The Three-Loop Splitting Functions in QCD:
The Helicity-Dependent Case

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Abstract

We present the next-to-next-to-leading order (NNLO) contributions to the main splitting functions for the evolution of longitudinally polarized parton densities of hadrons in perturbative QCD. The quark-quark and gluon-quark splitting functions have been obtained by extending our previous all Mellin-N calculations to the structure function g_1 in electromagnetic deep-inelastic scattering (DIS). Their quark-gluon and gluon-gluon counterparts have been derived using third-order fixed-N calculations of structure functions in graviton-exchange DIS, relations to the unpolarized case and mathematical tools for systems of Diophantine equations. The NNLO corrections to the splitting functions are small outside the region of small momentum fractions x where they exhibit a large double-logarithmic enhancement, yet the corrections to the evolution of the parton densities can be unproblematic down to at least $x \approx 10^{-4}$.

1 Introduction

The splitting functions for the scale dependence (evolution) of parton densities [1–3], or anomalous dimensions of twist-2 operators [4–10] in the light-cone operator-product expansion (OPE) [11], are important universal (process independent) quantities in perturbative QCD. A little more than ten years ago, we completed the calculation of the third-order (next-to-next-to-leading order, NNLO) corrections $P_{ik}^{(2)}$, $i,k = \mathrm{q,g}$ for the helicity-averaged (unpolarized) case [12, 13].

These calculations were performed in the approach of Ref. [14, 15] where physical quantities, specifically structure functions in inclusive deep-inelastic scattering (DIS), are calculated via forward amplitudes in dimensional regularization [16–19]. In order to access also the lower row of the NNLO flavour-singlet splitting-function matrix, i.e., $P_{\mathrm{gq}}^{(2)}$ and $P_{\mathrm{gg}}^{(2)}$, in a third-order calculation, this procedure requires the inclusion of a process other than standard gauge-boson exchange DIS. The method of choice, cf. Ref. [20], was to include DIS via a scalar ϕ coupling directly only to gluons via $\phi\, G_a^{\mu\nu} G_{a,\mu\nu}$, where $G_a^{\mu\nu}$ is the gluon field strength tensor, as realized in the Standard Model by the Higgs boson in the limit of a heavy top quark and five massless flavours [21, 22].

A corresponding calculation was performed six years ago for the structure function g_1 in polarized photon-exchange DIS, which is sufficient to extend the determination of the helicity-dependent (polarized) splitting functions [23–25] to NNLO for the upper-row quantities ΔP_{qq} and ΔP_{qg}. Since we had no access to the corresponding lower-row splitting functions, these results were only briefly discussed in Ref. [26]. There is no helicity-sensitive analogue to the above Higgs-boson exchange in the Standard Model or an effective theory derived from it (initially a pseudoscalar χ with a $\chi\,\varepsilon_{\mu\nu\rho\sigma} G_a^{\mu\nu} G_a^{\rho\sigma}$ coupling to gluons was tried, which however cannot probe spin information either, as also χ is a scalar under the rotation group).

This leaves only working in supersymmetry, as in Ref. [27] for the determination of the NNLO quark-gluon antenna function, or considering DIS by graviton exchange. We have chosen to adopt the second option, which is easier to implement in our setup and offers additional information and checks by accessing all four splitting functions ΔP_{ik} as well as their unpolarized counterparts and a full set of physical evolution kernels for both the unpolarized and the polarized case.

The basic formalism for graviton-exchange DIS has been developed in Ref. [28]; for a recent application see also Ref. [29]. There are three structure functions $H_{1,2,3}$ in the unpolarized case, of which three combinations can be formed which are analogous to F_2 (no gluon contribution at order α_{s}^0), F_ϕ (no quark contribution at order α_{s}^0) and F_L (neither) in gauge-boson and scalar DIS. In the polarized case there are two structure functions, H_4 and H_6, where $H_{\bar{4}} = H_4 - H_6$ and H_6 involve only the quark and gluon distributions, respectively, at the leading order, in perfect analogy with the system (F_2, F_ϕ) that we employed for obtaining the unpolarized splitting functions.

We have performed complete second-order calculations of all these quantities. At three loops, however, gravition exchange leads to a large number of integrals with a higher numerator complexity than encountered in the calculations for Refs. [12, 13, 26]. Hence repeating the step from fixed-N Mellin moments [14, 15] to all-N results would require a lot of time and/or considerably improved algorithms. We have therefore resorted to calculating $\Delta P_{\mathrm{gq}}^{(2)}$ and $\Delta P_{\mathrm{gg}}^{(2)}$ for fixed

(odd) values of N. Substantial improvement in our diagram handling and in the FORM [30–32] implementation of the MINCER program [33, 34], see Ref. [35], together with the availability of sufficient computing resources, have enabled us to completely determine $\Delta P_{\text{gq}}^{(2)}(N)$ for $3 \leq N \leq 27$ and $\Delta P_{\text{gg}}^{(2)}(N)$ for $3 \leq N \leq 25$ (the $N = 1$ moments are not accessible in this calculation [28]), and both for specific colour factors up to $N = 29$.

Initially the extension to high moments was intended to facilitate approximate x-space results, analogous to but much more accurate than those obtained in Ref. [36] based on the moments of Ref. [37] for the unpolarized case, which would suffice at all x-values relevant to 'spin physics' in the foreseeable future. Similar to the somewhat simpler case of transverse polarization in Ref. [38], however, it turned out that it is possible to reach values of N for which even the most complicated parts could be determined completely from the moments and additional endpoint information, in particular the suppression of $P_{ik}(x) - \Delta P_{ik}(x)$ by two powers of $(1-x)$ in the threshold limit $x \rightarrow 1$ in a suitable factorization scheme. The crucial step in this determination is the solution of systems of Diophantine equations for which we have, besides in-house tools coded in FORM, made use of a publicly available program [39] using the LLL-based [40] algorithm described in Ref. [41].

Consequently we are now in the position to present the complete NNLO contributions $\Delta P_{ik}^{(2)}$ to the helicity-difference splitting functions in perturbative QCD. The remainder of this article is organized as follows: In Section 2 we set up our notations and discuss aspects of the second-order calculations and results relevant to our determination of the third-order corrections which we turn to in Section 3. Our N-space results for $\Delta P_{ik}^{(2)}$ are presented in Section 4, and the corresponding x-space expressions in Section 5, where we also briefly illustrate the numerical size of the NNLO contributions to the evolution of polarized parton densities. We summarize our results in Section 6. Some additional information on scheme transformations and graviton-exchange DIS is collected in the Appendix. A brief account of this research has been presented before in Ref. [42].

2 Notations and second-order results

The unpolarized and polarized parton densities of a longitudinally polarized nucleon are given by

$$f_i(x,\mu^2) \; = \; f_i^+(x,\mu^2) + f_i^-(x,\mu^2) \tag{2.1}$$

and

$$\Delta f_i(x,\mu^2) \; = \; f_i^+(x,\mu^2) - f_i^-(x,\mu^2) \tag{2.2}$$

where f_i^+ and f_i^- represent the number distributions of the parton type i with positive and negative helicity, respectively, in a nucleon with positive helicity. Here x denotes the fraction of the nucleon's momentum carried by the parton, and μ the mass-factorization scale which can be identified with the coupling-constant renormalization scale without loss of information.

