# EFFICIENT PRECONDITIONERS for ITERATIVE SOLUTION of THE BOUNDARY ELEMENT EQUATIONS for THE THREE DIMENSIONAL HELMHOLTZ EQUATION

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**Abstract**

In this paper two types of local sparse preconditioners are generalised to solve three-dimensional Helmholtz problems iteratively. The iterative solvers considered are the conjugate gradient normal method (CGN) and the generalised minimal residual method (GMRES).

Both types of preconditioners can ensure a better eigenvalue clustering for the normal equation matrix and thus a faster convergence of CGN. Clustering of the eigenvalues of the preconditioned matrix is also observed. We consider a general surface configuration approximated by piecewise quadratic elements defined over unstructured triangular partitions. We present some promising numerical results.

# 1 Introduction

The problem of time harmonic acoustic radiation or scattering by a three dimensional (3D) structure immersed in an *infinite* homogeneous acoustic medium can be modelled by a Helmholtz equation. The differential equation in such a 3D infinite domain $\Omega_+$ is usually re-formulated as a boundary integral equation over the finite two-dimensional surface $S$ of the structure that is in turn solved by a boundary element method [2, 15]. For the exterior problem that we consider here, the usually popular method of the finite elements is not suitable for discretising an infinite domain of interest.

Boundary element techniques generate fully populated and in general unsymmetric linear systems of equations when collocation is used. As is known, the alternative and mathematically attractive approach is the Galerkin method but this approach is much more expensive to implement. Therefore the collocation method has been the practical choice of the engineering community and our preconditioners can be potentially combined with many existing software with ease.

For matrices with large sizes and from discretized singular operators, iterative methods with suitable preconditioners are necessary. Here we consider two conjugate gradient type solvers – the conjugate gradient normal method (CGN) [6] and the generalised minimal residual method (GMRES) [19]. There exist many choices of sparse preconditioners for preconditioning boundary element equations [3, 9, 20]. However most have only been analysed or tested on two dimensional (2D planar) problems. We concentrate on 3D exterior Helmholtz problems with Neumann's boundary condition defined in the exterior $\Omega_+$ of a general and closed 3D surface $S$.

As we know, for a given matrix $A$, $\lambda(A^\top A) = \sigma(A)^2$ ie. the singular values and the eigenvalues of its normal matrix are related. A clustering distribution of the singular values will ensure fast convergence of CGN and is also observed to give fast convergence of GMRES [17, 19, 18]. Theoretically, clustering eigenvalues of the preconditioned matrix are only sufficient to guarantee fast convergence of GMRES if the matrix can be assumed to be diagonalizable. In practice, we must consider the sensitivity of eigenvalues of the underlying unsymmetric matrix under small perturbations by studying its pseudo-eigenvalues. We have found that our preconditioned matrices do not have sensitive eigenvalues. In this paper, we shall mainly use preconditioning to achieve such a clustering of the eigenvalues and singular values.

Two types of local preconditioners, based on generalising some 2D results, are proposed here: the operator splitting type and the approximate inverse type. By 'local' we mean that preconditioners are constructed using local elements and geometrical information and involving inexpensive matrix inversion. With the surface configuration approximated by piecewise quadratic (or linear) elements, over unstructured triangular partitions, the former type admits a direct operator splitting and the latter admits an indirect one. Both types of preconditioners can ensure a better eigenvalue clustering for the normal matrix as well as the preconditioned matrix, and thus a faster convergence of iterative methods. In terms of implementation, both edge and vertex based splittings correspond to the tridiagonal preconditioner in the 2D case. The effectiveness of our proposed preconditioners is illustrated in Figures 4-5. Numerical experiments for some typical wavenumbers [2] will be presented.

We remark that a potentially useful application of local preconditioners is to combine them with other fast methods such as the fast multiple method (FMM) where the available 'near-field' elements provide sufficient information for preconditioner construction. It would also be of interest to explore the inherent parallelism of collocation method, iterative solvers and our local preconditioners. As our aim is to achieve fast convergence (i.e. fewer number of convergence steps), pursuit of these applications is beyond this paper.

# 2 BEM for the 3D Helmholtz equation

Consider the problem of solving the Helmholtz (or reduced wave) equation

$$\nabla^2 \phi + k^2 \phi = 0 \tag{1}$$

where $k$ is a known constant, called the wavenumber, in the region $\Omega_+$ exterior to some closed, simply connected domain $\Omega_-$ and subject to a Neumann boundary condition on $S$, the surface of $\Omega_-$, and the Sommerfeld radiation condition

$$\lim_{r=|\underline{r}|\to\infty} r \left\{ \frac{\partial \phi(r)}{\partial r} - ik\phi(r) \right\} = 0. \tag{2}$$

In modelling the sound field radiated or scattered by a structure immersed in an acoustic medium, where the wavenumber $k$ is the ratio of the angular frequency to the speed of sound in the medium, one needs to solve (1) for a range of $k$ values and this requires a fast solver.

