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Higher-Order Corrections in Threshold Resummation

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Abstract

We extend the threshold resummation exponents G^N in Mellin-N space to the fourth logarithmic ($\mathrm{N^3LL}$) order collecting the terms $\alpha_s^2(\alpha_s \ln N)^n$ to all orders in the strong coupling constant α_s. Comparing the results to our previous three-loop calculations for deep-inelastic scattering (DIS), we derive the universal coefficients B_q and B_g governing the final-state jet functions to order α_s^3, extending the previous quark and gluon results by one and two orders. A curious relation is found at second order between these quantities, the splitting functions and the large-angle soft emissions in Drell-Yan type processes. We study the numerical effect of the $\mathrm{N^3LL}$ corrections using both the fully exponentiated form and the expansion of the coefficient function in towers of logarithms.

1 Introduction

Coefficient functions, or partonic cross sections, form the backbone of perturbative QCD. These quantities are defined in terms of power expansions in the strong coupling constant α_s. In general, only a few terms in this expansion can be calculated. It is however possible, and necessary, to resum the dominant contributions to all orders in α_s close to exceptional kinematic points. Close to threshold, for example, where real emissions are kinematically suppressed, the resummation takes the form of an exponentiation in Mellin-N space [1–4], with the moments N defined with respect to the appropriate scaling variable, like Bjorken-x in deep-inelastic scattering (DIS) and $x_T = 2p_T/\sqrt{S}$ in direct photon and inclusive hadron production.

The resummation exponents are given by integrals over functions in turn defined by a power series in α_s. Besides by dedicated calculations, the corresponding expansion coefficients can be obtained by expanding the exponentials and comparing to the results of fixed-order calculations. Hence progress in the latter sector also facilitates improved resummation predictions. At present the next-to-leading order (NLO) is the standard approximation for many important observables, facilitating a resummation with next-to-leading logarithmic (NLL) accuracy. For recent introductory overviews see, for instance, Refs. [5, 6]. The next-to-next-to-leading order (NNLO) corrections have been completed so far only for the coefficient functions for inclusive lepton-proton DIS [7–11], the Drell-Yan process [12–14] and the related Higgs boson production [13, 15–17] in proton-proton collisions. Consequently, the threshold resummation has been carried out at the next-to-next-to-leading logarithmic (NNLL) accuracy only for these processes [18, 19].

Recently we have computed the complete three-loop coefficient functions for inclusive photon-exchange DIS [11, 20]. Moreover, in the course of the calculation of the third-order splitting functions governing the NNLO evolution of the parton distributions [21, 22], we have also computed DIS by exchange of a scalar directly coupling only to gluons. Together these results enable us to extend two more universal functions entering the resummation exponents, the quark and gluon jet functions B_q and B_g collecting final-state collinear emissions, to the third order in α_s. In fact, already the second-order coefficient for B_g represents a new result, relevant for future NNLL resummations of processes with final-state gluons at the Born level. Making use also of the results of Refs. [23, 24] we can furthermore effectively, i.e., up to the small contribution of the four-loop cusp anomalous dimension, extend the threshold resummation for inclusive DIS to an unprecedented next-to-next-to-next-to-leading logarithmic (N^{3}LL) accuracy.

The remainder of this article is organized as follows: after recalling the general structure of the resummation exponents in Section 2, we extend the required integrations in Section 3 to the fourth logarithmic (N^{3}LL) order. In Section 4 we determine the relevant expansion coefficients by comparison to our three-loop results for DIS and illustrate the numerical effect of the N^{3}LL contributions to the resummation exponent. In Section 5 we present the resulting predictions for the leading seven large-x terms of the four-loop coefficient function and discuss higher-order effects in terms of the expansion in towers of threshold logarithms. Our results are briefly summarized in Section 6. Some basic relations for the integrations of Section 2 can be found in the Appendix.

2 The general structure

For processes with only one colour structure at the Born level, the resummed Mellin-space coefficient functions C^N (defined in the $\overline{\mathrm{MS}}$ scheme) are given by a single exponential [1, 2]

$$C^N(Q^2)/C^N_{\mathrm{LO}}(Q^2) \;=\; g_0(Q^2)\cdot \exp\left[G^N(Q^2)\right] \,+\, O(N^{-1}\ln^n N)\;. \tag{2.1}$$

Here C^N_{LO} denotes the lowest-order coefficient function for the process under consideration, e.g., $C^N_{\mathrm{LO}} = 1$ for DIS. The prefactor g_0 collects, order by order in the strong coupling constant α_{s}, all N-independent contributions. The exponent G^N contains terms of the form $\ln^k N$ to all orders in α_{s}. Besides the physical hard scale Q^2 ($= -q^2$ in DIS, with q the four-momentum of the exchanged gauge boson), both functions also depend on the renormalization scale μ_r and the mass-factorization scale μ_f. The reference to these scales will be often suppressed for brevity.

The exponential in Eq. (2.1) is build up from universal radiative factors for each initial- and final-state parton p, Δ_{p} and J_{p}, together with a process-dependent contribution Δ^{int}. For example, the resummation exponents for inclusive deep-inelastic scattering, Drell-Yan (DY) lepton-pair production and direct photon production via $q\bar{q} \to g\gamma$ and $qg \to q\gamma$ [25] take the form

$$\begin{aligned}
G^N_{\mathrm{DIS}} \;&=\; \ln\Delta_{\mathrm{q}} \,+\, \ln J_{\mathrm{q}} \,+\, \ln\Delta^{\mathrm{int}}_{\mathrm{DIS}}\;, \\
G^N_{\mathrm{DY}} \;&=\; 2\,\ln\Delta_{\mathrm{q}} \,+\, \ln\Delta^{\mathrm{int}}_{\mathrm{DY}}\;, \\
G^N_{\mathrm{ab}\to\mathrm{c}\gamma} \;&=\; \ln\Delta_{\mathrm{a}} \,+\, \ln\Delta_{\mathrm{b}} \,+\, \ln J_{\mathrm{c}} \,+\, \ln\Delta^{\mathrm{int}}_{\mathrm{ab}\to\mathrm{c}\gamma}\;.
\end{aligned} \tag{2.2}$$

The radiation factors are given by integrals over functions of the running coupling. Specifically, the effects of collinear soft-gluon radiation off an initial-state parton $p = q, g$ are collected by

$$\ln\Delta_{\mathrm{p}}(Q^2,\mu_f^2) \;=\; \int_0^1 dz\, \frac{z^{N-1}-1}{1-z} \int_{\mu_f^2}^{(1-z)^2Q^2} \frac{dq^2}{q^2}\, A_{\mathrm{p}}(\alpha_{\mathrm{s}}(q^2))\;. \tag{2.3}$$

Collinear emissions from an ‘unobserved’ final-state parton lead to the so-called jet function,

$$\ln J_{\mathrm{p}}(Q^2) \;=\; \int_0^1 dz\, \frac{z^{N-1}-1}{1-z} \left[\int_{(1-z)^2Q^2}^{(1-z)Q^2} \frac{dq^2}{q^2}\, A_{\mathrm{p}}(\alpha_{\mathrm{s}}(q^2)) + B_{\mathrm{p}}(\alpha_{\mathrm{s}}([1-z]Q^2))\right]\;. \tag{2.4}$$

Finally the process-dependent contributions from large-angle soft gluons are resummed by

$$\ln\Delta^{\mathrm{int}}(Q^2) \;=\; \int_0^1 dz\, \frac{z^{N-1}-1}{1-z}\, D(\alpha_{\mathrm{s}}([1-z]^2Q^2))\;. \tag{2.5}$$

The functions g_0 in Eq. (2.1) and A_{p}, B_{p} and D in Eqs. (2.3) – (2.5) are given by the expansions

$$F(\alpha_{\mathrm{s}}) \;=\; \sum_{l=l_0}^{\infty} F_l\, \frac{\alpha_{\mathrm{s}}^l}{4\pi} \;\equiv\; \sum_{l=l_0}^{\infty} F_l\, a_{\mathrm{s}}^l\;, \tag{2.6}$$

where $l_0 = 0$ with $g_{00} = 1$ for $F = g_0$, and $l_0 = 1$ else. Here we have also taken the opportunity to specify the reduced coupling a_{s} employed for the rest of this article. The extent to which these functions are known defines the accuracy to which the threshold logarithms can be resummed.

The situation for inclusive DIS is actually simpler than indicated in Eq. (2.2), as the function (2.5) is found to vanish to all orders in α_s [23, 24],

$$D_k^{\rm DIS} \,=\, 0 \; , \quad \Delta_{\rm DIS}^{\rm int} \,=\, 1 \; . \tag{2.7}$$

On the other hand, the resummation of processes with four or more partons at the Born level, like inclusive hadron production in pp collisions [26], is more complicated than Eq. (2.1) due to colour interferences and correlations in the large-angle soft gluon emissions [3, 4]. However, the process-independent functions $A_{\rm p}$ and $B_{\rm p}$ retain their relevance also for such cases.

3 The resummation exponent to fourth logarithmic order

After performing the integrations in Eqs. (2.3) – (2.5), the function G^N in Eq. (2.2) takes the form

$$G^N(Q^2) \,=\, \ln N \cdot g_1(\lambda) \,+\, g_2(\lambda) \,+\, a_s g_3(\lambda) \,+\, a_s^2 g_4(\lambda) \,+\, \dots \; , \tag{3.1}$$

where $\lambda = \beta_0 a_s \ln N$. For the actual computation of the functions g_i it is convenient to employ the following representation for the scale dependence of a_s up to N^{3}LO:

$$\begin{aligned} a_s(q^2) \;=\;& \frac{a_s}{X} - \frac{a_s^2}{X^2}\frac{\beta_1}{\beta_0}\ln X + \frac{a_s^3}{X^3}\left[\frac{\beta_1^2}{\beta_0^2}(\ln^2 X - \ln X - 1 + X) + \frac{\beta_2}{\beta_0}(1-X)\right] \\ &+ \frac{a_s^4}{X^4}\left[\frac{\beta_1^3}{\beta_0^3}\left(2(1-X)\ln X + \frac{5}{2}\ln^2 X - \ln^3 X - \frac{1}{2} + X - \frac{1}{2}X^2\right)\right. \\ &\left. + \frac{\beta_3}{2\beta_0}(1-X^2) + \frac{\beta_1\beta_2}{\beta_0^2}(2X\ln X - 3\ln X - X(1-X))\right] \,+\, O(a_s^5) \end{aligned} \tag{3.2}$$

with $a_s \equiv a_s(\mu_r^2)$ and the abbreviation $X = 1 + a_s\beta_0 \ln(q^2/\mu_r^2)$. The terms up to the n-th order in a_s in Eq. (3.2) contribute to g_n in Eq. (3.1). Thus the calculation of g_4 requires the highest known coefficient of the beta function of QCD, β_3 [27, 28].