The scale dependence of the quantities in Eqs. (2.1) and Eqs. (2.2) is governed by the renormalization-group evolution equations

$$\frac{d}{d\ln\mu^2} (\Delta)f_i(x,\mu^2) \; = \; \left[(\Delta)P_{ik}(\alpha_{\text{s}}(\mu^2)) \otimes (\Delta)f_k(\mu^2)\right](x) \tag{2.3}$$

where $\otimes$ stands for the Mellin convolution in the momentum variable, given by

$$[a \otimes b](x) = \int_x^1 \frac{dy}{y} a(y) b\left(\frac{x}{y}\right) \tag{2.4}$$

if no $1/(1-x)_+$-distribution are involved. The splitting functions $(\Delta)P_{ik}$ in Eq. (2.3) admit an expansion in powers of the strong coupling constant α_s which we write as

$$(\Delta)P_{ik}(x,\mu^2) = \sum_{n=0} a_s^{n+1} (\Delta)P_{ik}^{(n)}(x) \tag{2.5}$$

with

$$a_s \equiv \frac{\alpha_s(\mu^2)}{4\pi} . \tag{2.6}$$

Using symmetries, the system (2.3) of $2n_f + 1$ coupled integro-differential equations, where n_f denotes the numbers of effectively massless flavours, can be reduced to $2n_f - 1$ scalar flavour non-singlet equations and the 2×2 system

$$\frac{d}{d\ln\mu^2} \begin{pmatrix} \Delta f_q \\ \Delta f_g \end{pmatrix} = \begin{pmatrix} \Delta P_{qq} & \Delta P_{qg} \\ \Delta P_{gq} & \Delta P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Delta f_q \\ \Delta f_g \end{pmatrix} \equiv \Delta P \otimes \Delta f \tag{2.7}$$

for the polarized gluon density $\Delta f_g(x,\mu^2)$ and the flavour-singlet quark distribution

$$\Delta f_q(x,\mu^2) = \sum_{i=1}^{n_f} \left\{ \Delta f_{q_i}(x,\mu^2) + \Delta f_{\bar{q}_i}(x,\mu^2) \right\} . \tag{2.8}$$

The quark-quark splitting function ΔP_{qq} in Eq. (2.7) can be decomposed as

$$\Delta P_{qq}^{(n)}(x) = \Delta P_{ns}^{+(n)}(x) + \Delta P_{ps}^{(n)}(x) \tag{2.9}$$

into non-singlet and pure singlet components. The former is related by $\Delta P_{ns}^+ = P_{ns}^-$ to an unpolarized quantity calculated in Ref. [12], the latter starts only at $n = 1$ and is specific to the present polarized case. It is often convenient to consider the Mellin transforms of all quantities, given by

$$a(N) = \int_0^1 dx \, x^{N-1} a(x) \tag{2.10}$$

and an obvious generalization for plus-distributions, since the convolutions (2.4) correspond to simple products in N-space, $[a \otimes b](N) = a(N) b(N)$.

The complete next-to-leading order (NLO) contributions $\Delta P_{ik}^{(1)}$ for the quantities in Eq. (2.7) have been derived almost 20 years ago in Ref. [23] in N-space using the OPE and in Refs. [24,25] in x-space, using the lightlike axial-gauge approach of Refs. [2,3]. Some years ago, we have checked these results, and obtained $\Delta P_{qq}^{(2)}$ and $\Delta P_{qg}^{(2)}$, by extending the calculations for Refs. [13,43] to the structure function g_1 in polarized DIS which was first addressed beyond the first order in Ref. [44]. All these calculations used dimensional regularization, and thus needed to address the issue of the Dirac matrix γ_5 in $D \neq 4$ dimensions which enters via the quark helicity-difference projector.

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The calculations in Ref. [23] used the 'reading-point' scheme for γ_5 [45]; those in Refs. [24, 25] were carried out primarily with the 't Hooft/Veltman prescription [46, 47], but included checks also using the so-called Larin scheme [48, 49],

$$\not{p}\,\gamma_{5,L} = \frac{1}{6}\,\varepsilon_{\mu\nu\rho\sigma}\,p^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma\,, \tag{2.11}$$

where the resulting contractions of two ε-tensors are evaluated in terms of the D-dimensional metric. All our calculations have been carried out using the Larin scheme which is equivalent to the 't Hooft/Veltman prescription for the present massless case.

Quantities calculated using Eq. (2.11) need to be subjected to a factorization scheme transformation in order to arrive at expressions in the standard $\overline{\text{MS}}$ scheme [50, 51], for example

$$g_1 = C_{g_1,L}\,\Delta_L = (C_{g_1,L}\,Z^{-1})\,(Z\,\Delta f_L) = C_{g_1}\,\Delta f \tag{2.12}$$

where we have switched to a matrix notation in N-space and suppressed all function arguments. Denoting the perturbative expansion of the transformation matrix by

$$Z(x,\mu^2) = 1 + \sum_{n=1} a_{\mathrm{s}}^n\,Z^{(n)}(x) = 1 + \sum_{n=1} a_{\mathrm{s}}^n \begin{pmatrix} z_{\mathrm{qq}}^{(n)}(x) & z_{\mathrm{qg}}^{(n)}(x) \\ z_{\mathrm{gq}}^{(n)}(x) & z_{\mathrm{gg}}^{(n)}(x) \end{pmatrix}, \tag{2.13}$$

the transformation (2.12) of the coefficient functions C_{g_1} and the parton densities Δf leads to

$$\begin{aligned}
\Delta P = {}& a_{\mathrm{s}}\,\Delta P^{(0)} \\
& + a_{\mathrm{s}}^2\left\{ \Delta P_L^{(1)} + [Z^{(1)}, \Delta P^{(0)}] - \beta_0\,Z^{(1)} \right\} \\
& + a_{\mathrm{s}}^3\left\{ \Delta P_L^{(2)} + [Z^{(2)}, \Delta P^{(0)}] + [Z^{(1)}, \Delta P_L^{(1)}] - [Z^{(1)}, \Delta P^{(0)}]Z^{(1)} \right. \\
& \left. \qquad + \beta_0\left((Z^{(1)})^2 - 2Z^{(2)}\right) - \beta_1\,Z^{(1)} \right\} + O(a_{\mathrm{s}}^4)
\end{aligned} \tag{2.14}$$

for the splitting functions in the $\overline{\text{MS}}$ scheme, where $[a, b]$ denotes the standard matrix commutator. Here β_0 and β_1 are the leading two coefficients in the expansion of the beta function of QCD,

$$\frac{da_{\mathrm{s}}}{d\ln\mu^2} = \beta(a_{\mathrm{s}}) = -\sum_{\ell=0} a_{\mathrm{s}}^{\ell+2}\,\beta_\ell\,, \tag{2.15}$$