One of the most successful methods for solving this problem is the boundary element method where the problem of determining the solution to the differential equation in the unbounded three-dimensional region exterior to the structure is reformulated as an integral equation over the two-dimensional finite surface of the structure. There are two methods for reformulating the Helmholtz equation as an integral equation, the layer potential method and the Green's theorem method, which is used in this work.

Using Green's second theorem it is possible to show that $\phi$ and its normal derivative $\frac{\partial \phi}{\partial n}$ satisfy [2, 5, 11, 15]

$$\int_S \left\{ \phi(q) \frac{\partial G_k(p,q)}{\partial n_q} - G_k(p,q) \frac{\partial \phi(q)}{\partial n_q} \right\} dS_q = \begin{cases} 0, & p \in \Omega_- \\ \frac{1}{2}\phi(p), & p \in S \\ \phi(p), & p \in \Omega_+ \end{cases} \tag{3}$$

where

$$G_k(p,q) = \frac{e^{ik|p-q|}}{4\pi|p-q|} \tag{4}$$

is the free-space Green's function for Helmholtz equation. The equation for $p \in S$ can be written in operator notation as

$$\left(-\frac{1}{2}\mathcal{I} + M_k\right)\phi = L_k \frac{\partial \phi}{\partial n} \tag{5}$$

where

$$\begin{aligned} L_k\sigma(p) &= \int_S G_k(p,q)\sigma(q)dS_q, \\ M_k\sigma(p) &= \int_S \frac{\partial G_k(p,q)}{\partial n_q}\sigma(q)dS_q. \end{aligned} \tag{6}$$

However, it is well known that for certain discrete values of the wavenumber $k$, called characteristic values at which resonance occurs for the corresponding interior problem, equation (5) does not have a unique solution, although it can be shown that the underlying differential equation does have a unique solution for all real and positive values of $k$ [5].

By differentiating (5) along the normal at $p$ it is possible to get

$$N_k\phi = \left(\frac{1}{2}\mathcal{I} + M_k^T\right)\frac{\partial \phi}{\partial n} \tag{7}$$

where

$$\begin{aligned} M_k^T\sigma(p) &= \int_S \frac{\partial G_k(p,q)}{\partial n_p}\sigma(q)dS_q, \\ N_k\sigma(p) &= \int_S \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q}\sigma(q)dS_q, \end{aligned} \tag{8}$$

which has a different set of characteristic wavenumbers from (5). Burton and Miller [4] showed that the integral equation

$$\mathcal{A}\phi = \left(-\frac{1}{2}\mathcal{I} + M_k + \alpha N_k\right)\phi = \left[L_k + \alpha\left(\frac{1}{2}\mathcal{I} + M_k^T\right)\right]\frac{\partial \phi}{\partial n} \tag{9}$$

has a unique solution for all real and positive $k$ provided that the imaginary part of the coupling parameter $\alpha$ is non-zero. In fact, it is possible to show that the almost-optimal value of $\alpha$ (which minimises the norm of the operator on the left hand side of (9)) is [1, 2]

$$\alpha = \begin{cases} i & k \leq 1 \\ \frac{i}{k} & k > 1 \end{cases} . \tag{10}$$

However, problems arise in the numerical solution of (9) as the integral operator $N_k$ is hyper-singular, unlike the other operators which are at worst weakly singular. Meyer et al [16] showed that

$$\begin{aligned} \int_S \phi(q) \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} dS_q &= \int_S \{\phi(q) - \phi(p)\} \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} dS_q \\ &\quad + k^2 \phi(p) \int_S G_k(p,q) \underline{n}_p . \underline{n}_q dS_q \end{aligned}$$

and if $\phi$ is approximated by a piecewise constant function then the first integral on the right-hand side will be zero in the neighbourhood of the singularity. The second integral in the right hand side is only weakly singular and can be evaluated using an appropriate quadrature rule [15].

In order to discretise (9), first the surface $S$ is approximated by $N$ surface elements $S_1, S_2, \ldots, S_N$. In this work each surface element is a quadratically curved triangular element where points $p$ in the element are interpolated using vertexes and mid-points (see Figure 1)

$$p = \sum_{j=1}^{6} p_j \Phi_j(u,v)$$

and $\Phi_j(u,v)$ are the standard quadratic basis functions. We then calculate quantities such as the unit normal using the interpolated surface [2]. Further, each element can be mapped onto a standard element which simplifies evaluating the various integrals. The collocation points $p_1, p_2, \ldots, p_N$ are chosen to be the centroid of each element, and both $\phi$ and $\frac{\partial \phi}{\partial n}$ are approximated by a constant value over each element. This leads to the discrete equivalent of (9) being

$$A\underline{\phi} = \left( -\frac{1}{2} I + \tilde{M} + \alpha \tilde{N} \right) \underline{\phi} = \left[ \tilde{L} + \alpha \left( \frac{1}{2} I + \tilde{M}^T \right) \right] \underline{v} \tag{11}$$