Generalizing the approach of Ref. [18], the functions $g_i(\lambda)$ can be obtained using well-known methods for Mellin transforms based on properties of harmonic sums and harmonic polylogarithms [29, 30] in addition to algorithms for the evaluation of nested sums [31]. The basic relations for this approach, suitable for the evaluation of Eq. (3.1) to any accuracy, are presented in the Appendix. As a check we have also carried out the integrations along the lines of Ref. [19].

For the convenience of the reader, we first recall the known results for g_1, g_2 and g_3 [1, 2, 18, 19]. For brevity suppressing factors of β_0 (see below) and using the short-hand notations $L_{\rm qr} = \ln(Q^2/\mu_r^2)$ and $L_{\rm fr} = \ln(\mu_f^2/\mu_r^2)$, these functions can be written as

$$g_1^{\rm DIS}(\lambda) \;=\; A_1(1 - \ln(1-\lambda) + \lambda^{-1}\ln(1-\lambda)) \; , \tag{3.3}$$

$$\begin{aligned} g_2^{\rm DIS}(\lambda) \;=\;& (A_1\beta_1 - A_2)(\lambda + \ln(1-\lambda)) + \frac{1}{2}A_1\beta_1\ln^2(1-\lambda) \\ & - (A_1\gamma_e - B_1)\ln(1-\lambda) + L_{\rm qr}A_1\ln(1-\lambda) + L_{\rm fr}A_1\lambda \; , \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
g_3^{\mathrm{DIS}}(\lambda) &= \frac{1}{2}(A_1\beta_2 - A_1\beta_1^2 + A_2\beta_1 - A_3)\Big(1+\lambda-\frac{1}{1-\lambda}\Big) \\
&\quad + A_1\beta_1^2\Big(\frac{\ln(1-\lambda)}{1-\lambda} + \frac{1}{2}\frac{\ln^2(1-\lambda)}{1-\lambda}\Big) + \Big(A_1\beta_2 - A_1\beta_1^2\Big)\ln(1-\lambda) \\
&\quad + (A_1\beta_1\gamma_e + A_2\beta_1 - B_1\beta_1)\Big(1-\frac{1}{1-\lambda}-\frac{\ln(1-\lambda)}{1-\lambda}\Big) \\
&\quad - \Big(A_1\beta_2 + \frac{1}{2}A_1(\gamma_e^2+\zeta_2) + A_2\gamma_e - B_1\gamma_e - B_2\Big)\Big(1-\frac{1}{1-\lambda}\Big) \\
&\quad + L_{\mathrm{qr}}\Big[(A_1\gamma_e - A_1\beta_1 + A_2 - B_1)\Big(1-\frac{1}{1-\lambda}\Big) + A_1\beta_1\Big(\frac{\ln(1-\lambda)}{1-\lambda}\Big)\Big] \\
&\quad + L_{\mathrm{fr}}A_2\lambda - L_{\mathrm{qr}}^2\frac{1}{2}A_1\Big(1-\frac{1}{1-\lambda}\Big) - L_{\mathrm{fr}}^2\frac{1}{2}A_1\lambda\,. \qquad (3.5)
\end{aligned}
$$

The dependence on β_0 is recovered by $A_k \to A_k/\beta_0^k$, $B_k \to B_k/\beta_0^k$, $\beta_k \to \beta_k/\beta_0^{k+1}$ and multiplication of g_3 by β_0. In the same notation the new function g_4 (to be multiplied by β_0^2) is given by

$$
\begin{aligned}
g_4^{\mathrm{DIS}}(\lambda) &= -\frac{1}{6}A_1\beta_1^3\frac{\ln^3(1-\lambda)}{(1-\lambda)^2} + \frac{1}{2}(A_1\beta_1^2\gamma_e + A_2\beta_1^2 - B_1\beta_1^2)\frac{\ln^2(1-\lambda)}{(1-\lambda)^2} + \frac{1}{2}(A_1\beta_1^3 - A_1\beta_1\beta_2 \\
&\quad - A_1\beta_1(\gamma_e^2+\zeta_2) + A_2\beta_1^2 - 2A_2\beta_1\gamma_e - A_3\beta_1 + 2B_1\beta_1\gamma_e + 2B_2\beta_1)\frac{\ln(1-\lambda)}{(1-\lambda)^2} \\
&\quad - (A_1\beta_1^3 - A_1\beta_1\beta_2)\frac{\ln(1-\lambda)}{1-\lambda} + \Big(\frac{1}{2}A_1\beta_1^3 - A_1\beta_1\beta_2 + \frac{1}{2}A_1\beta_3\Big)\ln(1-\lambda) + (A_1\beta_1^3 \\
&\quad - A_1\beta_1\beta_2 - A_1\beta_1^2\gamma_e + A_1\beta_2\gamma_e - A_2\beta_1^2 + A_2\beta_2 + B_1\beta_1^2 - B_1\beta_2)\Big(\frac{1}{2}-\frac{1}{1-\lambda} \\
&\quad + \frac{1}{2}\frac{1}{(1-\lambda)^2}\Big) + \frac{1}{2}\Big(\frac{1}{3}A_1\beta_1^3 - \frac{1}{6}A_1\beta_1\beta_2 - \frac{1}{6}A_1\beta_3 - \frac{1}{3}A_1(3\gamma_e\zeta_2 + \gamma_e^3 + 2\zeta_3) \\
&\quad + A_2\beta_1\gamma_e - A_2(\gamma_e^2+\zeta_2) - \frac{5}{6}A_2\beta_1^2 + \frac{1}{3}A_2\beta_2 + \frac{5}{6}A_3\beta_1 - A_3\gamma_e - \frac{1}{3}A_4 - B_2\beta_1 \\
&\quad + B_1(\gamma_e^2+\zeta_2) + 2B_2\gamma_e + B_3\Big)\Big(1-\frac{1}{(1-\lambda)^2}\Big) + \frac{1}{3}\Big(A_1\beta_1^3 - 2A_1\beta_1\beta_2 + A_1\beta_3 \\
&\quad + A_2\beta_2 - A_2\beta_1^2 + A_3\beta_1 - A_4\Big)\lambda + L_{\mathrm{qr}}\Big[(A_1\beta_1^2 - A_1\beta_2)\Big(\frac{1}{2}-\frac{1}{1-\lambda}+\frac{1}{2}\frac{1}{(1-\lambda)^2}\Big) \\
&\quad + \Big(\frac{1}{2}A_1(\gamma_e^2+\zeta_2) - \frac{1}{2}A_2\beta_1 + A_2\gamma_e + \frac{1}{2}A_3 - B_1\gamma_e - B_2\Big)\Big(1-\frac{1}{(1-\lambda)^2}\Big) \\
&\quad + (A_1\beta_1\gamma_e + A_2\beta_1 - B_1\beta_1)\frac{\ln(1-\lambda)}{(1-\lambda)^2} - \frac{1}{2}A_1\beta_1^2\frac{\ln^2(1-\lambda)}{(1-\lambda)^2}\Big] - L_{\mathrm{qr}}^2\Big[\frac{1}{2}(A_1\gamma_e \\
&\quad + A_2 - B_1)\Big(1-\frac{1}{(1-\lambda)^2}\Big) + \frac{1}{2}A_1\beta_1\frac{\ln(1-\lambda)}{(1-\lambda)^2}\Big] + L_{\mathrm{qr}}^3\frac{1}{6}A_1\Big(1-\frac{1}{(1-\lambda)^2}\Big) \\
&\quad + L_{\mathrm{fr}}A_3\lambda - L_{\mathrm{fr}}^2\Big(A_2 + \frac{1}{2}A_1\beta_1\Big)\lambda + L_{\mathrm{fr}}^3\frac{1}{3}A_1\lambda\,. \qquad (3.6)
\end{aligned}
$$

The results g_i^{DY} for the Drell-Yan process and, with slightly different coefficients A_i and D_i, Higgs production via gluon-gluon fusion, are related to Eqs. (3.3) – (3.6) as follows: the function corresponding to Eq. (3.3) reads $g_1^{\mathrm{DY}}(\lambda) = 2g_1^{\mathrm{DIS}}(2\lambda)$, while the functions $g_2^{\mathrm{DY}}, g_3^{\mathrm{DY}}$ and g_4^{DY} are obtained from Eqs. (3.4) – (3.6) by replacing $\lambda \to 2\lambda$ everywhere and substituting $B_i \to D_i/2$ in all terms. Finally, the constants have to be changed according to $\gamma_e \to 2\gamma_e$ and $\zeta_n \to 2^n\zeta_n$. The generalization of Eqs. (3.3) – (3.6) to other processes involving Eqs. (2.3) – (2.5) is obvious.

The functions $g_1(\lambda)$ collecting the leading logarithms $L(a_{\mathrm{s}}L)^k$ depend on A_1 only and are finite for all λ. The N^{n-1}LL contributions $g_{n>1}(\lambda)$ to Eq. (3.1) including A_n, B_{n-1} and D_{n-1}, on the other hand, exhibit Landau poles at the moments $N = \exp[1/(\beta_0 a_s)]$ for DIS and $N = \exp[1/(2\beta_0 a_s)]$ for the DY case (and at both values in general). As obvious from Eq. (3.3) – (3.6) the strength of these singularities increases with the logarithmic order, reaching $[1-(2)\lambda]^{-2}$ at the N^3LL level.