which to NNLO is given by [4, 5, 52–55]

$$\begin{aligned}
\beta_0 &= \tfrac{11}{3}\,C_A - \tfrac{2}{3}\,n_f\,, \\
\beta_1 &= \tfrac{34}{3}\,C_A^2 - \tfrac{10}{3}\,C_A\,n_f - 2C_F\,n_f\,, \\
\beta_2 &= \tfrac{2857}{54}\,C_A^3 - \tfrac{1415}{54}\,C_A^2\,n_f - \tfrac{205}{18}\,C_F\,C_A\,n_f + C_F^2\,n_f + \tfrac{79}{54}\,C_A\,n_f^2 + \tfrac{11}{9}\,C_F\,n_f^2
\end{aligned} \tag{2.16}$$

with $C_A = n_c = 3$ and $C_F = (n_c^2 - 1)/(2n_c) = 4/3$ in $SU(n_c = 3)$. β_0 and β_1 are scheme-independent in massless perturbative QCD; β_2 is given in the $\overline{\text{MS}}$ scheme adopted in this article.

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The transformation matrix has been determined to NNLO in Ref. [56] as

$$Z_{ik} = \delta_{ik} + \delta_{iq}\delta_{kq}\left(a_s z_{ns}^{(1)} + a_s^2\left\{z_{ns}^{(2)} + z_{ps}^{(2)}\right\}\right) + O(a_s^3).$$ (2.17)

Its non-singlet entries can be fixed by the relation between the corresponding coefficient functions for g_1 and the structure function F_3 which is known to order α_s^3 [57]; the critical part is the pure-singlet part for which, as far as we know, only that one calculation has been performed so far. For the convenience of the reader the results are included in Appendix A. For $z_{qg}^{(n)} = z_{gg}^{(n)} = 0$, Eq. (2.14) leads to the following transformations of the NLO and NNLO splitting functions:

$$\Delta P_{qq}^{(1)} = \Delta P_{qq,L}^{(1)} - \beta_0 z_{qq}^{(1)} - \Delta P_{qg}^{(0)} z_{gq}^{(1)},$$

$$\Delta P_{qg}^{(1)} = \Delta P_{qg,L}^{(1)} + \Delta P_{qg}^{(0)} z_{qq}^{(1)},$$

$$\Delta P_{gq}^{(1)} = \Delta P_{gq,L}^{(1)} - \Delta P_{gq}^{(0)} z_{qq}^{(1)} + \left(\Delta P_{qq}^{(0)} - \Delta P_{gg}^{(0)} - \beta_0\right) z_{gq}^{(1)},$$

$$\Delta P_{gg}^{(1)} = \Delta P_{gg,L}^{(1)} + \Delta P_{qg}^{(0)} z_{gq}^{(1)}$$ (2.18)

and

$$\Delta P_{qq}^{(2)} = \Delta P_{qq,L}^{(2)} + \beta_0\left((z_{qq}^{(1)})^2 - 2 z_{qq}^{(2)}\right) - \beta_1 z_{qq}^{(1)} - \Delta P_{qg,L}^{(1)} z_{gq}^{(1)} - \Delta P_{qg}^{(0)} z_{gq}^{(2)},$$

$$\Delta P_{qg}^{(2)} = \Delta P_{qg,L}^{(2)} + \Delta P_{qg,L}^{(1)} z_{qq}^{(1)} + \Delta P_{qg}^{(0)} z_{qq}^{(2)},$$

$$\Delta P_{gq}^{(2)} = \Delta P_{gq,L}^{(2)} - \left(\Delta P_{gq,L}^{(1)} - \Delta P_{gq}^{(0)} z_{qq}^{(1)}\right) z_{qq}^{(1)} - \Delta P_{gq}^{(0)} z_{qq}^{(2)}$$
$$+ \left(\Delta P_{gg}^{(0)} - \Delta P_{qq}^{(0)} + \beta_0\right) z_{qq}^{(1)} z_{gq}^{(1)} - \left(\Delta P_{gg}^{(0)} - \Delta P_{qq}^{(0)} + 2\beta_0\right) z_{gq}^{(2)}$$
$$+ \left(\Delta P_{qq,L}^{(1)} - \Delta P_{gg,L}^{(1)} - \beta_1 - \Delta P_{qg}^{(0)} z_{gq}^{(1)}\right) z_{gq}^{(1)},$$

$$\Delta P_{gg}^{(2)} = \Delta P_{gg,L}^{(2)} + \Delta P_{qg,L}^{(1)} z_{gq}^{(1)} + \Delta P_{qg}^{(0)} z_{gq}^{(2)}.$$ (2.19)

These expressions are reduced to the standard scheme transformation of Refs. [23–25, 56] by dropping all contributions with $z_{gq}^{(1)}$ or $z_{gq}^{(2)}$; it will become clear below why these terms have been included in Eqs. (2.18) and (2.19).

It is instructive to consider the $x \to 1$ threshold limit of the splitting functions. It is expected that the physical probability of a helicity flip is suppressed by two powers in $(1-x)$ in this limit [58]. Hence the differences

$$\delta_{ik}^{(n)} \equiv P_{ik}^{(n)} - \Delta P_{ik}^{(n)}$$ (2.20)

should be suppressed, in a 'physical' factorization scheme, by a factor of $(1-x)^2$, or $1/N^2$ in N-space, relative to the respective sums which behave (modulo logarithms) as $(1-x)^{-1}$ or N^0 for $ik = qq$, gg and $(1-x)^0$ or N^{-1} for $ik = qg$, gq. For the scheme-independent leading-order (LO) splitting functions, the differences (2.20) read

$$\delta_{qq}^{(0)}(x) = 0,$$

$$\delta_{ik}^{(0)}(x) = \text{const} \cdot (1-x)^2 + \dots \quad \text{for} \quad ik = qg, gq, gg.$$ (2.21)

The corresponding NLO results for the $\overline{\text{MS}}$ splitting functions [23–25] are given by

$$\delta_{ik}^{(1)}(x) = O((1-x)^a) \quad \text{for} \quad ik = \text{qq, gg } (a=1), \text{ qg } (a=2),\tag{2.22}$$

$$\delta_{gq}^{(1)}(x) = 8C_F(C_A-C_F)\ln(1-x) + \tfrac{44}{3}C_F C_A - 6C_F^2 - \tfrac{8}{3}C_F n_f$$

$$- (1-x)\left\{8C_F(C_A-C_F)\ln(1-x) + \left(\tfrac{20}{3}C_F C_A + 2C_F^2 - \tfrac{8}{3}C_F n_f\right)\right\}\tag{2.23}$$

$$+ O((1-x)^2) \ .$$

Interestingly, as already noted in Ref. [26], all 10 terms in Eq. (2.23) can be removed by including the simple additional term $z_{gq}^{(1)} = -\Delta P_{gq}^{(0)}$ in the NLO scheme transformation (2.18). The splitting functions $\Delta P_{qg}^{(1)}(x)$ and $\Delta P_{gq}^{(1)}(x)$ are shown, together with their unpolarized counterparts, in Fig. 1 in the standard scheme, from now on denoted by 'M' wherever required, that uses only Eq. (2.17) and an alternative scheme ('A') that also includes this additional term.