where $\underline{\phi}$ and $\underline{v}$ are the vectors of the values of $\phi$ and $\frac{\partial \phi}{\partial n}$ at the collocation points, and

$$
\begin{aligned}
&\tilde{L}_{ij} = \int_{S_j} G(p_i, q) dS_q, \qquad \tilde{M}_{ij} = \int_{S_j} \frac{\partial G(p_i, q)}{\partial n_q} dS_q, \\
&\tilde{M^T}_{ij} = \int_{S_j} \frac{\partial G(p_i, q)}{\partial n_p} dS_q, \\
&\tilde{N}_{ij} = \begin{cases} \displaystyle\int_{S_j} \frac{\partial^2 G_k(p_i, q)}{\partial n_p \partial n_q} dS_q, & i \neq j \\ \displaystyle k^2 \int_{S_j} G_k(p_i, q) \underline{n}_p . \underline{n}_q dS_q + \sum_{\ell=1, \ell \neq i}^{N} \tilde{N}_{i\ell}, & i = j \end{cases} .
\end{aligned}
\tag{12}
$$

Note that $\tilde{M^T}$ is not simply the matrix transpose of $\tilde{M}$ and the matrix $A$ is full and nonsymmetric. All the integrals in the definitions of the above matrices are weakly singular and so can be readily evaluated by appropriate quadrature rules. A complete discussion on the choice of quadrature rules is given in Amini et al [2].

# 3 Solution of an equivalent linear system

Denote the dense linear system (11) by

$$Ax = b.$$

We seek a preconditioning matrix $P^{-1}$ (with either $P$ or $P^{-1}$ sparse) such that the following system

$$P^{-1}Ax = P^{-1}b$$

can be solved efficiently by iterative methods. For unsymmetric conjugate gradient solvers, fast convergence is often seen with a clustering distribution of the eigenvalues and singular values [17, 18]. Using the operator splitting idea [9], we hope to split the operator $\mathcal{A}$ as $\mathcal{A} = \mathcal{D} + \mathcal{C}$ by domain decomposition such that $\mathcal{D}$ is a suitable bounded operator and $\mathcal{C}$ is a compact operator [8]. To proceed, as in [7, 8], we use the surface domain partition $S = \bigcup_{j=1}^{N} S_j$ to decompose the operator

$$
\mathcal{A} = \begin{pmatrix}
\mathcal{A}_{1,1} & \mathcal{A}_{1,2} & \mathcal{A}_{1,3} & \cdots & \mathcal{A}_{1,N} \\
\mathcal{A}_{2,1} & \mathcal{A}_{2,2} & \mathcal{A}_{2,3} & \cdots & \mathcal{A}_{2,N} \\
\mathcal{A}_{3,1} & \mathcal{A}_{3,2} & \mathcal{A}_{3,3} & \cdots & \mathcal{A}_{3,N} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\mathcal{A}_{N,1} & \mathcal{A}_{N,2} & \mathcal{A}_{N,3} & \cdots & \mathcal{A}_{N,N}
\end{pmatrix}
\tag{13}
$$

where in element $S_i$ operator $\mathcal{A}_{i,\ell}$ is the restriction of $\mathcal{A}$ over surface $S_\ell$.

We shall choose a bounded operator splitting $\mathcal{D}$ in order for it to give rise to a sparse matrix $D$ on discretisation. Then the new operator $\mathcal{D}^{-1}\mathcal{A} = \mathcal{I} + \mathcal{D}^{-1}\mathcal{C}$ will be a compact perturbation of the identity operator because a product of a bounded operator with a compact operator is still compact. Since compact operators have all eigenvalues clustered at most at point 0, eigenvalues of operator $\mathcal{D}^{-1}\mathcal{A}$ will cluster at 1. Furthermore eigenvalues of its normal operator also cluster at 1 because $\mathcal{D}^{-1}\mathcal{A}(\mathcal{D}^{-1}\mathcal{A})^* = \mathcal{I} + \mathcal{D}^{-1}\mathcal{C} + \mathcal{C}^*\mathcal{D}^{-*} + \mathcal{D}^{-1}\mathcal{C}\mathcal{C}^*\mathcal{D}^{-*}$. The reader is referred to Figures 4-5 for an illustration of this.

The properties of these continuous operators are inherited by the discrete operators if a consistent discretization scheme such as collocation is used. On discretization, with $P = D$, the preconditioned system $P^{-1}Ax = P^{-1}b$ has a new matrix with clustering eigenvalues at 1. Moreover, the singular values and the eigenvalues of the normal of this new matrix $P^{-1}A$ are also clustered at 1. Thus conjugate gradient methods will be expected to exhibit fast convergence. This differs from the idea of a bounded condition number which may not describe the convergence.

