., Carlson R. E. and Chejne F., *Radial basis functions: A class of grip-free, scattered data approximations*, Comput. Fluid Dyn. J., 3/4, pp. 479-496, 1995.

[7] Karur S. R. and Ramachandran P. A., *Radial Basis Function Approximation in Dual Reciprocity Method*, Math. Comput. Modelling, Vol. 20, No. 7, pp. 59-70, 1994.

Table 1: The error values for u and q, indicated by e_u and e_q, is given for Example 1 in the case of using different overlap with different iterations.

		Overlap (a)		Overlap (b)	
No. Of nodes	Iteration	e_u	e_q	e_u	e_q
115	1	2.2E-1	3.0E0	1.6E-1	2.4E0
	2	1.8E-2	7.5E-2	9.5E-4	2.4E-2
	3	1.2E-3	3.4E-2	9.0E-4	2.5E-2
	4	9.8E-4	3.0E-2	9.0E-4	2.5E-2
220	1	3.5E-1	3.6E0	2.4E-1	3.35E0
	2	4.1E-2	1.3E-1	8.1E-3	2.0E-2
	3	4.4E-3	2.4E-2	3.1E-4	1.9E-2
	4	3.3E-4	2.1E-2	2.4E-4	1.9E-2

Table 2: The error values for potential function u in the case of using different number of subdomains with the number of iterations are listed for Example 2.

No. of nodes: 520				
No. Of Iterations	No. of subdomains	New method	Old method	Single domain
6	2	1.5E-2	6.1E-2	1.2E-2
10	4	6.2E-3	3.6E-2	
14	8	6.0E-3	5.8E-2	

Table 3: The relative error for potential function u on the boundary in the case of using different number of subdomains are listed for Example 3.

No. of nodes: 1080				
No. Of Iterations	No. of subdomains	New method	Old method	Single domain
5	2	4.7E-3	6.9E-3	1.0E-2
6	3	2.7E-3	4.7E-3	
7	4	6.0E-3	8.1E-3	

[8] Li J. C., Hon Y. C. and Chen C. S., *Numerical comparisons of two meshless methods using radial basis functions*, Engineering Analysis with Boundary Elements, 26, pp. 205-225, 2002.

[9] Partridge P. W., Brebbia C. A. and Wrobel L. C., *The Dual Reciprocity Boundary Element Method*, Computational Mechanics Publications, UK, 1992.

[10] Popov V. and Power H., *A domain decomposition in the dual reciprocity approach*, Bound. Elem. Commun., 7, pp.1-5, 1996.

[11] Popov V. and Power H., *DRM-MD approach for the numerical solution of gas flow in porous media with application to landfill*, Engineering Analysis with boundary Elements, 1998.

[12] Popov V. and Power H., *The DRM-MD integral equation method*, Int. J. Numer. Meth. Engng., 44, pp.327-353, 1999.

[13] Powell M. J. D., *The Theory of Radial Basis Function Approximation in 1990, in:* Advances in Numerical Analysis, Vol. II, Oxford Science Publications. Oxford. 1992.

[14] Power H. and Mingo R., *The DRM subdomain decomposition approach to solve the two-dimensional Navier-Stokes system of equations*, Engineering Analysis with boundary Elements, 2000.

[15] Smith B. F., Bjorstad P. E. and Gropp W. D., *Domain Decomposition parallel multilevel methods for elliptic partial differential equations*, Cambridge University Press, 1996.

[16] Xu J. C., *Iterative methods by space decomposition and subspace correction*, SIAM Rev., 34, pp.581-613, 1992.