4 Resummation coefficients and numerical stability

Since observables are independent, order by order in α_{s}, of the factorization scale, the functions A_{p} in Eqs. (2.3) and (2.4) are given by the large-N coefficients of the diagonal splitting functions for $\mu_r = \mu_f$,

$$P_{\mathrm{pp}}(\alpha_{\mathrm{s}}) \;=\; -A_{\mathrm{p}}(\alpha_{\mathrm{s}})\,\ln N + O(1)\;, \tag{4.1}$$

which in turn are identical to the anomalous dimension of a Wilson line with a cusp [32]. The first and second order coefficients have been known for a long time, the third order has been recently completed by us. The expansion coefficients (2.6) for the quark case read [21, 33]

$$\begin{aligned}
A_{\mathrm{q},1} &= 4C_F \\
A_{\mathrm{q},2} &= 8C_F\left[\left(\frac{67}{18}-\zeta_2\right)C_A-\frac{5}{9}n_f\right] \\
A_{\mathrm{q},3} &= 16C_F\left[C_A^2\left(\frac{245}{24}-\frac{67}{9}\zeta_2+\frac{11}{6}\zeta_3+\frac{11}{5}\zeta_2^2\right)+C_Fn_f\left(-\frac{55}{24}+2\zeta_3\right)\right. \\
&\qquad\left.+\,C_An_f\left(-\frac{209}{108}+\frac{10}{9}\zeta_2-\frac{7}{3}\zeta_3\right)+n_f^2\left(-\frac{1}{27}\right)\right]\;.
\end{aligned} \tag{4.2}$$

Here n_f denotes the number of effectively massless quark flavours, and C_F and C_A are the usual colour factors, with $C_F = 4/3$ and $C_A = 3$ in QCD. The gluonic quantities are given by

$$A_{\mathrm{g,i}} \;=\; C_A/C_F\,A_{\mathrm{q,i}}\;. \tag{4.3}$$

The perturbative expansion of A_{p} is very benign. For $n_f = 4$, for example, Eqs. (4.2) lead to

$$A_{\mathrm{q}}(\alpha_{\mathrm{s}}) \;\cong\; 0.4244\,\alpha_{\mathrm{s}}\,(1+0.6381\,\alpha_{\mathrm{s}}+0.5100\,\alpha_{\mathrm{s}}^2+\ldots)\;. \tag{4.4}$$

Consequently, already the effect of A_3 on the resummed coefficient functions is very small [18, 19], and a simple estimate suffices for the presently unknown fourth-order coefficients A_4 entering g_4. With the [0/2] results differing by less than 10%, we will employ the [1/1] Padé approximants

$$A_{\mathrm{q},4} \;\approx\; 7849\,,\; 4313\,,\; 1553 \quad \text{for} \quad n_f \;=\; 3\,,\,4\,,\,5\;, \tag{4.5}$$

corresponding to an estimate of $+0.4075\,\alpha_{\mathrm{s}}^3$ for the next term in Eq. (4.4).

For the determination of the coefficients B_i we also need the constant-N piece, g_0 in Eq. (2.1), for inclusive DIS. The corresponding expansion coefficients g_{0k} can be obtained by Mellin inverting the +-distribution and $\delta(1-x)$ parts of the k-th order coefficient function $c^{(k)}_{2,\mathrm{q}}$. Again using the expansion parameter $a_\mathrm{s} = \alpha_\mathrm{s}/(4\pi)$, the presently known terms [7, 10, 20] are given by

$$
\begin{aligned}
g_{01}^{\mathrm{DIS}} &= C_F \left(-9 - 2\zeta_2 + 2\gamma_e^2 + 3\gamma_e\right) \ , \qquad (4.6) \\
g_{02}^{\mathrm{DIS}} &= C_F^2 \left(\frac{331}{8} - \frac{51}{2}\gamma_e - \frac{27}{2}\gamma_e^2 + 6\gamma_e^3 + 2\gamma_e^4 + \frac{111}{2}\zeta_2 - 18\gamma_e\zeta_2 - 4\gamma_e^2\zeta_2 \right.\\
&\quad \left. -66\zeta_3 + 24\gamma_e\zeta_3 + \frac{4}{5}\zeta_2^2\right) + C_A C_F \left(-\frac{5465}{72} + \frac{3155}{54}\gamma_e + \frac{367}{18}\gamma_e^2 \right.\\
&\quad \left. + \frac{22}{9}\gamma_e^3 - \frac{1139}{18}\zeta_2 - \frac{22}{3}\gamma_e\zeta_2 - 4\gamma_e^2\zeta_2 + \frac{464}{9}\zeta_3 - 40\gamma_e\zeta_3 + \frac{51}{5}\zeta_2^2\right)\\
&\quad + C_F n_f \left(\frac{457}{36} - \frac{247}{27}\gamma_e - \frac{29}{9}\gamma_e^2 - \frac{4}{9}\gamma_e^3 + \frac{85}{9}\zeta_2 + \frac{4}{3}\gamma_e\zeta_2 + \frac{4}{9}\zeta_3\right) \qquad (4.7)
\end{aligned}
$$

and

$$
\begin{aligned}
g_{03}^{\mathrm{DIS}} &= C_F^3 \left(-\frac{7255}{24} + \frac{1001}{8}\gamma_e + \frac{187}{4}\gamma_e^2 - \frac{93}{2}\gamma_e^3 - 9\gamma_e^4 + 6\gamma_e^5 + \frac{4}{3}\gamma_e^6 - \frac{6197}{12}\zeta_2 \right.\\
&\quad + \frac{579}{2}\gamma_e\zeta_2 + 66\gamma_e^2\zeta_2 - 36\gamma_e^3\zeta_2 - 4\gamma_e^4\zeta_2 - 411\zeta_3 - 346\gamma_e\zeta_3 - 60\gamma_e^2\zeta_3\\
&\quad + 48\gamma_e^3\zeta_3 - \frac{1791}{5}\zeta_2^2 + 84\gamma_e\zeta_2^2 + \frac{8}{5}\gamma_e^2\zeta_2^2 + 556\zeta_2\zeta_3 - 80\gamma_e\zeta_2\zeta_3 + 1384\zeta_5\\
&\quad \left. -240\gamma_e\zeta_5 + \frac{8144}{315}\zeta_2^3 - \frac{176}{3}\zeta_3^2\right) + C_A C_F^2 \left(\frac{9161}{12} - \frac{16981}{24}\gamma_e - \frac{5563}{36}\gamma_e^2 \right.\\
&\quad + \frac{8425}{54}\gamma_e^3 + \frac{433}{9}\gamma_e^4 + \frac{44}{9}\gamma_e^5 + \frac{191545}{108}\zeta_2 - \frac{28495}{54}\gamma_e\zeta_2 - \frac{592}{3}\gamma_e^2\zeta_2 - 8\gamma_e^4\zeta_2\\
&\quad - \frac{284}{9}\gamma_e^3\zeta_2 - \frac{49346}{27}\zeta_3 + 752\gamma_e\zeta_3 + \frac{640}{9}\gamma_e^2\zeta_3 - 80\gamma_e^3\zeta_3 + \frac{11419}{27}\zeta_2^2\\
&\quad + \frac{299}{3}\gamma_e\zeta_2^2 + \frac{142}{5}\gamma_e^2\zeta_2^2 - 828\zeta_2\zeta_3 + 96\gamma_e\zeta_2\zeta_3 - \frac{3896}{9}\zeta_5 + 120\gamma_e\zeta_5\\
&\quad \left. - \frac{23098}{315}\zeta_2^3 + \frac{536}{3}\zeta_3^2\right) + C_A^2 C_F \left(-\frac{1909753}{1944} + \frac{599375}{729}\gamma_e + \frac{50689}{162}\gamma_e^2 \right.\\
&\quad + \frac{4649}{81}\gamma_e^3 + \frac{121}{27}\gamma_e^4 - \frac{78607}{54}\zeta_2 - \frac{18179}{81}\gamma_e\zeta_2 - \frac{778}{9}\gamma_e^2\zeta_2 - \frac{88}{9}\gamma_e^3\zeta_2\\
&\quad + \frac{115010}{81}\zeta_3 + \frac{121}{27}\gamma_e^4 - \frac{78607}{54}\zeta_2 - \frac{18179}{81}\gamma_e\zeta_2 - \frac{778}{9}\gamma_e^2\zeta_2 - \frac{88}{9}\gamma_e^3\zeta_2\\
&\quad \left. + \frac{3496}{9}\zeta_2\zeta_3 + \frac{176}{3}\gamma_e\zeta_2\zeta_3 - \frac{416}{3}\zeta_5 + 232\gamma_e\zeta_5 - \frac{12016}{315}\zeta_2^3 - \frac{248}{3}\zeta_3^2\right)\\
&\quad + C_F^2 n_f \left(-\frac{341}{36} + \frac{2003}{108}\gamma_e + \frac{83}{18}\gamma_e^2 - \frac{683}{27}\gamma_e^3 - \frac{70}{9}\gamma_e^4 - \frac{8}{9}\gamma_e^5 - \frac{10733}{54}\zeta_2 \right.\\
&\quad + \frac{2177}{27}\gamma_e\zeta_2 + \frac{112}{3}\gamma_e^2\zeta_2 + \frac{32}{9}\gamma_e^3\zeta_2 + \frac{10766}{27}\zeta_3 - \frac{20}{9}\gamma_e\zeta_3 + \frac{8}{9}\gamma_e^2\zeta_3 - \frac{8}{3}\gamma_e\zeta_2^2
\end{aligned}
$$

$$
-\frac{10802}{135}\zeta_2^2-\frac{40}{3}\zeta_2\zeta_3-\frac{784}{9}\zeta_5\bigg)+C_AC_Fn_f\bigg(\frac{142883}{486}-\frac{160906}{729}\gamma_e
-\frac{7531}{81}\gamma_e^2-\frac{1552}{81}\gamma_e^3-\frac{44}{27}\gamma_e^4+\frac{33331}{81}\zeta_2+\frac{5264}{81}\gamma_e\zeta_2+\frac{56}{3}\gamma_e^2\zeta_2+\frac{16}{9}\gamma_e^3\zeta_2
-\frac{21418}{81}\zeta_3+\frac{1976}{27}\gamma_e\zeta_3+8\gamma_e^2\zeta_3+\frac{164}{135}\zeta_2^2-\frac{128}{15}\gamma_e\zeta_2^2-\frac{64}{9}\zeta_2\zeta_3+\frac{8}{3}\zeta_5\bigg)
+C_Fn_f^2\bigg(-\frac{9517}{486}+\frac{8714}{729}\gamma_e+\frac{470}{81}\gamma_e^2+\frac{116}{81}\gamma_e^3+\frac{4}{27}\gamma_e^4-\frac{2110}{81}\zeta_2-\frac{8}{9}\gamma_e^2\zeta_2
-\frac{116}{27}\gamma_e\zeta_2+\frac{80}{81}\zeta_3+\frac{64}{27}\gamma_e\zeta_3-\frac{292}{135}\zeta_2^2\bigg)+\frac{d^{abc}d_{abc}}{n_c}fl_{11}\,(64+160\zeta_2
+\frac{224}{3}\zeta_3-\frac{32}{5}\zeta_2^2-\frac{1280}{3}\zeta_5\bigg)\,. \tag{4.8}
$$

Note the new flavour structure fl_{11} [20,34] in g_{03}. This contribution, introducing the colour factor $d^{abc}d_{abc}/n_c$, for the first time leads to a difference between the flavour-singlet and non-singlet coefficient functions for the photon-exchange structure function F_2 in the soft-gluon limit, with $fl_{11}^{\rm ns}=3\langle e\rangle$ and $fl_{11}^{\rm s}=\langle e\rangle^2/\langle e^2\rangle$, where $\langle e^k\rangle$ represents the average of the charge e^k for the active quark flavours, $\langle e^k\rangle=n_f^{-1}\sum_{i=1}^{n_f}e_i^k$. Correspondingly, the large-N coefficient functions for Z- and W-exchange DIS will differ from each other and from $F_2^{\rm e.m.}$ at order α_s^3. Based on the size of the fl_{11} term in Eq. (4.8), however, we expect these differences to be numerically insignificant.