Based on previous 2D work [9], we propose the construction of $D$ by two detecting methods (§3.1) which in turn give rise to two preconditioning methods — local preconditioners (§3.2). Further we consider the approximate inverse based preconditioners using the results of these detecting methods (§3.3).

## 3.1 Determining the mesh neighbours

In this work the collocation points for the integral equation are the centroids of the elements, and so the problem of finding adjacent collocation points is the same as the problem of finding adjacent elements. For any given element we need to determine which other elements are adjacent to it. Here we have employed two strategies for finding the adjacent elements. The first is to find the elements which share an edge with the element of interest. Since we are working with quadratic triangular elements there will always be exactly 3 adjacent elements using this method (see Figure 1). The second method is to find the elements which have a vertex in common. This method will yield a much larger number of adjacent elements (see Figure 2 where there are 13 adjacent elements). In general, the exact number of adjacent elements will depend on a number of factors, including the mesh configuration and the way in which the mesh was originally generated. When using either strategy described above, elments adjacent to a given element can be found by simply searching through the list of elements.

Figure 1: Strategy 1 (a typical element on a 3D surface). Note: the centriod of the inner triangle is the collocation point and marked as "♦".

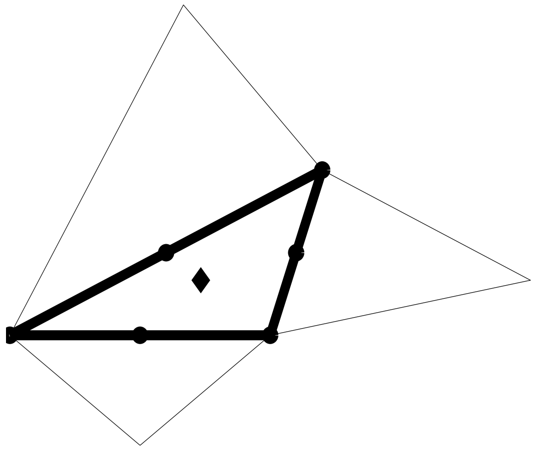


Figure 2: Strategy 2 (a typical element on a 3D surface)

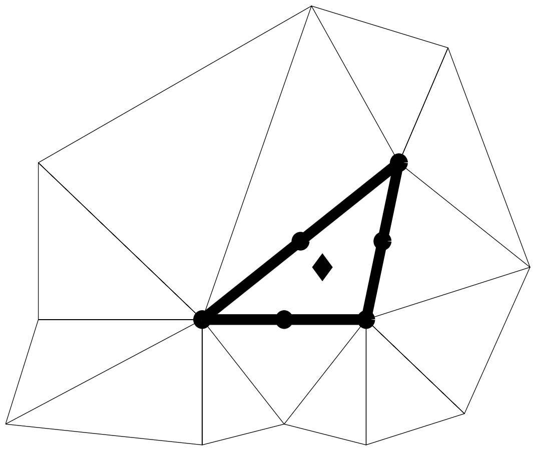
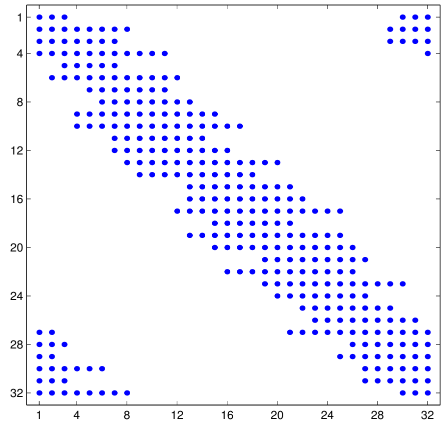


Figure 3: Illustration of matrix $P = D$ of Preconditioner II



## 3.2 Preconditioning methods I and II

The two detection methods described in §3.1 can be used to define two preconditioners.

**Preconditioner I.** Using the first detection method, each element has 3 neighbours (see Figure 1). The preconditioning matrix $P = D$ will be a band matrix having 4 nonzeros on each row and the exact sparse structure depends on the element ordering. For example one best possibility ($n = 8$) will be the following

$$D = \begin{bmatrix} x & x & x & 0 & 0 & 0 & 0 & x \\ x & x & x & x & 0 & 0 & 0 & 0 \\ 0 & x & x & x & x & 0 & 0 & 0 \\ 0 & 0 & x & x & x & x & 0 & 0 \\ 0 & 0 & 0 & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & x & x & x & x \\ x & 0 & 0 & 0 & 0 & x & x & x \\ x & x & 0 & 0 & 0 & 0 & x & x \end{bmatrix}.$$

**Preconditioner II.** Using the second detection method, each element $k$ will have a fixed number $\mu_k$ of neighbouring elements (see Figure 2 with $\mu = 13$). In the general case where obtuse triangles are allowed, we can verify that $\mu_k \geq 9$ while in the special case where only acute triangles are allowed $\mu_k \geq 12$. Therefore the preconditioning matrix $P = D$ will be a band matrix with a varying number of nonzeros on each row as illustrated in Figure 3 for $n = 32$.