The issue of the physical large-x behaviour of the helicity-dependent quark-gluon splitting can be addressed by studying suitable flavour-singlet physical evolution kernels (or physical anomalous dimensions) for structure functions in unpolarized and polarized DIS. Graviton-exchange DIS, for which the basic formalism was worked out in Ref. [28], provides a sufficiently large set of structure functions. It is convenient to combine and normalize four of these functions as

$$H_{\text{u}} = \begin{pmatrix} H_{\bar{2}} \\ H_3 \end{pmatrix} = \begin{pmatrix} C_{\bar{2},q} & C_{\bar{2},g} \\ C_{3,q} & C_{3,g} \end{pmatrix} \begin{pmatrix} f_q \\ f_g \end{pmatrix} \equiv C_{\text{u}} f\tag{2.24}$$

with $H_{\bar{2}} = H_2 - 4H_3$ in the unpolarized case, and

$$H_{\text{p}} = \begin{pmatrix} H_{\bar{4}} \\ H_6 \end{pmatrix} = \begin{pmatrix} C_{\bar{4},q} & C_{\bar{4},g} \\ C_{6,q} & C_{6,g} \end{pmatrix} \begin{pmatrix} f_q \\ f_g \end{pmatrix} \equiv C_{\text{p}} \Delta f\tag{2.25}$$

with $H_{\bar{4}} = 2(H_4 - H_6)$ in the polarized case, where we have changed the x^n prefactors relative to Eq. (31) of Ref. [28] such that $(C_{\text{u}})_{ij} = (C_{\text{p}})_{ij} = \delta_{ij}$ at LO. The corresponding NLO coefficient functions can be found in Appendix B. The physical-kernel matrices K_a, $a = \text{u, p}$ (for the renormalization scale $\mu_R^2 = Q^2$) are obtained from the coefficient functions, the beta function (2.15) and the respective unpolarized ($P_{\text{u}} = P$) and polarized ($P_{\text{p}} = \Delta P$) splitting functions, cf. Eq. (2.7), by

$$\frac{dH_a}{d\ln Q^2} = \left(\beta(a_s)\frac{dC_a}{da_s} + C_a P_a\right)C_a^{-1} H_a \equiv K_a H_a \ .\tag{2.26}$$

The expansion of this result to order a_s^3 can be read off from Eq. (2.14) for $Z = C_a$.

We have performed complete two-loop calculations of these structure functions, recovering both the unpolarized and polarized NLO flavour-singlet splitting functions from graviton-exchange DIS, and used these results to obtain the NLO physical kernels $K_{\text{u}}^{(1)}(x)$ and $K_{\text{p}}^{(1)}(x)$. The respective off-diagonal elements for the systems (2.24) and (2.25) are compared in Fig. 2. It is clear, also from the corresponding analytical results, that also the large-x limits of the kernels $K_{32}^{(1)}(x)$ and $K_{64}^{(1)}(x)$ corresponding to the splitting functions $(\Delta)P_{gq}^{(1)}$ are consistent with the expectation of Ref. [58]; hence Eq. (2.23) is indeed a unphysical feature of the standard transformation to the $\overline{\text{MS}}$ scheme.
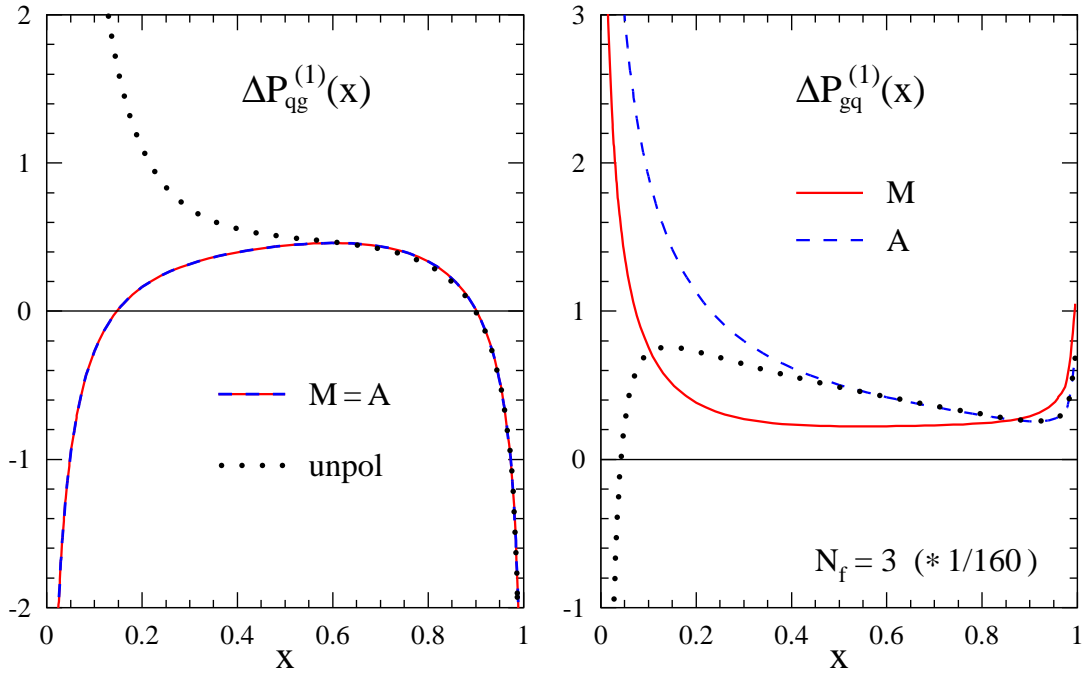
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Figure 1: The NLO contributions to the off-diagonal splitting functions in Eq. (2.7), compared to their unpolarized counterparts. The polarized results are shown as published in Refs. [23–25] ('M') and after including an additional term $z_{gq}^{(1)} = -\Delta P_{gq}^{(0)}$ in the transformation (2.14) from the Larin scheme ('A'), which removes all $(1-x)^{0,1}$ terms from the quantity $\delta_{gq}^{(1)}(x)$ in Eq. (2.23).
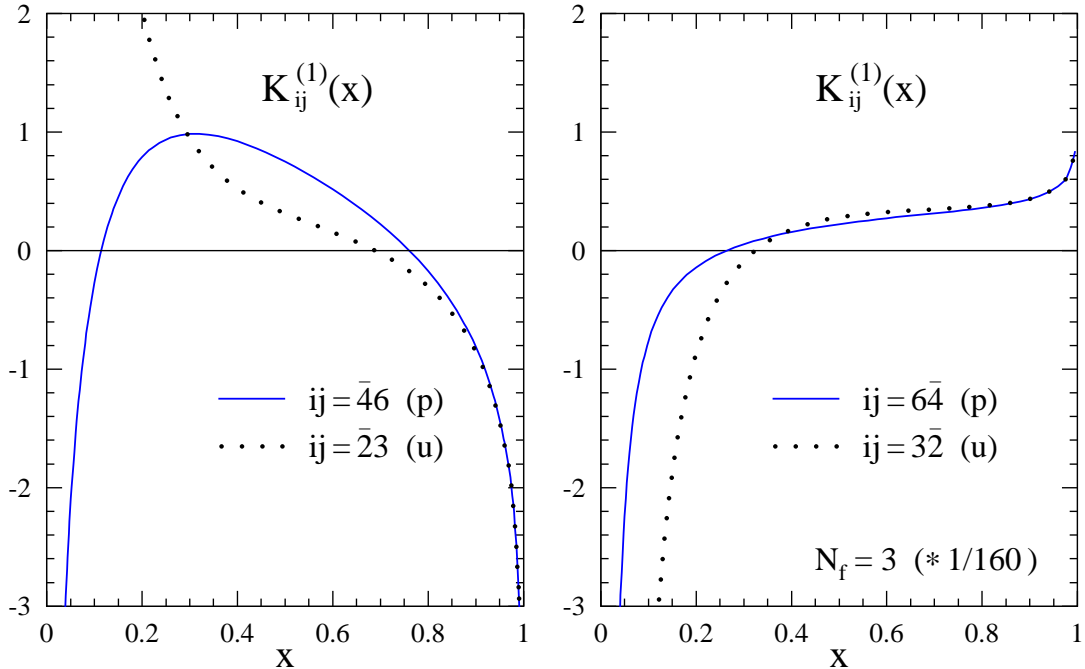