Now the coefficients $B_{{\rm q},k}$ entering the jet function (2.4) can be derived successively from the $\ln N$ terms of the k-loop DIS coefficient functions $c_{2,\rm q}^{(k)}$. Expansion of Eqs. (3.1) and (3.3) – (3.6) in powers of a_s yields

$$
\begin{aligned}
c_{2,\rm q}^{(1)}\Big|_{\ln N} &= A_1\gamma_e-B_1\\
c_{2,\rm q}^{(2)}\Big|_{\ln N} &= \frac{1}{2}A_1\beta_0\,(\gamma_e^2+\zeta_2)+A_2\,\gamma_e-B_1\beta_0\gamma_e-B_2+g_{01}(A_1\gamma_e-B_1)\\
c_{2,\rm q}^{(3)}\Big|_{\ln N} &= \frac{1}{3}A_1\beta_0^2(\gamma_e^3+3\gamma_e\zeta_2+2\zeta_3)+\frac{1}{2}A_1\beta_1(\gamma_e^2+\zeta_2)+A_2\beta_0(\gamma_e^2+\zeta_2)\\
&\quad +A_3\gamma_e-B_1(\beta_1\gamma_e+\beta_0^2\zeta_2+\beta_0^2\gamma_e^2)-2B_2\beta_0\gamma_e-B_3\\
&\quad +g_{02}(A_1\gamma_e-B_1)+g_{01}\bigg(\frac{1}{2}A_1\,\beta_0(\gamma_e^2+\zeta_2)+A_2\,\gamma_e-B_1\beta_0\gamma_e-B_2\bigg)\,.
\end{aligned}\tag{4.9}
$$

The coefficients of $\ln^l N\,, 2\le l\le 2k$ in $c_{2,\rm q}^{(k)}$, on the other hand, are completely fixed by lower-order resummation coefficients, thus providing an explicit k-loop check of the exponentiation formula. Comparison of the relations (4.9) with the corresponding results from the fixed-order calculations of Refs. [7,10,20], using Eqs. (4.2), (4.6) and (4.7), leads to

$$
B_{\rm q,1} = -3\,C_F\;, \tag{4.10}
$$

$$
B_{\rm q,2} = C_F^2\left[-\frac{3}{2}+12\,\zeta_2-24\,\zeta_3\right]+C_FC_A\left[-\frac{3155}{54}+\frac{44}{3}\,\zeta_2+40\,\zeta_3\right]
+C_Fn_f\left[\frac{247}{27}-\frac{8}{3}\,\zeta_2\right]\;, \tag{4.11}
$$

$$
\begin{aligned}
B_{\mathrm{q},3} \;=\;& C_F^3\left[-\frac{29}{2} - 18\,\zeta_2 - 68\,\zeta_3 - \frac{288}{5}\,\zeta_2^2 + 32\,\zeta_2\zeta_3 + 240\,\zeta_5\right] \\
&+ C_A C_F^2\left[-46 + 287\,\zeta_2 - \frac{712}{3}\,\zeta_3 - \frac{272}{5}\,\zeta_2^2 - 16\,\zeta_2\zeta_3 - 120\,\zeta_5\right] \\
&+ C_A^2 C_F\left[-\frac{599375}{729} + \frac{32126}{81}\,\zeta_2 + \frac{21032}{27}\,\zeta_3 - \frac{652}{15}\,\zeta_2^2 - \frac{176}{3}\,\zeta_2\zeta_3 - 232\,\zeta_5\right] \\
&+ C_F^2 n_f\left[\frac{5501}{54} - 50\,\zeta_2 + \frac{32}{9}\,\zeta_3\right] + C_F n_f^2\left[-\frac{8714}{729} + \frac{232}{27}\,\zeta_2 - \frac{32}{27}\,\zeta_3\right] \\
&+ C_A C_F n_f\left[\frac{160906}{729} - \frac{9920}{81}\,\zeta_2 - \frac{776}{9}\,\zeta_3 + \frac{208}{15}\,\zeta_2^2\right] \; . \qquad (4.12)
\end{aligned}
$$

Eq. (4.10) is, of course, a well-known result [1,2]. Eq. (4.11) has been derived by us before [35], establishing $D_2^{\mathrm{DIS}} = 0$ from the $n_f \ln^2 N$ term at three loops. For our new result (4.12), on the other hand, we have to rely on the subsequent all-order proofs of Eq. (2.7) in Refs. [23,24]. The QCD expansion of B_{q} analogous to Eq. (4.4) appears far less stable than that for A_{q},

$$
B_{\mathrm{q}}(\alpha_{\mathrm{s}}, n_f\!=\!4) \;\cong\; -0.3183\,\alpha_{\mathrm{s}}\,(1 \,-\, 1.227\,\alpha_{\mathrm{s}} \,-\, 3.405\,\alpha_{\mathrm{s}}^2 \,+\, \ldots) \; . \qquad (4.13)
$$

The ingredients for the resummation of inclusive DIS are now complete, and in the left part of Fig. 1 we show the corresponding LL, NLL, N^{2}LL and N^{3}LL approximations to the exponent (3.1) resulting, for $\alpha_{\mathrm{s}} = 0.2$ and three flavours, from Eqs. (3.3) – (3.6), (4.2), (4.5) and (4.10) – (4.12). For these parameters the expansion (3.1) is stable in the N-range shown in the figure. For example, the relative N^{3}LL corrections amount to 2% at $N = 10$ ($\lambda = 0.33$) and 4% at $N = 40$ ($\lambda = 0.53$), whereas the corresponding N^{2}LL figures read 9% and 12%. The large third-order contribution to B_{q} actually stabilizes $g_4(\lambda)$: for $B_{\mathrm{q},3} = 0$ the N^{3}LL term at $N = 40$ would instead reach 12%, i.e., the size of the previous order. The effect of both $A_{\mathrm{q},4}$ and β_3, on the other hand, is very small, as their respective nullification would change the result even at $N = 40$ by only 0.6% and 0.1%.

In the right part of Fig. 1 and in Fig. 2 the exponentiated results are convoluted with the typical input shape $xf = x^{0.5}(1-x)^3$ for a couple of values for α_{s} and n_f. The Mellin inversion is in principle ambiguous due to the Landau poles briefly addressed at the end of Section 3. We employ the standard 'minimal prescription' (thus adopting the usual fixed-order contour) of Ref. [36], to which the reader is referred for a detailed discussion. For total soft-gluon enhancements up to almost an order of magnitude, as shown in the figures, the resulting N^{3}LL corrections remain far smaller than their N^{2}LL counterparts and amount to less than 10% even for $\alpha_{\mathrm{s}} = 0.3$. Note that the dependence on n_f is larger than the effect of g_4. Thus, at this level of accuracy, a reliable understanding of heavy-quark mass effects is called for also in the limit $x \to 1$.

The gluonic coefficients corresponding to Eqs. (4.10) – (4.12) can be obtained in the same manner from DIS by exchange of a scalar ϕ with a pointlike coupling to gluons, like the Higgs boson in limit of a heavy top quark. We have derived the corresponding coefficient functions $c_{\phi,\mathrm{p}}^{(k)}$ up to $k = 3$ already during the calculations for Ref. [22], as a process of this type is required to access the lower row of the flavour-singlet splitting function matrix. Comparing those results to

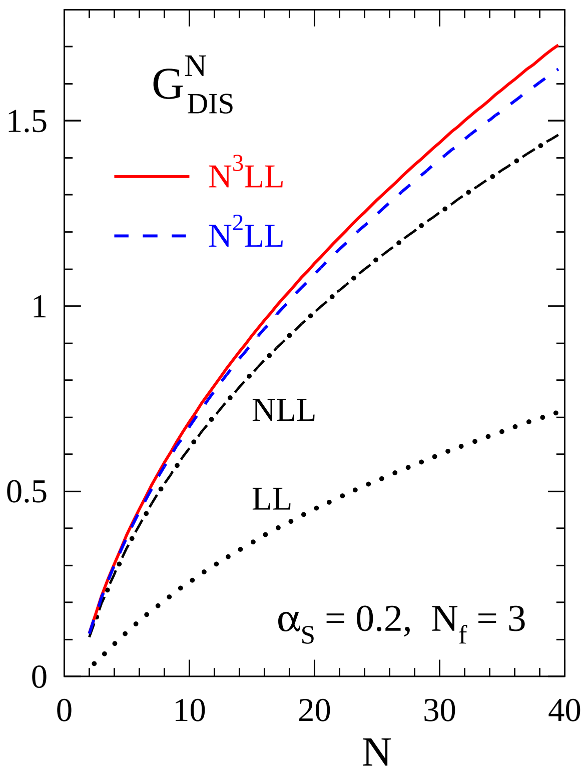

Figure 1: Left: the LL, NLL, N^{2}LL and N^{3}LL approximations for the resummation exponent for standard DIS. Right: the convolutions of the exponentiated results with a typical input shape.

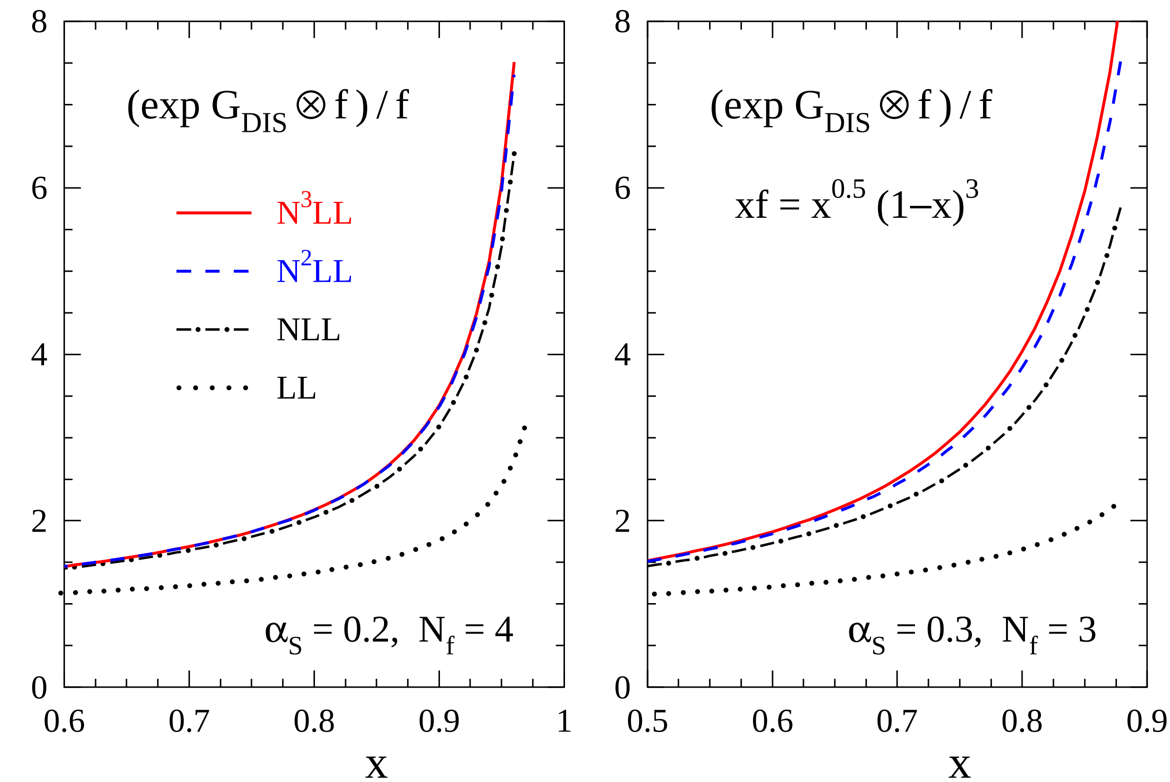

Figure 2: As the right part of the previous figure, but for a different value of n_f (left) and α_s (right).