*Remark.* With both preconditioning methods (I and II), it is worth noting that the ordering of the elements is important in order for the preconditioning matrix to be tightly banded. If possible the elements should be ordered so that the band-width of this matrix is minimised and hence making the process which is used to find its inverse as efficient as possible. One strategy that we found useful is to locate the *first* new element and then order elements according to their centroidal distance from the new element 1. For elements within the same distance, order them according to angular orientations with respect to a fixed coordinate direction.

## 3.3 Preconditioning methods III and IV

Sparse approximate inverse preconditioners have been proposed for preconditioning sparse linear systems in the literature; see [18] and the references therein for more details. Briefly the objective is to construct approximations, $P^{-1}$, to the inverse of matrix $A$ for which $\|AP^{-1} - I\|$ is small in some norm. Then $P^{-1}$ is used as a preconditioner. Let $P^{-1} = [m_1, \cdots, m_n]$ and $I = [e_1, \cdots, e_n]$. Using the usual Frobenius norm,

$$\min_{P^{-1}} \|AP^{-1} - I\|_F^2 = \sum_{j=1}^{n} \min_{m_j} \|Am_j - e_j\|_2^2,$$

where we let $\mathcal{S}$ define the sparsity pattern of $P^{-1}$. For each column $j$, $\min_{m_j} \|Am_j - e_j\|_2^2$ is a least squares problem for a $n \times k$ system where $k$ is the number of columns of $A$ involved via pattern $\mathcal{S}$. Following [20], we shall solve the approximate problem $\min_{m_j} \|\widetilde{A}_j m_j - e_j\|_2^2$ where $\widetilde{A}_j$ is the $k \times k$ principal submatrix of $A$ induced by the columns of $A$ specified by $\mathcal{S}$. As is known, the most important step is a specification of the sparsity pattern of $P^{-1}$. Here we use the two detection methods of §3.1 to define this pattern $\mathcal{S}$.

**Preconditioner III.** Using the first detection method, the sparse pattern $\mathcal{S}$ will be determined by matrix $D$ with 4 nonzeros in each column. Our preconditioner III will be $P^{-1} = [m_1, \cdots, m_n]$ found by solving $n$ $4 \times 4$ systems.

**Preconditioner IV.** Using the second detection method, the sparse pattern $\mathcal{S}$ will be determined by matrix $D$ with $\mu_k$ nonzeros in column $k$. Our preconditioner IV will be $P^{-1} = [m_1, \cdots, m_n]$ found by solving $n$ $\mu_k \times \mu_k$ systems for $k = 1, 2, \ldots, n$. For example, for the element illustrated in Figure 2, we solve a $13 \times 13$ system to find a column of $P^{-1}$.

## 3.4 Computational complexity of preconditioners

Different from the 2D case, the unstructured triangular meshes in 3D complicate the estimation of computational complexity of computing $P^{-1}\underline{y}$. However the total number of nonzeros — more precisely the nonzeros in matrix $P$ for I and II and in $P^{-1}$ for III and IV — in all of our 4 preconditioners can be predicted. For Preconditioners I and II, working out $P$ takes almost no effort, as observed in our experiments, provided that the elements have been suitably ordered. But we need to do an one-off sparse LU decomposition before using it for preconditioning and this should be done in consideration of the sparse nature of $P$ otherwise Preconditioners I and II can be more expensive than III-IV. For III and IV, working out $P^{-1}$ takes $O(n) = c_1 n$ operations but forming $P^{-1}\underline{y}$ takes about $O(n) = c_2 n$ operation with $c_2$ much less than $c_1$. For example, for III, $c_1 \approx 4^3/3 = 21$ and $c_2 = 4$.

# 4 Numerical results

Test problems with exact solutions can be generated by considering problems that are equivalent to those having acoustic point sources in the interior domain $\Omega_-$. The acoustic pressure at a point $p$ due to a set of points sources located at $q_j \in \Omega_-$ with strengths $A_j$, $j = 1, 2, \ldots Q$ in the absence of the structure is

$$\phi(p) = \sum_{j=1}^{Q} A_j \frac{e^{ik|p-q_j|}}{4\pi|p-q_j|} \quad \text{and} \quad \frac{\partial \phi}{\partial n_p} = \sum_{j=1}^{n} A_j \frac{\partial}{\partial n_p} \left( \frac{e^{ik|p-q_j|}}{4\pi|p-q_j|} \right) \tag{14}$$

where the second equation is obtained by differentiating the first one in the normal direction. The acoustic problem with this distribution of $\frac{\partial \phi}{\partial n}$ on the surface $S$ is equivalent to the point source problem for which the solution is known.