Figure 2: The NLO contributions to the off-diagonal elements of the physical-kernel matrices for the systems $(H_{\bar{2}}, H_3)$ and $(H_{\bar{4}}, H_6)$ of structure functions in unpolarized and polarized graviton-exchange DIS [28] as defined in and below Eqs. (2.24) and (2.25). The factor $1/160 \simeq 1/(4\pi)^2$ approximately converts the results from our small expansion parameter (2.6) to a series in α_s.

3 Determination of the third-order corrections

As before, we have calculated inclusive DIS via the optical theorem, which relates the probe (q)-parton (p) total cross sections (with $Q^2 = -q^2 > 0$ and $p^2 = 0$) to forward amplitudes, and a dispersion relation in x that provides the N-th moments from the coefficient of $(2p \cdot q)^N$ [14,15]. For the splitting functions $\Delta P_{qq}^{(2)}$ and $\Delta P_{qg}^{(2)}$ we have extended the three-loop all-N calculations of Refs. [12, 13] to the photon-exchange structure function g_1. As discussed in Ref. [26], a large number of additional integrals, arising from a fairly small set of top-level integrals with higher numerator powers, had to be calculated for this extension; their determination took several months.

The situation is far worse in the case of graviton-exchange DIS, which is our means to access also $\Delta P_{gq}^{(2)}$ and $\Delta P_{gg}^{(2)}$, in terms of both the complexity and the number of new top-level integrals. We have therefore not tried a direct all-N calculation in this case, but managed to set up a two-step procedure with the same result. The first step is a calculation of fixed-N moments for the structure functions in polarized graviton-exchange DIS, as in Refs. [14, 15] using the MINCER program [33, 34], but up to much higher moments in particular for H_6, cf. Eq. (2.25). The second step is the determination of the all-N expressions for $\Delta P_{gq}^{(2)}$ and $\Delta P_{gg}^{(2)}$ from the moments calculated in the first step together with insight into the structure of these functions.

In order to drive the first step to a point where the second became possible, and its results could be verified by one or two yet higher moments, improvements had to be made in our diagram preparation and the MINCER code, see also Ref. [35]. The diagrams were generated, as before, with a special version of QGRAF [59]. Unlike in our previous calculations, however, the diagrams with the same group-invariant colour factor, the same topology and subtopology (see below), and the same flavour structure have been combined in the 'diagram' files which are managed, as before, using the database program MINOS [60]. In this way the number of third-order diagrams has been reduced from 5176 to 1142 and from 15208 to 1249 for the quark and gluon contributions, respectively, to H_4 and H_6. The combined diagrams take roughly as much time as the most difficult individual diagram in the set, which leads to an overall gain in speed by a factor of three to five.

The overall most demanding subtopology, in terms of execution time and required disk space, is NO_{25} (see Fig. 3), i.e, the most difficult p-flow in the most difficult three-loop topology. Also notable are the LA_{14} (also shown in Fig. 3), $O4_{57}$, $O2_{26}$ cases, where the momentum p flows through four internal lines, and the three-line BE_{57} and BE_{28} 'Benz' cases. The largest diagram calculated took about 10^7 CPU seconds and required 6.7 TB of disk space for the projection on N.
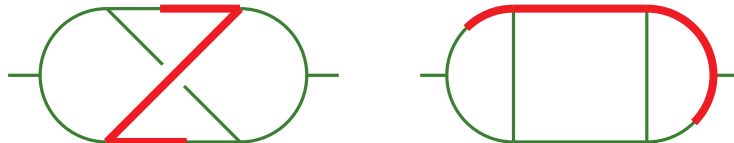

Figure 3: The NO_{25} (left) and LA_{14} (right) subtopologies for the forward probe-parton amplitudes. The momentum q of the probe, with $q^2 < 0$, enters the diagram from the right and leaves on the left. The parton momentum p, with $p^2 = 0$, flows through the fat (in the coloured version: red) lines.

The results for $3 \leq N \leq 25$ were employed for obtaining the all-N expressions for $\Delta P_{\text{gq}}^{(2)}$ and $\Delta P_{\text{gg}}^{(2)}$. For checking these expressions, the quark case was computed completely at $N = 27$ and in the 'planar limit' $C_A - 2C_F \rightarrow 0$ at $N = 29$, and the gluon case for the C_A^3 terms at $N = 27$ and $N = 29$. The latter was possible since most of the slowest and largest diagrams do not contribute to this colour factor, which is the most complicated one in terms of the structure of the splitting function.

Most of the diagram calculations were performed on the `ulgqcd` cluster in Liverpool, using TFORM [31,32] with 16 workers on more than 200 cores; the hardest diagrams at the highest values of N were calculated on a new high-end computer at NIKHEF. For the previous optimization of MINCER we were also able to use a multi-core workstation at DESY-Zeuthen.