Eqs. (4.9) for $c^{(k)}_{\phi,\mathrm{g}}$ yields

$$
\begin{aligned}
B_{\mathrm{g},1} &= -\frac{11}{3}C_A + \frac{2}{3}n_f \;=\; -\beta_0 \;, && (4.14)\\
B_{\mathrm{g},2} &= C_A^2\left[-\frac{611}{9} + \frac{88}{3}\zeta_2 + 16\,\zeta_3\right] + C_A n_f\left[\frac{428}{27} - \frac{16}{3}\zeta_2\right] + 2\,C_F n_f - \frac{20}{27}n_f^2 \;, && (4.15)\\
B_{\mathrm{g},3} &= C_A^3\left[-\frac{1492081}{1458} + \frac{60875}{81}\zeta_2 + \frac{13796}{27}\zeta_3 - \frac{2596}{15}\zeta_2^2 - \frac{128}{3}\zeta_2\zeta_3 - 112\,\zeta_5\right]\\
&\quad + C_A^2 n_f\left[\frac{498329}{1458} - \frac{21014}{81}\zeta_2 - \frac{296}{9}\zeta_3 + \frac{568}{15}\zeta_2^2\right] - C_F^2 n_f\\
&\quad + C_A C_F n_f\left[\frac{8579}{54} - 16\,\zeta_2 - \frac{832}{9}\zeta_3 - \frac{32}{5}\zeta_2^2\right] + C_F n_f^2\left[-\frac{47}{3} + \frac{32}{3}\zeta_3\right]\\
&\quad + C_A n_f^2\left[-\frac{48829}{1458} + \frac{716}{27}\zeta_2 - \frac{176}{27}\zeta_3\right] + n_f^3\left[\frac{200}{243} - \frac{8}{9}\zeta_2\right] \;, && (4.16)
\end{aligned}
$$

where Eqs. (4.15) and (4.16) are new results. For $n_f = 4$ the numerical expansion of B_{g} reads

$$
B_{\mathrm{g}}(\alpha_{\mathrm{s}}) \;\cong\; -0.6631\,\alpha_{\mathrm{s}}\,(1 - 0.7651\,\alpha_{\mathrm{s}} - 2.696\,\alpha_{\mathrm{s}}^2 + \ldots) \;, \qquad (4.17)
$$

exhibiting an enhanced third order correction similar to that of B_{q} in Eq. (4.13).

The gluonic threshold resummation resulting from Eqs. (4.3) and (4.14) – (4.16) is illustrated in Fig. 3 using the practically irrelevant scalar-exchange process, with same parameters as in Fig. 1 for direct comparison. The soft and collinear radiation effects are much larger here due to the larger colour charge of the gluons, but the qualitative pattern is rather similar to 'normal' inclusive DIS.

As mentioned above, the (closely related) Drell-Yan process and Higgs boson production via gluon-gluon fusion presently represent the only other processes for which the NNLL threshold resummation is known, with $D_1 = 0$ and [18, 19]

$$
D_2^{\{\mathrm{DY},\mathrm{H}\}} \;=\; \{C_F, C_A\}\left[C_A\left(-\frac{1616}{27} + \frac{176}{3}\zeta_2 + 56\,\zeta_3\right) + n_f\left(\frac{224}{27} - \frac{32}{3}\zeta_2\right)\right] \;. \qquad (4.18)
$$

Extending a result given in Ref. [18], we notice the following conspicuous relation between these coefficients and $B_{\mathrm{p},2}$:

$$
\begin{aligned}
\frac{1}{2}D_2^{\mathrm{DY}} - B_{\mathrm{q},2} - P^{(1)}_{\mathrm{q},\delta} &= 7\,\beta_0 C_F\\
\frac{1}{2}D_2^{\mathrm{H}} \;\; - B_{\mathrm{g},2} - P^{(1)}_{\mathrm{g},\delta} &= \frac{1}{3}\beta_0\left(4C_A + 5\,\beta_0\right) \;, && (4.19)
\end{aligned}
$$

where $P^{(1)}_{\mathrm{p},\delta}$ denotes the coefficients of $\delta(1-x)$ in the diagonal two-loop splitting functions, and the colour structures on the right-hand sides are those of $A_{\mathrm{p},1}$ and $B_{\mathrm{p},1} = -P^{(0)}_{\mathrm{p},\delta}$, multiplied by β_0. Note especially the non-trivial cancellation of all ζ-function terms between the three contributions on the left-hand sides of Eqs. (4.19) and the vanishing of the right-hand sides for $\beta_0 \to 0$.

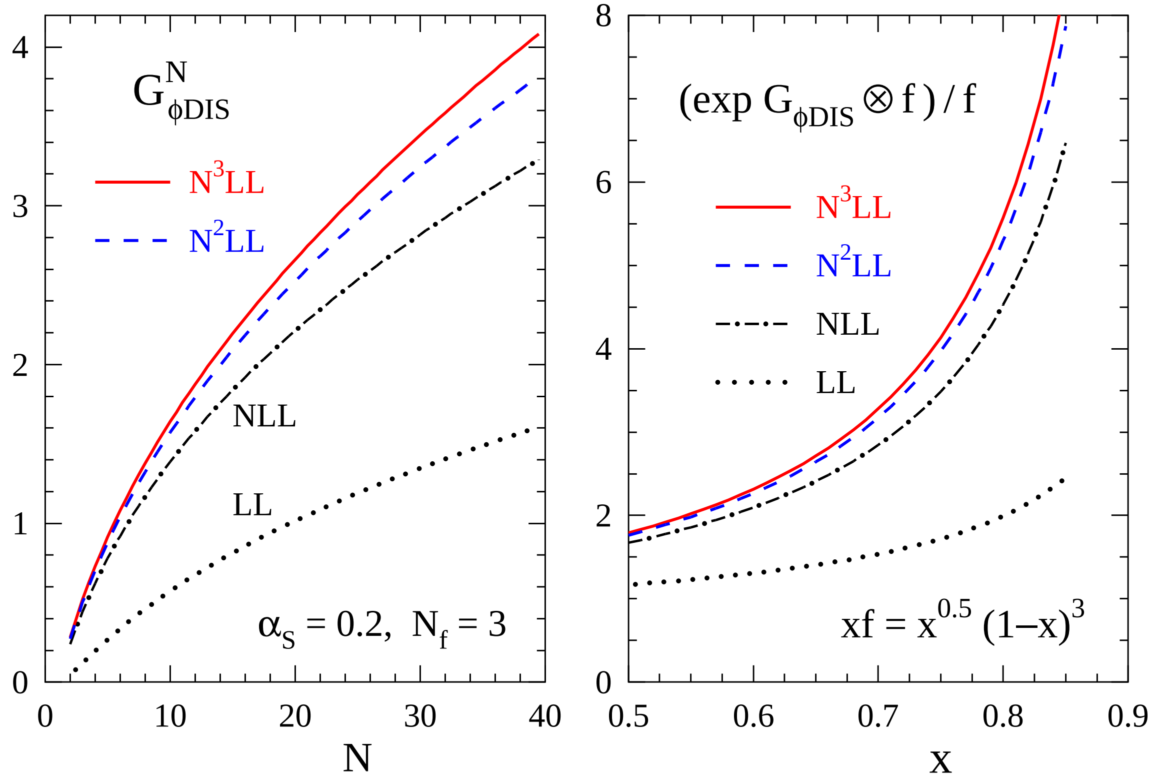

Figure 3: As Fig. 1, but for inclusive DIS by exchange of a scalar ϕ directly coupling to gluons.

5 Fourth-order predictions and tower expansion

Another manner to organize the all-order information encoded in Eqs. (2.1) – (2.5) is to re-expand the exponential,

$$C^N(Q^2)/C^N_{\rm LO}(Q^2) \;=\; 1 + \sum_{k=1}^{\infty} a_{\rm s}^k \sum_{l=1}^{2k} c_{kl} \ln^{2k-l+1} N \; , \tag{5.1}$$

and retain only those terms in the second sum which are completely fixed by the available information on the expansion coefficients in Eq. (2.6). Using the notation

$$g_i(\lambda) \;=\; \sum_{k=1}^{\infty} g_{ik} \lambda^k \tag{5.2}$$

for the expansion of Eqs. (3.3) – (3.6) together with Eq. (2.6) for g_0, the quantities c_{kl} in Eq. (5.1) receive contributions from the following coefficients:

$$\begin{array}{lclll}
c_{k1} & : & \;\; g_{11} & & \\
c_{k2} & : & + \, g_{12} \, , & g_{21} & \\
c_{k3} & : & + \, g_{13} \, , & g_{22} \, , & g_{01} \\
c_{k4} & : & + \, g_{14} \, , & g_{23} \, , & g_{31}
\end{array}$$

$$
\begin{array}{lllllll}
c_{k5} & : & +\, g_{15}\,, & g_{24}\,, & g_{32}\,, & g_{02} & \\
c_{k6} & : & +\, g_{16}\,, & g_{25}\,, & g_{33}\,, & g_{41} & \\
c_{k7} & : & +\, g_{17}\,, & g_{26}\,, & g_{34}\,, & g_{42}\,, & g_{03} \\
c_{k8} & : & +\, g_{18}\,, & g_{27}\,, & g_{35}\,, & g_{43}\,, & g_{51} \quad \ldots \quad .
\end{array}
\tag{5.3}
$$

The complete relations for the first four terms $c_{k1} \ldots c_{k4}$ can be found in Ref. [37] in a slightly different notation, $g_{31} \to g_{32}$ (i.e., the second index denoting the total power of $a_{\rm s}$ in Eq. (3.1)). Note that the quantities c_{kl} vanish factorially for $k \to \infty$ and fixed l.