The problems considered here are for 2 domains: (i) Problem 1, $\Omega_-$: a unit sphere and (ii) Problem 2, $\Omega_-$: a cylinder of radius 0.6 and height 1.8. For both structures, taking $Q = 2$, the point sources are located at $(0, 0, 0.5)$ and $(0.25, 0.25, 0.25)$ with strengths $2 + 3i$ and $4 - i$ respectively.

Numerical experiments were carried out for the above two problems with 4 wavenumbers $k = 1, 5, 10, 20$ and results of convergence steps to reduce the residual to below $\tau$ from using the CGN and GMRES(5), restarted GMRES with 5 vectors, are presented in Tables 1-2. We have chosen $\tau = 10^{-4}, \ldots, 10^{-7}$ for $n = 144, \ldots, 3840$ respectively. The local preconditioners I–IV (denoted by $P_1, P_2, P_3, P_4$ respectively) are those introduced in §3.2–3.3. Although we have used the same $n$ for different wavenumbers, in practice, one would increase $n$ for large $k$ to obtain the same accuracy. To illustrate the effectiveness of all four preconditioners, we plot in Figures 4-5 (the moduli of) eigenvalue and singular value distributions of the original matrix ($\lambda(A)$ and $\sigma(A)$) and preconditioned matrices ($\lambda(C)$ and $\sigma(C)$) for the case $k = 1$ and $n = 36$ and 144. The relatively better clustering patterns of $P_2$ and $P_4$ are seen with faster convergence in the following experiments.

The two tables show the CPU time (in seconds) required for each converged solution with iteration steps displayed in round brackets. For all preconditioners the CPU time

Figure 4: Eigenvalue and singular value distributions of the original matrix and preconditioned matrices ($k = 1$ and $n = 36$).

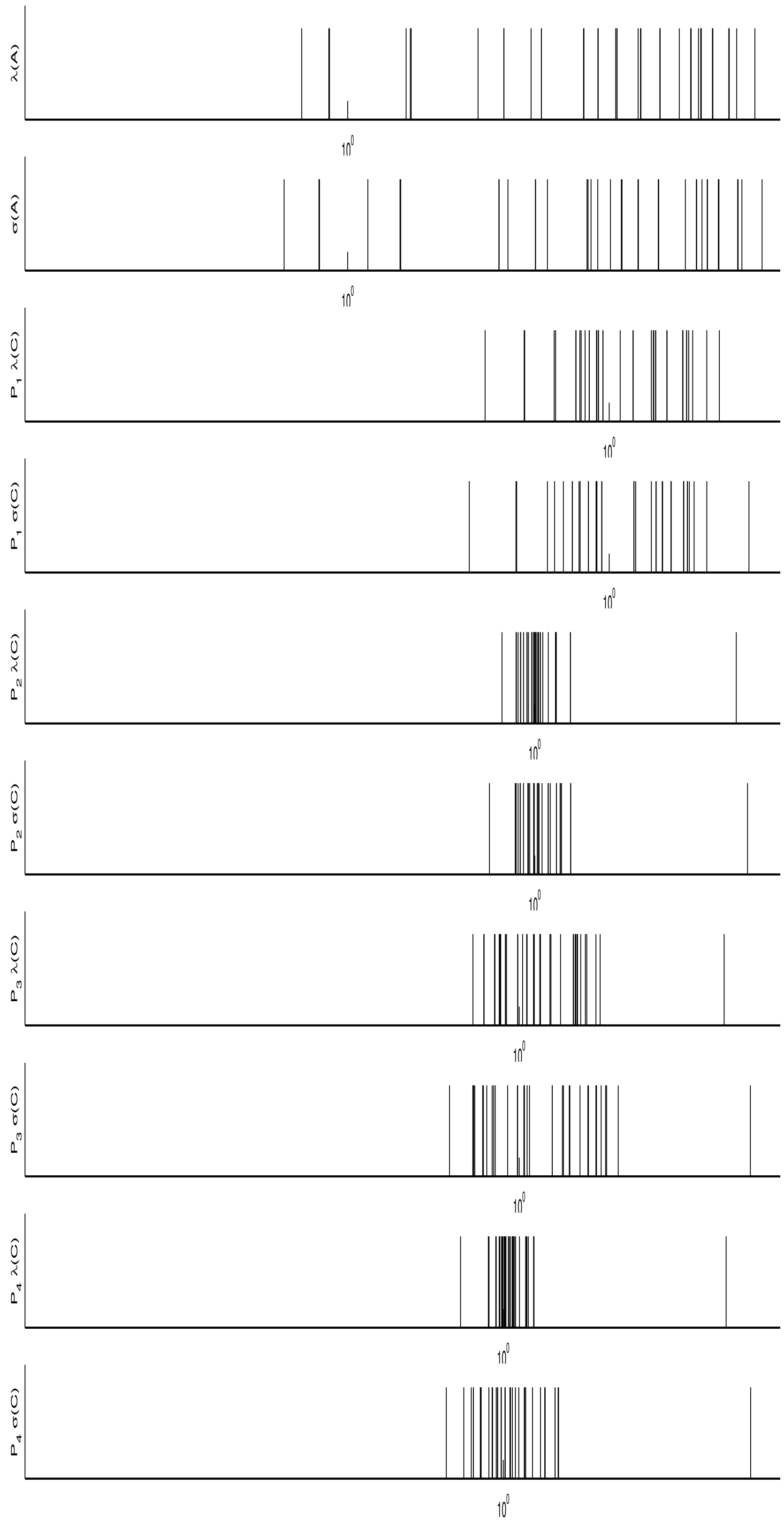


Figure 5: Eigenvalue and singular value distributions of the original matrix and preconditioned matrices ($k = 1$ and $n = 144$).