As an example, we show the non-ζ_3 parts of the moments $3 \leq N \leq 25$ of the C_F^3 part of $\Delta P_{\text{gq}}^{(2)}$ in the Larin scheme, i.e., before the transformation of the output of the mass factorization to $\overline{\text{MS}}$:

$N = 3$: $186505/(3^5\, 2^5)$

$N = 5$: $9473569/(5^5\, 3^5\, 2^2)$

$N = 7$: $-509428539731/(7^5\, 5^4\, 3^2\, 2^{11})$

$N = 9$: $-266884720969207/(7^4\, 5^5\, 3^{10}\, 2^7)$

$N = 11$: $-3349566589170829651/(11^5\, 7^4\, 5^4\, 3^9\, 2^7)$

$N = 13$: $-751774767290148022507/(13^5\, 11^4\, 7^3\, 5^3\, 3^7\, 2^8)$ $\qquad$ (3.1)

$N = 15$: $-23366819019913026454180147/(13^4\, 11^4\, 7^4\, 5^5\, 3^9\, 2^{16})$

$N = 17$: $-3052142278186280906801741 70947/(17^5\, 13^4\, 11^4\, 7^4\, 5^4\, 3^{10}\, 2^{10})$

$N = 19$: $-5706796486846568075781997 91973487/(19^5\, 17^4\, 13^4\, 11^4\, 7^3\, 5^5\, 3^7\, 2^9)$

$N = 21$: $-20443040920892357622791 48843319979/(19^4\, 17^4\, 13^4\, 11^4\, 7^5\, 5^3\, 3^9\, 2^{11})$

$N = 23$: $-289119840113761409530260 333250139823739/(23^5\, 19^4\, 17^4\, 13^4\, 11^4\, 7^4\, 5\, 3^9\, 2^{13})$

$N = 25$: $-18904732552838029376788307 45102921869938637/(23^4\, 19^4\, 17^4\, 13^5\, 11^4\, 7^4\, 5^{10}\, 3^5\, 2^{12})$

In order to obtain, with certainty, the analytical forms of $\Delta P_{\text{gq}}^{(2)}(N)$ and $\Delta P_{\text{gg}}^{(2)}(N)$ from only 12 moments, we need to make use of additional constraints on the structure of these functions. At least up to NNLO, the splitting functions can be expressed in terms of harmonic sums [61], see also Ref. [62], which can be recursively defined by

$$S_{\pm m}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^m} \qquad (3.2)$$

and

$$S_{\pm m_1, m_2, \ldots, m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2, \ldots, m_k}(i) . \qquad (3.3)$$

The sum of the absolute values of the indices m_k defines the weight of the harmonic sum. Assigning a weight m to the un-summed denominators

$$D_k^m \equiv \frac{1}{(N+k)^m} \qquad (3.4)$$

which can be expressed as differences of two harmonic sums of weight m, the N^{n}LO splitting

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functions include terms up to weight $2n+1$. For example, the $C_F^2 n_f$ contribution to $\Delta P_{\rm qg}^{(2)}(N)$ reads

$$
\begin{aligned}
\tfrac{1}{8}\,\Delta P_{\rm qg}^{(2)}(N)\Big|_{C_F^2 n_f} &= 2\Delta p_{\rm qg}\big(-S_{-4}+2S_{-2,-2}+4S_{1,-3}+2S_{1,1,1,1}-S_{1,1,2}-5S_{1,2,1}\\
&\qquad\quad +4S_{1,3}+2S_{2,-2}-6S_{2,1,1}+6S_{2,2}+7S_{3,1}-3S_4\big)\\[4pt]
&+4S_{-3}\,(D_0^2-2D_0+2D_1)+8S_{1,-2}\,(2D_1^2-D_0+D_1)\\
&+S_{1,1,1}\,(5D_0^2-2D_1^2-21/2\,D_0+12D_1)-2S_{1,2}\,(2D_0^2-2D_1^2-5D_0+5D_1)\\
&-2S_{2,1}\,(4D_0^2+2D_1^2-11D_0+11D_1)+2S_3\,(3D_0^2+6D_1^2-11D_0+11D_1)\\
&-3\zeta_3\,(2D_0^2+4D_1^2-9D_0+12D_1)-6D_2\,(S_{-2}+1)\\[4pt]
&+2S_{-2}\,(8D_1^3-5D_0^3-6D_1^2+10D_0-9D_1)-S_{1,1}\,(10D_0^3+6D_1^3-35/2\,D_0^2\\
&\quad -5D_1^2+29D_0-36D_1)+2S_2\,(4D_0^3+6D_1^3-10D_0^2-4D_1^2+17D_0-22D_1)\\[4pt]
&+S_1\,(7D_0^4+4D_1^4-43/2\,D_0^3-15D_1^3+99/2\,D_0^2+18D_1^2-78D_0+329/4\,D_1)\\
&+32D_1^5-15/2\,D_0^4-3D_1^4+59/8\,D_0^3+53/4\,D_1^3+77/8\,D_0^2+213/8\,D_1^2\\
&-1357/32\,D_0+777/16\,D_1
\end{aligned}
\tag{3.5}
$$

in the standard $\overline{\rm MS}$ scheme [56], where all harmonic sums are understood to be taken at argument N. Here we have also made used of the first of the abbreviations

$$
\Delta p_{\rm qg} = 2D_1-D_0 \,, \quad \Delta p_{\rm gq} = 2D_0-D_1
\tag{3.6}
$$

for the N-dependence of the lowest-order splitting functions, cf. Eq. (4.2) below.

If the unpolarized counterpart of Eq. (3.5) is written down in the same notation, the first two lines are the same except for the replacement of $\Delta p_{\rm qg}$ by $p_{\rm qg} = 2D_2-2D_1+D_0$. The same holds for the $C_A C_F n_f$ and $C_A^2 n_f$ contributions. As in other results in massless perturbative QCD, the number of harmonic sums is reduced by the absence of sums with index -1. This leaves seven sums of weight 3, of which one is missing in Eq. (3.5) but not the corresponding $C_A C_F n_f$ and $C_A^2 n_f$ expressions. Half of their in principle 28 coefficients with $D_{0,1}$ and $D_{0,1}^2$ are fixed by the $1/N^2$ suppression of the difference $\delta_{\rm qg}^{(2)}$ in Eq. (2.20), which is found to hold separately for each harmonic sum. Taking into account the lower-weight sums, this large-N behaviour relates as many as 24 coefficients to the unpolarized result for each of the three non-n_f colour factors.