Taking into account Eq. (2.7) and considering the coefficient A_i as either known or irrelevant, the function g_i for inclusive DIS is completely specified by its leading term, obtained by matching to the i-th order calculation of the coefficient functions as in Eqs. (4.9). The same holds for other processes, at least for cases like Eqs. (2.2), once the required coefficients B_i are known from DIS.

The leading three towers of logarithms, c_{kl} for any k and $l = 1,\, 2,\, 3$, are fixed by a one-loop calculation (providing g_{i1} for $i = 0,\, 1,\, 2$) together with the NLL resummation (adding g_{1k} and g_{2k} for $k \geq 2$). This is the status for many important observables [5,6]. Correspondingly, a two-loop computation of the process under consideration specifies g_{31} and g_{02} and hence fixes, together with the NNLL resummation, also the next two towers. This is the accuracy reached for the Drell-Yan process and Higgs production [18,19]. Finally a three-loop computation combined with the N^{3}LL resummation fixes the first seven towers, c_{kl} for $l = 1,\, \ldots,\, 7$. With the results of Ref. [20] and Sections 3 and 4, we have now reached this point for the structure function F_2 in DIS.

The resulting four-loop predictions, in x-space expressed in terms of the coefficients of the +-distributions $\mathcal{D}_k = [(1-x)^{-1}\ln(1-x)]_+$, for the six highest terms read

$$
c^{(4)}_{2,\rm q}\Big|_{\mathcal{D}_7} \;=\; \frac{16}{3}\, C_F^4 \;, \tag{5.4}
$$

$$
c^{(4)}_{2,\rm q}\Big|_{\mathcal{D}_6} \;=\; -28\, C_F^4 - \frac{308}{9}\, C_A C_F^3 + \frac{56}{9}\, C_F^3 n_f \;, \tag{5.5}
$$

$$
\begin{aligned}
c^{(4)}_{2,\rm q}\Big|_{\mathcal{D}_5} \;=\;& C_F^4\Big[-18 - 128\,\zeta_2 \Big] + C_A C_F^3 \left[\frac{998}{3} - 48\,\zeta_2 \right] + \frac{1936}{27}\, C_A^2 C_F^2 \\
& - \frac{164}{3}\, C_F^3 n_f - \frac{704}{27}\, C_A C_F^2 n_f + \frac{64}{27}\, C_F^2 n_f^2 \;,
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
c^{(4)}_{2,\rm q}\Big|_{\mathcal{D}_4} \;=\;& C_F^4 \left[210 + 600\,\zeta_2 + \frac{400}{3}\,\zeta_3 \right] + C_A C_F^3 \left[-\frac{27835}{27} + \frac{6800}{9}\zeta_2 + 400\,\zeta_3 \right] \\
& + C_A^2 C_F^2 \left[-\frac{24040}{27} + \frac{440}{3}\,\zeta_2 \right] - \frac{1331}{27}\, C_A^3 C_F + C_F^3 n_f \left[\frac{4630}{27} - \frac{1040}{9}\,\zeta_2 \right] \\
& + C_A C_F^2 n_f \left[\frac{8120}{27} - \frac{80}{3}\,\zeta_2 \right] + \frac{242}{9}\, C_A^2 C_F n_f - \frac{640}{27}\, C_F^2 n_f^2 - \frac{44}{9}\, C_A C_F n_f^2 \\
& + \frac{8}{27}\, C_F n_f^3 \;,
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
c^{(4)}_{2,\mathrm{q}}\Big|_{\mathcal{D}_3} \;=\;& C_F^4\left[\frac{113}{2}+264\,\zeta_2-1072\,\zeta_3+\frac{1392}{5}\,\zeta_2^2\right]+C_A^3C_F\left[\frac{55627}{81}-\frac{968}{9}\,\zeta_2\right]\\
&+C_AC_F^3\left[-\frac{1534}{3}-\frac{41824}{9}\,\zeta_2-\frac{8800}{9}\,\zeta_3+\frac{3128}{5}\,\zeta_2^2\right]\\
&+C_A^2C_F^2\left[\frac{2154563}{486}-\frac{52912}{27}\,\zeta_2-\frac{13024}{9}\,\zeta_3+\frac{864}{5}\,\zeta_2^2\right]\\
&+C_F^3n_f\left[-\frac{280}{3}+\frac{7216}{9}\,\zeta_2+\frac{1888}{9}\,\zeta_3\right]+C_A^2C_Fn_f\left[-\frac{9502}{27}+\frac{352}{9}\,\zeta_2\right]\\
&-C_AC_F^2n_f\left[\frac{339134}{243}-\frac{14096}{27}\,\zeta_2-\frac{1216}{9}\,\zeta_3\right]+C_AC_Fn_f^2\left[\frac{1540}{27}-\frac{32}{9}\,\zeta_2\right]\\
&+C_F^2n_f^2\left[\frac{24238}{243}-\frac{928}{27}\,\zeta_2\right]-\frac{232}{81}\,C_Fn_f^3\;,
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
c^{(4)}_{2,\mathrm{q}}\Big|_{\mathcal{D}_2} \;=\;& C_F^4\left[-\frac{1299}{2}-2808\,\zeta_2+1392\,\zeta_3-1836\,\zeta_2^2-640\,\zeta_2\zeta_3+4128\,\zeta_5\right]\\
&+C_AC_F^3\left[\frac{13990}{3}+\frac{30704}{3}\,\zeta_2+\frac{2716}{3}\,\zeta_3-\frac{12906}{5}\,\zeta_2^2-3648\,\zeta_2\zeta_3-720\,\zeta_5\right]\\
&-C_A^2C_F^2\left[\frac{2254339}{243}-\frac{86804}{9}\,\zeta_2-\frac{24544}{3}\,\zeta_3+\frac{4034}{3}\,\zeta_2^2+832\,\zeta_2\zeta_3+1392\,\zeta_5\right]\\
&+C_A^3C_F\left[-\frac{649589}{162}+\frac{4012}{3}\,\zeta_2+1452\,\zeta_3-\frac{968}{5}\,\zeta_2^2\right]\\
&+C_AC_F^2n_f\left[\frac{713162}{243}-\frac{82004}{27}\,\zeta_2-\frac{10600}{9}\,\zeta_3+\frac{3772}{15}\,\zeta_2^2\right]\\
&+C_A^2C_Fn_f\left[\frac{17189}{9}-\frac{5096}{9}\,\zeta_2-352\,\zeta_3+\frac{176}{5}\,\zeta_2^2\right]\\
&+C_F^3n_f\left[-\frac{145}{9}-\frac{5132}{3}\,\zeta_2-936\,\zeta_3+\frac{1032}{5}\,\zeta_2^2\right]+C_Fn_f^3\left[\frac{940}{81}-\frac{32}{9}\,\zeta_2\right]\\
&-C_AC_Fn_f^2\left[\frac{7403}{27}-\frac{688}{9}\,\zeta_2-16\,\zeta_3\right]-C_F^2n_f^2\left[\frac{52678}{243}-\frac{6104}{27}\,\zeta_2-\frac{304}{9}\,\zeta_3\right]\;.
\end{aligned}
\tag{5.9}
$$

The seventh term with $\mathcal{D}_1$ is not exactly known, since the fourth-order contribution to A_q has not been computed so far. Inserting the numerical values for the ζ-functions and the QCD colour factors, including $d^{abc}d_{abc}/n_c = 5n_f/18$, the resummation prediction is given by

$$
c^{(4)}_{2,\mathrm{q}}\Big|_{\mathcal{D}_1} = -286702+64219.0\,n_f-2019.24\,n_f^2+2.0166\,n_f^3-63.402\,fl_{11}\,n_f+A_{\mathrm{q},4}\;. \tag{5.10}
$$

As mentioned above, numerically insignificant are both the uncertainty due to $A_{\mathrm{q},4}$ (estimated in Eq. (4.5)) and the singlet / non-singlet difference introduced by the fl_{11} contribution of Eq. (4.8). It is also interesting to note that the fourth coefficient β_3 of the beta function [27,28] with its quartic group invariants d_F^{abcd} and d_A^{abcd} only enters the eighth tower, starting at the fifth order in α_s.

k	c_{k1}	c_{k2}	c_{k3}	c_{k4}	c_{k5}	c_{k6}	c_{k7}
1	2.66667	7.0784	—	—	—	—	—
2	3.55556	26.2834	40.760	−67.13	—	—	—
3	3.16049	44.9210	238.885	470.82	−620.3	−1639	—
4	2.10700	48.7090	477.854	2429.46	5240.0	−1824	−30318
5	1.12373	38.3254	581.518	5015.18	25150.5	48482	11268
6	0.49944	23.5617	505.972	6432.95	52129.7	225320	675012
7	0.19026	11.8592	340.954	5933.61	68602.9	485712	2494841
8	0.06342	5.0463	186.822	4244.45	65550.0	668223	4979993
9	0.01879	1.8583	86.041	2467.72	48805.8	666670	6718531
10	0.00501	0.6028	34.118	1204.34	29604.7	517490	6747332

Table 1: Numerical values of the four-flavour coefficients c_{kl} in Eq. (5.1) for the quark coefficient function in DIS. The first six columns are exact up to the numerical truncation, and the same for F_1, F_2 and F_3. The seventh column refers to F_1 and F_2 for the photon-exchange flavour-singlet case, $fl_{11} = 1/10$ [27], and uses the estimate (4.5) for the four-loop cusp anomalous dimension.

The numerical values of the N-space coefficients c_{kl} in Eq. (5.1) are presented in Table 1 for $l \leq 7$ and $k \leq 10$. Recall that also these coefficients refer to an expansion in $a_s = \alpha_s/(4\pi)$. Whatever the normalization of the expansion parameter, however, the coefficients in each column (tower) finally vanish for $k \to \infty$, as mentioned below Eq. (5.3). Thus the series (5.1) converges at all $N \neq 0$ for any finite number of towers $l_{\max}$, i.e., with the upper limit in the second sum replaced by $l_{\max}$. The Mellin inversion of the product with the parton distributions f^N is therefore well-defined, in contrast to the fully exponentiated result discussed above.

Before we turn to the higher-order predictions, it is instructive to compare the approximations by the leading large-x and large-N terms to the completely known two- and three-loop coefficient functions [7, 10, 20]. This is done in Fig. 4 for the successive approximations in terms of the +-distributions $\mathcal{D}_k$ defined above Eq. (5.4). The corresponding results for the expansion in powers of $\ln N$ are presented in Fig. 5. Obviously both expansions reproduce the exact large-x behaviour (up to terms not increasing as $x \to 1$ for the ratios shown in the figures) at order α_s^n once all enhanced terms, $\mathcal{D}_k$ with $k = 0, \ldots, 2n-1$ or $\ln^l N$ with $l = 1, \ldots, 2n$, have been taken into account. The x-space expansion, however, would lead to a gross overestimate if only the terms up to $k \simeq n+1$ were known. The convergence in the N-space approach, on the other hand, is much smoother, with a good approximation already reached after n terms.