include the set up time for a fair comparison. We have used the Fortran 77 programming language and taken the CPU timings from a Sun Ultra 60 workstation in double precision.

From these two tables, we can observe that all new sparse preconditioners out perform the direct method by a large factor (say up to 60). Among the 4 preconditioners, $P_2$ and $P_4$ are consistently better than $P_1$ and $P_3$. In other experiments, we have tried to compare with the following preconditioning cases: (i) no preconditioning; (ii) the diagonal preconditioner; (iii) the tridiagonal preconditioner, and found that these do not produce competitive results to our preconditioners. We have also experimented with the re-start parameter $K$ for GMRES($K$) in the range $5 \leq K \leq 50$ and found that our preconditioners are not sensitive to any change. This somewhat confirms our belief that in general preconditioners are important than iterative solvers in achieving fast convergence.

# 5 Conclusions

We have proposed two methods (4 local preconditioners) for preconditioning 3D boundary element equations over unstructured triangular surface elements. The first method is based on operator splitting and the second method uses the idea of constructing approximate sparse inverses that admits an implicit operator splitting. The effectiveness of the new preconditioners is confirmed by numerical experiments solving exterior Helmholtz equations. All preconditioners perform much better than the simple band preconditioners as well the direct solution. Further work will involve a combination with other types of fast methods [14, 10, 13] and tackle the more complex fluid structure interaction problems [2, 12].

# References


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Table 1: Performance of preconditioners with GMRES(5) versus a direct method

| Problem (`k`) | $n$ | Direct | $P_1$ (#) | $P_2$ (#) | $P_3$ (#) | $P_4$ (#) |
|---|---|---|---|---|---|---|
| 1 (`1`) | 3840 | 23496 | 2085 (31) | 1396 (19) | 2685 (42) | 1768 (27) |
| | 2304 | 5397 | 532 (21) | 307 (10) | 609 (26) | 437 (18) |
| | 576 | 84 | 14 (8) | 11 (5) | 18 (12) | 12 (7) |
| | 144 | 1.2 | 0.5 (4) | 0.4 (2) | 0.5 (3) | 0.5 (3) |
| 1 (`5`) | 3840 | 23496 | 907 (12) | 629 (7) | 1093 (16) | 732 (10) |
| | 2304 | 5397 | 240 (8) | 187 (5) | 258 (10) | 170 (6) |
| | 576 | 84 | 8 (4) | 7 (2) | 7 (4) | 6 (3) |
| | 144 | 1.2 | 0.3 (2) | 0.4 (2) | 0.3 (2) | 0.4 (2) |
| 1 (`10`) | 3840 | 23496 | 665 (8) | 640 (4) | 666 (9) | 490 (6) |
| | 2304 | 5397 | 167 (5) | 140 (3) | 171 (6) | 104 (3) |
| | 576 | 84 | 5 (2) | 7 (2) | 4 (2) | 6 (2) |
| | 144 | 1.2 | 0.3 (2) | 0.3 (1) | 0.3 (2) | 0.3 (2) |
| 1 (`20`) | 3840 | 23496 | 414 (4) | 376 (3) | 423 (5) | 368 (4) |
| | 2304 | 5397 | 124 (3) | 117 (2) | 105 (3) | 104 (3) |
| | 576 | 84 | 5 (2) | 6 (2) | 4 (2) | 5 (2) |
| | 144 | 1.2 | 0.2 (1) | 0.3 (1) | 0.2 (1) | 0.3 (1) |
| 2 (`1`) | 3840 | 23496 | 2585 (39) | 1472 (20) | 3793 (60) | 1991 (30) |
| | 2304 | 5397 | 647 (26) | 421 (15) | 815 (36) | 488 (21) |
| | 576 | 84 | 19 (12) | 13 (6) | 21 (14) | 16 (10) |
| | 144 | 1.2 | 0.5 (4) | 0.5 (3) | 0.6 (6) | 0.5 (3) |
| 2 (`5`) | 3840 | 23496 | 1044 (14) | 705 (8) | 1165 (17) | 867 (12) |
| | 2304 | 5397 | 299 (11) | 236 (7) | 244 (14) | 235 (9) |
| | 576 | 84 | 9 (5) | 8 (3) | 10 (6) | 7 (4) |
| | 144 | 1.2 | 0.3 (2) | 0.4 (2) | 0.4 (3) | 0.4 (2) |
| 2 (`10`) | 3840 | 23496 | 684 (9) | 518 (5) | 859 (12) | 443 (7) |
| | 2304 | 5397 | 211 (7) | 168 (4) | 235 (9) | 170 (6) |
| | 576 | 84 | 7 (3) | 7 (2) | 7 (4) | 6 (3) |
| | 144 | 1.2 | 0.3 (2) | 0.4 (2) | 0.3 (2) | 0.4 (2) |
| 2 (`20`) | 3840 | 23496 | 435 (5) | 412 (4) | 463 (6) | 341 (4) |
| | 2304 | 5397 | 182 (20) | 189 (18) | 175 (22) | 171 (21) |
| | 576 | 84 | 5 (2) | 7 (2) | 5 (3) | 5 (2) |
| | 144 | 1.2 | 0.3 (2) | 0.4 (2) | 0.3 (2) | 0.4 (2) |