Another crucial feature of Eq. (3.5) and all other available results for splitting functions is that all coefficients are integer in a suitable normalization. E.g., after eliminating all terms linear in D_0 and D_1 using the $1/N^3$ large-N behaviour, the remaining coefficients in Eq. (3.5) are integers once factors of 2^{w-3} have been bracketed out of the terms with sums of weight $w<3$. Consequently the equations relating the remaining coefficients to fixed-N moments are Diophantine equations, and far less that n equations are required to determine n unknown coefficients. While there are a few additional constraints, on the coefficient of the $D_{0,1}^5$ and D_1^4 terms corresponding to the $\ln^5 x$ and $x\ln^{5,4} x$ small-x logarithms and the remaining coefficients of $S_{1,1,1}(N)$, see below, it is clear that it is vital for the determination of $\Delta P_{\rm gg}^{(2)}(N)$ to have an extension of the A-scheme of Fig. 1 to NNLO, in order not to miss out on those 24 large-N constraints.
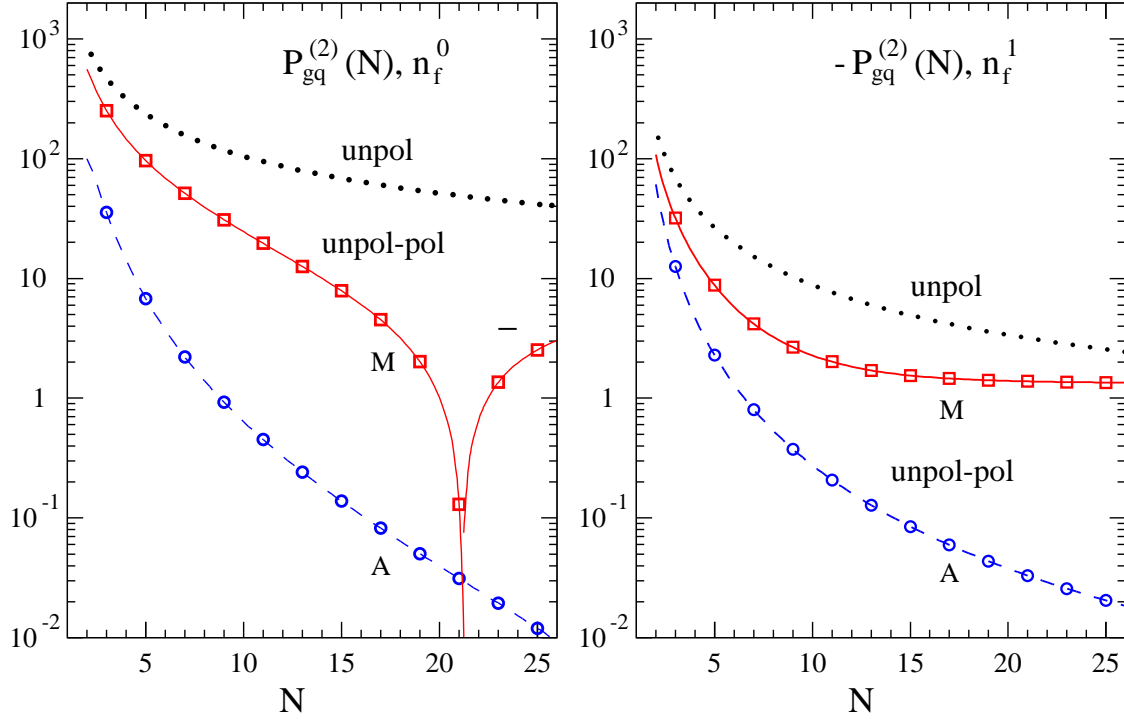
Figure 4: The NNLO differences $\delta_{gq}^{(2)}(N) = P_{gq}^{(2)}(N) - \Delta P_{gq}^{(2)}(N)$ for the non-n_f and n_f^1 terms in the M and A schemes for $C_A = 3$ and $C_F = 4/3$, compared to the unpolarized result. The symbols show moments calculated using MINCER, the solid and dashed lines the exact all-N results presented below. As at NLO, cf. Fig. 1, the M-scheme difference turns negative at large N.

The double-logarithmic $S_{1,1,1}$ and $S_{1,1,1,1}$ contributions to $\Delta P_{gq}^{(2)}(N)$ can be derived from the calculations of polarized graviton-exchange DIS, without any reference to the unpolarized results, from the single-log threshold enhancement of the physical kernel K_p in Eq. (2.26), cf. Ref. [63]. An additional scheme transformation that removes those contributions to δ_{gq} is found to be

$$Z_{gq,A} = -a_s \Delta P_{gq}^{(0)} - \frac{1}{2} a_s^2 \Delta P_{gq,L}^{(1)} + O(a_s^3) \, . \tag{3.7}$$

The assumption that this remarkably simple transformation leads to $\delta_{gq}(x) = O((1-x)^2)$ is consistent with the results for $N \leq 25$ is illustrated in Fig. 4 for the n_f^0 and n_f^1 contributions in QCD.

The physical kernels for the system $(H_{\bar{4}}, H_6)$ also allow to settle another issue observed in Ref. [26], the apparent partial disagreement of the leading small-x logarithm of $\Delta P_{qg}^{(2)}(x)$ with the old resummation result of Ref. [64]: the $\ln^4 x$ contribution to $K_{\bar{4}6}$ agrees perfectly with that prediction, which clarifies its proper interpretation, see also Refs. [65–67]. Consequently it should be possible to use the prediction of Ref. [64], via $K_{6\bar{4}}$, also for $\Delta P_{gq}^{(2)}(x)$. Furthermore the $x\ln^5 x$ and $x\ln^4 x$ terms of this function can be fixed by extending the analysis of the small-x limits of the unfactorized expressions in Ref. [68] to the present case, see also Ref. [69].

Finally we need to briefly address the issue of denominators other than D_0 and D_1, as occurring in the sixth line of Eq. (3.5), and with sums to weight 3 in its $C_A C_F n_f$ counterpart. Due to the different leading-order structure, there are far fewer such terms here than in the unpolarized case.

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Terms with D_2 in $\Delta P_{qg}^{(2)}(N)$, D_{-1} in $\Delta P_{gq}^{(2)}(N)$ and $D_{-1}D_2$ in $\Delta P_{gg}^{(2)}(N)$ do neither affect the prime-number decomposition of the denominators of the odd-N moments, e.g., the $N = 17$ moments do not involve a factor $1/19$, cf. Eq. (3.1), nor can they lead to an overall pole at $N = 1$.

We are now ready to turn to the determination of the all-N expressions. The structure of the critical C_F^3, $C_A C_F^2$ and $C_A^2 C_F$ parts of $\Delta P_{gq}^{(2)}$ is analogous to Eq. (3.5) discussed in detail above. With the coefficients of the weight-4 sums fixed by the unpolarized result [13], we are left with 2×32 coefficients of sums at weight 3 and below combined with powers of D_0 and D_1, recall Eq. (3.4), plus at most 11 sums combined with D_{-1}. The large-N suppression of $\delta_{gq}^{(2)}$ in the A-scheme and the other endpoint constraints fix 29 or 30 of these coefficients (depending whether or not $D_{-1} S_{1,1,1}(N)$ is included in the basis set), leaving up to 45 unknown integer parameters.

We have developed FORM tools for analyzing the prime-number structure of the moments, see Eq. (3.1), and deriving relations between the remaining parameters using the Chinese remainder theorem [70]. These tools have proved sufficient, sometimes together with a brute-force scan of a few variables, for simpler cases. It is however not easy to derive more than about ten relations for the three difficult n_f^0 parts of $\Delta P_{gq}^{(2)}$. For these cases we have employed the program provided in Ref. [39], see also Refs. [40,41] to solve the remaining system of linear Diophantine equations. Since this program looks for short vectors, it is best for our purposes to eliminate 4 to 6 'unpleasant' coefficients, in particular those of low-weight combinations such as D_0^2, D_1^2, $D_0^2 S_1$, $D_1^2 S_1$, using the moments to $N = 9$ or $N = 13$, and work with the remaining 6 to 8 equations.