A similar pattern is found for the fourth-order coefficient function $c_{2,\mathrm{q}}^{(4)}$ illustrated in the same manner in the left part of Fig. 6: the expansion in decreasing powers of $\ln N$ stabilizes after the fourth term. Based on these results and the higher-order coefficients shown in Table 1, we expect that the first l logarithms should provide a good estimate up to about the l-th order in α_s, but severely underestimate the effect of the coefficient functions of much higher orders. Consequently,

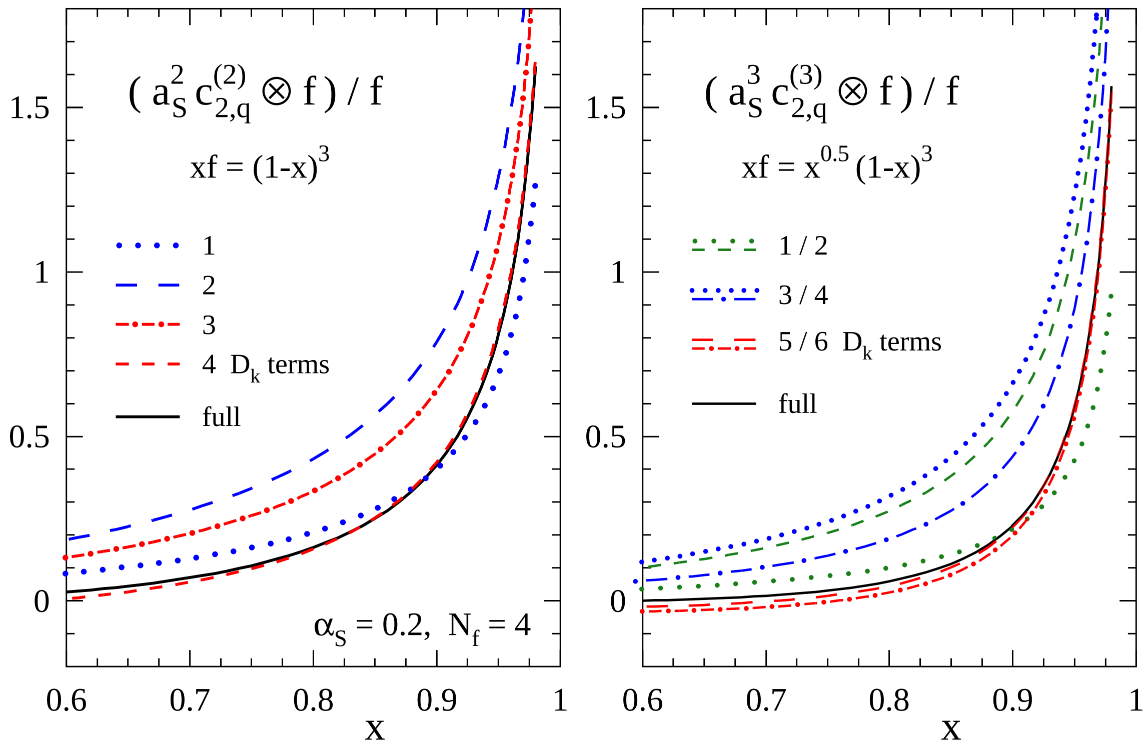

Figure 4: Convolution of the two-loop (left) and three-loop (right) contributions to the DIS coefficient functions $C_{2,q}$ with a typical input shape. Shown are the full results [7, 10, 20] and the large-x expansion by successively including the +-distributions $\mathcal{D}_k$, respectively starting with $\mathcal{D}_3$ and $\mathcal{D}_5$.

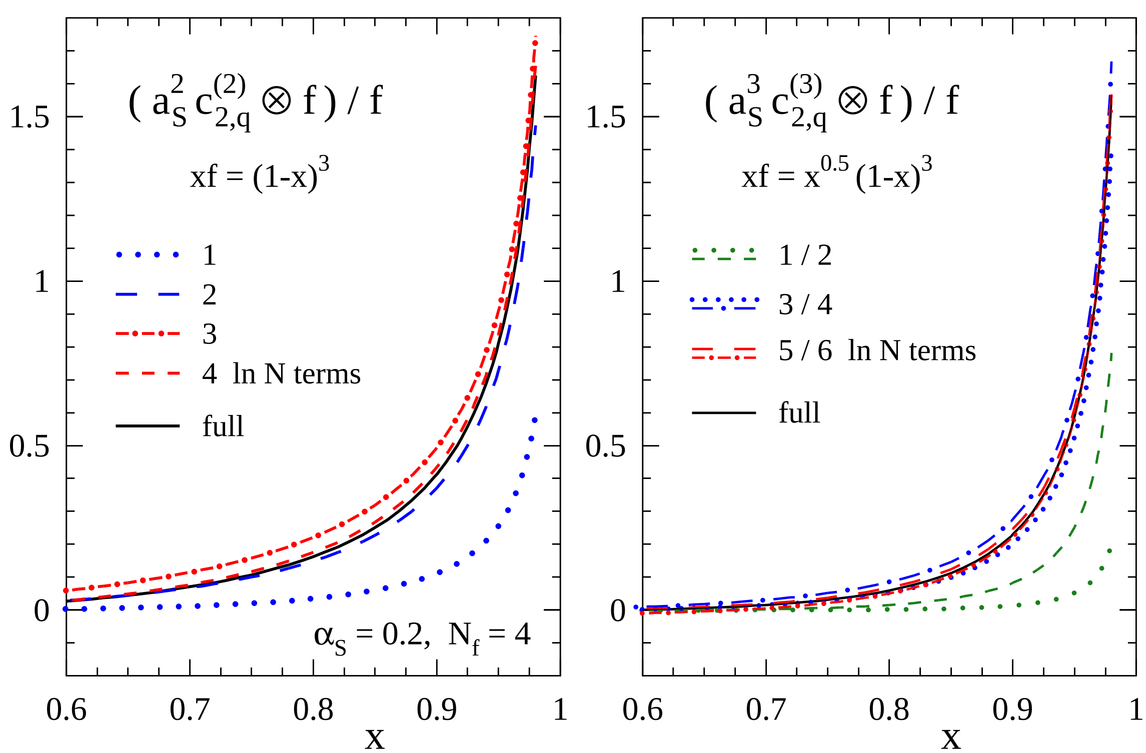

Figure 5: As the previous figure, but for large-N expansion in terms of decreasing powers of $\ln N$.

the tower expansion should underestimate the corrections towards $x \to 1$, where more and more orders become relevant. This is exactly the pattern shown in the right part of Fig. 6, where the predictions of all effects beyond order α_s^3 are compared between the tower expansion and the full exponentiation (for the latter again using a 'minimal-prescription' contour [36]). Both approaches agree very well, for the chosen input parameters, at $x < 0.93$, but start to diverge at $x \gtrsim 0.95$ where the exponentiation is also intrinsically more stable.

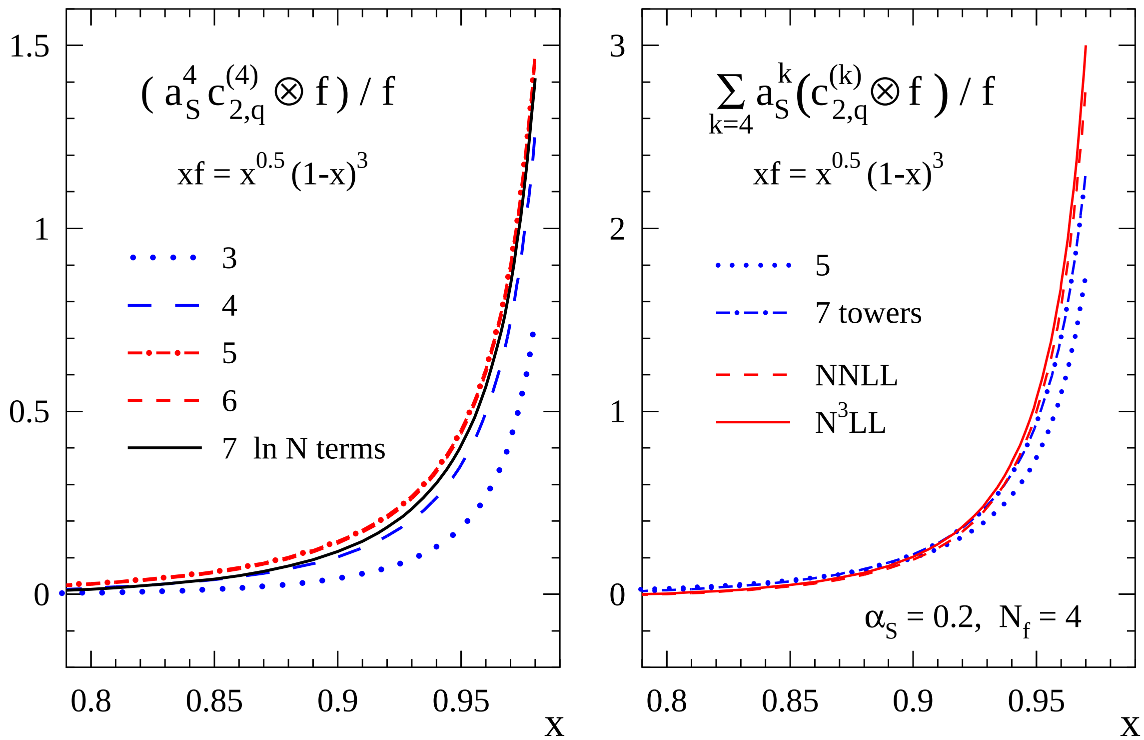

Figure 6: Left: the successive approximations of the four-loop coefficient function $c_{2,q}^{(4)}$ by the large-N terms specified in Table 1, illustrated by the convolution with a typical quark distribution. Right: corresponding results for the effect of all orders beyond α_s^3 as obtained from the tower expansion with up to seven towers and from the exponentiation up to N^{3}LL accuracy.

6 Summary

We have extended the threshold resummation exponents [1, 2, 18, 19] for few-parton processes to the fourth logarithmic (N^{3}LL) order collecting the terms $\alpha_s^2(\alpha_s \ln N)^k$ to all orders in α_s. For our reference process, inclusive deep-inelastic scattering (DIS), the N^{3}LL contributions are specified by two universal expansion parameters: the four-loop cusp anomalous dimension $A_{q,4}$ and the third-order quantity $B_{q,3}$ which defines the jet function resumming collinear radiation off an unobserved final-state quark. The former coefficient has not been computed so far, but can be safely expected to have a very small effect of less than 1%. In fact, the perturbative expansion up to

A_3 [21, 22] does not exhibit enhanced higher-order corrections, and A_4 can be estimated by Padé approximations. We have calculated the more important second coefficient $B_{q,3}$ by comparison of the expanded resummation result to our recent third-order calculation of electromagnetic DIS [20].