Table 2: Performance of preconditioners with CGN versus a direct method

| Problem (`k`) | $n$ | Direct | $P_1$ (#) | $P_2$ (#) | $P_3$ (#) | $P_4$ (#) |
|---|---|---|---|---|---|---|
| 1 (`1`) | 3840 | 23496 | 1611 (81) | 948 (42) | 2033 (109) | 1298 (67) |
| | 2304 | 5397 | 384 (50) | 242 (25) | 472 (67) | 299 (40) |
| | 576 | 84 | 11 (21) | 8 (11) | 12 (28) | 8 (17) |
| | 144 | 1.2 | 0.4 (9) | 0.3 (5) | 0.4 (12) | 0.4 (7) |
| 1 (`5`) | 3840 | 23496 | 639 (27) | 470 (16) | 763 (37) | 502 (22) |
| | 2304 | 5397 | 156 (16) | 149 (12) | 172 (21) | 123 (13) |
| | 576 | 84 | 6 (10) | 8 (9) | 5 (11) | 6 (11) |
| | 144 | 1.2 | 0.3 (7) | 0.4 (6) | 0.3 (9) | 0.3 (6) |
| 1 (`10`) | 3840 | 23496 | 478 (18) | 452 (15) | 480 (21) | 430 (18) |
| | 2304 | 5397 | 137 (13) | 149 (12) | 126 (14) | 129 (14) |
| | 576 | 84 | 6 (9) | 7 (8) | 6 (12) | 5 (9) |
| | 144 | 1.2 | 0.3 (6) | 0.4 (6) | 0.3 (7) | 0.3 (6) |
| 1 (`20`) | 3840 | 23496 | 424 (15) | 438 (14) | 430 (18) | 485 (21) |
| | 2304 | 5397 | 132 (12) | 142 (11) | 135 (15) | 125 (13) |
| | 576 | 84 | 6 (9) | 6 (8) | 5 (9) | 5 (9) |
| | 144 | 1.2 | 0.3 (5) | 0.3 (5) | 0.2 (5) | 0.3 (5) |
| 2 (`1`) | 3840 | 23496 | 1813 (91) | 1015 (45) | 2252 (120) | 1452 (74) |
| | 2304 | 5397 | 463 (63) | 283 (31) | 565 (83) | 363 (51) |
| | 576 | 84 | 13 (26) | 9 (13) | 15 (35) | 11 (22) |
| | 144 | 1.2 | 0.4 (11) | 0.4 (6) | 0.5 (15) | 0.4 (9) |
| 2 (`5`) | 3840 | 23496 | 738 (32) | 515 (18) | 856 (42) | 559 (25) |
| | 2304 | 5397 | 207 (24) | 172 (15) | 232 (31) | 159 (19) |
| | 576 | 84 | 7 (13) | 8 (12) | 7 (14) | 7 (13) |
| | 144 | 1.2 | 0.4 (10) | 0.4 (8) | 0.4 (12) | 0.5 (10) |
| 2 (`10`) | 3840 | 23496 | 515 (21) | 552 (21) | 526 (24) | 492 (22) |
| | 2304 | 5397 | 168 (18) | 185 (17) | 149 (18) | 159 (19) |
| | 576 | 84 | 6 (9) | 7 (8) | 6 (12) | 5 (9) |
| | 144 | 1.2 | 0.4 (11) | 0.5 (10) | 0.4 (12) | 0.5 (12) |
| 2 (`20`) | 3840 | 23496 | 570 (24) | 570 (22) | 546 (25) | 567 (26) |
| | 2304 | 5397 | 182 (20) | 189 (18) | 175 (22) | 171 (21) |
| | 576 | 84 | 9 (17) | 11 (17) | 8 (18) | 9 (19) |
| | 144 | 1.2 | 0.5 (13) | 0.6 (12) | 0.4 (13) | 0.5 (12) |

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