The perturbative expansion of B_q seems to indicate, as far as this can be judged from the first three terms, the onset of a factorial enhancement of the higher-order coefficients. However, the rather large size of the coefficient $B_{q,3}$ actually stabilizes the logarithmic expansion of the coefficient functions. In fact, the N^{3}LL corrections are very small at large scales Q^2, and even facilitate a reliable prediction of the soft-gluon effects at scales as low as $Q^2 \approx 4\,\mathrm{GeV}^2$ (corresponding to $\alpha_s \simeq 0.3$) down to very small invariant masses W of the hadronic final state in $ep \to eX$, $W^2 - m_p^2 \approx 0.5\,\mathrm{GeV}^2$. Thus we expect our results to be useful also for low-scale data analyses using parton-hadron duality concepts.

The threshold resummation can also be employed to predict, order by order in α_s, the leading $\ln N$ contributions to the higher-order coefficient functions. At the level of accuracy reached in the present article for inclusive DIS, the exponentiation fixes the seven highest terms, $\ln^n N$ with $2l-6 \leq n \leq 2k$, at all orders $k \geq 4$ of α_s. Already the highest k powers of $\ln N$ provide a good estimate of the soft-gluon enhancement of the k-loop coefficient functions at least for $k \leq 7$, in contrast to the (expected, see Ref. [36]) worse behaviour of the corresponding expansion in x-space +-distributions. Except very close to threshold, where too many orders in α_s become important, the summation of the above seven N-space logarithms to all orders yields a good agreement with the exponentiated coefficient function. This agreement further confirms the 'minimal prescription' [36] used for defining the in principle ambiguous Mellin inversion of the resummation exponential.

Besides the standard (gauge-boson exchange) process, we have also considered DIS by exchange of a scalar directly coupling to gluons. By comparison of the resummation to our unpublished three-loop coefficient function for this process we have derived, for the first time, the second and third order contributions to the coefficient B_g governing the jet function of a final-state gluon. The quantity $B_{g,2}$ will be employed to extend the NNLL resummation to more processes, once the corresponding NNLO results required to fix the process-dependent large-angle soft contribution become available. Finally we would like to draw attention to the curious relation (4.19) which connects, for both the quark and gluon channels, the second-order splitting functions, jet functions and the two-parton (Drell-Yan) large-angle soft emissions in a non-trivial manner.

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Appendix

Here we show some key elements of the calculation of the resummation exponents g_i presented in Section 3. Useful auxiliary relations (for $|x| < 1$) are

$$\frac{1}{(1-x)^{n-\varepsilon}} = \sum_{i=0}^{\infty} \frac{\Gamma(n-\varepsilon+i)}{\Gamma(n-\varepsilon)} \frac{x^i}{i!} \,, \tag{A.1}$$

$$\frac{\ln^k(1-x)}{(1-x)^{n-\varepsilon}} = \left(\frac{\partial}{\partial\varepsilon}\right)^k \frac{1}{(1-x)^{n-\varepsilon}} \,. \tag{A.2}$$

The Mellin transforms of the +-distributions follow from the results for harmonic polylogarithms [30] and are given by

$$\int_0^1 dz \frac{z^N-1}{1-z} \ln^k(1-z) = (-1)^{k+1} k! \, S_{\underbrace{1,\ldots,1}_{k+1}}(N) \,, \tag{A.3}$$

where $S_{m_1,\ldots,m_k}(N)$ denotes the harmonic sums [29]. Eqs. (A.1) – (A.3) lead to the master formula for the derivation of the functions g_i,

$$\int_0^1 dz \frac{z^N-1}{1-z} \frac{1}{(1+a\ln(1-z))^{n-\varepsilon}} =$$
$$-\sum_{i=0}^{\infty} \frac{\Gamma(n-\varepsilon+i)}{\Gamma(n-\varepsilon)} (a_s\beta_0)^i S_{\underbrace{1,\ldots,1}_{i+1}}(N) \sum_{j=0}^{\infty} \begin{pmatrix} i+j-1 \\ j \end{pmatrix} \left(-a_s\beta_0 \ln\frac{Q^2}{\mu_r^2}\right)^j \tag{A.4}$$

with $a = (a_s\beta_0)/(1+a_s\beta_0\ln Q^2/\mu_r^2)$. The double sum in Eq. (A.4) can be solved to the desired logarithmic accuracy with the algorithms for the summation of nested sums [31] coded in FORM [38]. The expansion of the Gamma function in powers of ε for positive integers n reads

$$\begin{aligned}\frac{\Gamma(n+1+\varepsilon)}{n!\,\Gamma(1+\varepsilon)} = {} & 1+\varepsilon S_1(n)+\varepsilon^2(S_{1,1}(n)-S_2(n))+\varepsilon^3(S_{1,1,1}(n)-S_{1,2}(n) \\ & -S_{2,1}(n)+S_3(n))+\varepsilon^4(S_{1,1,1,1}(n)-S_{1,1,2}(n)-S_{1,2,1}(n) \\ & +S_{1,3}(n)-S_{2,1,1}(n)+S_{2,2}(n)+S_{3,1}(n)-S_4(n))+O(\varepsilon^5) \,. \end{aligned} \tag{A.5}$$

Finally, with $\theta_{ij} = 1$ for $i \geq j$ and $\theta_{ij} = 0$ else, the sums $S_{1,\ldots,1}(N)$ are factorized according to

$$\begin{aligned} i!\, S_{\underbrace{1,\ldots,1}_{i}}(N) = {} & (S_1(N))^i + \frac{1}{2}i(i-1)S_2(N)(S_1(N))^{i-2} + \frac{1}{3}i(i-1)(i-2)S_3(N)(S_1(N))^{i-3} \\ & + \frac{1}{4}i(i-1)(i-2)(i-3)\left(S_4(N)+\frac{1}{2}(S_2(N))^2\right)(S_1(N))^{i-4} + \ldots \end{aligned} \tag{A.6}$$

$$\begin{aligned} \simeq {} & \theta_{i1}\ln^i\tilde{N} + \frac{1}{2}\theta_{i3}i(i-1)\zeta_2\ln^{i-2}\tilde{N} + \frac{1}{3}\theta_{i4}i(i-1)(i-2)\zeta_3\ln^{i-3}\tilde{N} \\ & + \frac{1}{4}\theta_{i5}i(i-1)(i-2)(i-3)\left(\zeta_4+\frac{1}{2}{\zeta_2}^2\right)\ln^{i-4}\tilde{N} + \ldots \,, \end{aligned} \tag{A.7}$$

where $\tilde{N} = Ne^{\gamma_e}$ and the algebraic properties of harmonic sums have been used [29].

References

[1] G. Sterman, Nucl. Phys. B281 (1987) 310

[2] S. Catani and L. Trentadue, Nucl. Phys. B327 (1989) 323

[3] N. Kidonakis and G. Sterman, Nucl. Phys. B505 (1997) 321, hep-ph/9705234

[4] R. Bonciani, S. Catani, M.L. Mangano and P. Nason, Nucl. Phys. B529 (1998) 424, hep-ph/9801375

[5] E. Laenen, Pramana 63 (2004) 1225

[6] E. Laenen, AIP Conf. Proc. 753 (2005) 45

[7] W.L. van Neerven and E.B. Zijlstra, Phys. Lett. B272 (1991) 127

[8] E.B. Zijlstra and W.L. van Neerven, Phys. Lett. B273 (1991) 476

[9] E.B. Zijlstra and W.L. van Neerven, Phys. Lett. B297 (1992) 377

[10] S. Moch and J.A.M. Vermaseren, Nucl. Phys. B573 (2000) 853, hep-ph/9912355

[11] S. Moch, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B606 (2005) 123, hep-ph/0411112

[12] R. Hamberg, W.L. van Neerven and T. Matsuura, Nucl. Phys. B359 (1991) 343 [Erratum ibid. B644 (2002) 403]

[13] R.V. Harlander and W.B. Kilgore, Phys. Rev. Lett. 88 (2002) 201801, hep-ph/0201206

[14] C. Anastasiou, L.J. Dixon, K. Melnikov and F. Petriello, Phys. Rev. D69 (2004) 094008, hep-ph/0312266

[15] C. Anastasiou and K. Melnikov, Nucl. Phys. B646 (2002) 220, hep-ph/0207004

[16] V. Ravindran, J. Smith and W.L. van Neerven, Nucl. Phys. B665 (2003) 325, hep-ph/0302135

[17] C. Anastasiou, K. Melnikov and F. Petriello, hep-ph/0501130

[18] A. Vogt, Phys. Lett. B497 (2001) 228, hep-ph/0010146

[19] S. Catani, D. de Florian, M. Grazzini and P. Nason, JHEP 0307 (2003) 028, hep-ph/0306211

[20] J.A.M. Vermaseren, A. Vogt and S. Moch, hep-ph/0504242, to appear in Nucl. Phys. B

[21] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B688 (2004) 101, hep-ph/0403192

[22] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. B691 (2004) 129, hep-ph/0404111

[23] S. Forte and G. Ridolfi, Nucl. Phys. B650 (2003) 229, hep-ph/0209154

[24] E. Gardi and R. G. Roberts, Nucl. Phys. B653 (2003) 227, hep-ph/0210429

[25] S. Catani, M.L. Mangano and P. Nason, JHEP 9807 (1998) 024, hep-ph/9806484

[26] D. de Florian and W. Vogelsang, Phys. Rev. D71 (2005) 114004, hep-ph/0501258

[27] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys. Lett. B400 (1997) 379, hep-ph/9701390

[28] M. Czakon, Nucl. Phys. B710 (2005) 485, hep-ph/0411261

[29] J.A.M. Vermaseren, Int. J. Mod. Phys. A14 (1999) 2037, hep-ph/9806280

[30] E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. A15 (2000) 725, hep-ph/9905237

[31] S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. 43 (2002) 3363, hep-ph/0110083
S. Moch and P. Uwer, to appear

[32] G. P. Korchemsky, Mod. Phys. Lett. A4 (1989) 1257

[33] J. Kodaira and L. Trentadue, Phys. Lett. B112 (1982) 66

[34] S.A. Larin, P. Nogueira, T. van Ritbergen and J.A.M. Vermaseren, Nucl. Phys. B492 (1997) 338, hep-ph/9605317

[35] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B646 (2002) 181, hep-ph/0209100

[36] S. Catani, M.L. Mangano, P. Nason and L. Trentadue, Nucl. Phys. B478 (1996) 273, hep-ph/9604351

[37] A. Vogt, Phys. Lett. B471 (1999) 97, hep-ph/9910545

[38] J.A.M. Vermaseren, math-ph/0010025