For example, using the moments (3.1) this procedure leads to the result

$$
\begin{aligned}
\frac{1}{8} \Delta P_{gq}^{(2)}(N)\Big|_{C_F^3} = {} & 2\Delta p_{gq}(-S_{-4} + 6 S_{-2,-2} + 4 S_{1,-3} + 2 S_{1,1,1,1} + S_{1,1,2} \\
& + 3 S_{1,2,1} - 3 S_{1,3} + 2 S_{2,-2} + 2 S_{2,1,1} - 2 S_{2,2}) \\
& - 4 S_{-3}(2 D_0^2 - D_0 + D_1) - 8 S_{1,-2}(D_1^2 - 2 D_0 + 2 D_1) \\
& + S_{1,1,1}(2 D_0^2 - 5 D_1^2 - 6 D_0 - 3/2 D_1) - 2 S_{1,2}(D_1^2 + 4 D_0 - D_1) \\
& - S_{2,1}(4 D_0^2 + 4 D_1^2 - 4 D_0 + 7 D_1) + S_3(2 D_0^2 + D_1^2 + 6 D_0 - 3/2 D_1) \\
& + 6\zeta_3(2 D_0 - D_1)(2 S_1 - 3) - 6 D_{-1}(S_{-2} + 1) \\
& - S_{-2}(8 D_1^3 + 4 D_0^2 + 18 D_1^2 - 26 D_0 + 24 D_1) - S_{1,1}(6 D_0^3 + 6 D_1^3 + 4 D_0^2 \\
& + 5 D_1^2 + 2 D_0 - 7/4 D_1) + 2 S_2(D_1^3 + 2 D_1^2 + 10 D_0 - 4 D_1) \\
& - S_1(6 D_0^4 + 7 D_1^4 + 4 D_0^3 + 23/2 D_1^3 - 27/2 D_0^2 + 39/4 D_1^2 - 8 D_0 + 23/4 D_1) \\
& - 8 D_0^5 - 12 D_1^5 + 23 D_0^4 - 28 D_1^4 - 39/4 D_0^3 - 427/8 D_1^3 - 341/8 D_0^2 \\
& - 767/8 D_1^2 + 2427/16 D_0 - 4547/32 D_1
\end{aligned}
\tag{3.8}
$$

in the standard (M) definition of the $\overline{\text{MS}}$ scheme [56], where we have again used the abbreviations (3.4) and (3.6) and suppressed the argument N of the harmonic sums. The corresponding expressions for the $C_A C_F^2$ and $C_A^2 C_F$ parts are somewhat longer, see below. The n_f-dependent terms are much shorter; their determination does not require the $N = 23$ and $N = 25$ moments.

Note the simplicity of the coefficients in Eq. (3.8), in particular those of the terms with overall weights of 5 and 4 and sums of weight 2 or higher, which strongly indicates that the result is correct even without further checks. In fact, if any erroneous information is entered for an externally fixed parameters, e.g., a wrong coefficient of D_1^5, or if the set of functions is too small, e.g., by omitting the term with D_{-1}, then either no solution exists for the system of Diophantine equations, or only solutions with nonsensically large coefficients (also) for the high-weight terms.

Nevertheless it is, of course, necessary to validate the resulting all-N formulae. For this purpose their predictions at higher values of N have been compared to additional MINCER moments such as

$$-\Delta P_{\mathrm{gq},L}^{(2)}(N=27) = 460977038358760543281329153084972633526481072 7/$$
$$\left(23^4\,19^4\,17^4\,13^4\,11^4\,7^5\,5^8\,3^{15}\,2^{13}\right)C_F^3 + \dots \tag{3.9}$$

The diagram calculations for the corresponding result at $N = 29$ have been carried out only in the planar limit $C_A - 2C_F \to 0$ at $n_f = 0$. As this result combines the three difficult all-N expressions for the C_F^3, $C_A C_F^2$ and $C_A^2 C_F$ colour factors, which have been obtained independently from each other, it provides another strong check of all these results including Eq. (3.8). Perfect agreement is found for the not entirely trivial fractions at both values of N.

The overall most difficult case was the n_f-independent, i.e., C_A^3 part of $\Delta P_{\mathrm{gg}}^{(2)}$. Also here the harmonic sums beyond weight 3 can be determined from the unpolarized case; the same holds for all terms not involving any un-summed denominators: these contribute to either the $1/(1-x)_+$ of the $\delta(1-x)$ terms the large-x limit which are the same for P_{gg} and ΔP_{gg}. This reduces the problem to the same basis set as in the case of $\Delta P_{\mathrm{gq}}^{(2)}$ at $n_f = 0$. The $1/N^2$ suppression of δ_{gg} with respect to P_{gg}, however, only removes one instead two coefficients for each harmonic sum up to weight 3.

Taking into account our additional knowledge of the coefficients of D_0^5 from Ref. [64] (this coefficient is the same for $K_{66}^{(2)}$ and $\Delta P_{\mathrm{gg}}^{(2)}$, unlike for the off-diagonal cases), D_1^5 and D_1^4, cf. Refs. [68, 69], and of $S_{1,1,1}$, cf. Ref. [63], this leaves 49 terms with D_0 and D_1 plus the functions with the 'extra' denominator $D_{-1}D_2$ corresponding to D_{-1} in the previous case of $\Delta P_{\mathrm{gq}}^{(2)}$. The non-$C_F$ parts of ΔP_{gg} are non-singlet like quantities, e.g., they are not affected by scheme transformations with $z_{\mathrm{gg}} = 0$, see Eqs. (2.18) and (2.19). Hence we could use some non-singlet heuristics, see Ref. [38], to reduce the overall basis to 52 functions, which we were able to determine using our own programs and, in the final step, Ref. [39] with 8 equations at $11 \leq N \leq 25$ for 41 unknowns.

Quite a few of the resulting coefficients are far less simple than those in Eq. (3.8), see Eq. (4.12) below; on the other hand seven coefficients put in are zero, and there are some expected relations. The result has been checked against the MINCER calculations at $N = 27$ and $N = 29$ which were finished only after we had obtained $\Delta P_{\mathrm{gg}}^{(2)}(N)$. Another important check is the first moment which is not accessible directly [28], but can be obtained by Mellin-inverting to x-space expressions in terms of harmonic polylogarithms [71] from which arbitrary moments can be calculated. The results is

$$\Delta P_{\mathrm{gg}}^{(2)}(N=1) = \beta_2^{\overline{\mathrm{MS}}} \tag{3.10}$$

see Eq. (2.16), as expected from the two previous orders. This result is the same in all factorization schemes considered here also for the C_F terms due to $\Delta P_{\mathrm{qg}}^{(n)}(N=1) = 0$ in Eq. (2.19), cf. Ref. [72].

4 The NNLO splitting functions in Mellin space

The analytical odd-N expressions of the splitting functions to NNLO can be written in terms of harmonic sums [61] as recalled in Eqs. (3.2) and (3.3) above. Our notation is different from section 3 of Refs. [12, 13]: here all sums are taken at argument N (which we usually suppress), for the additional un-summed denominators we employ the abbreviations (3.4), (3.6) and

$$\eta \equiv \{N(N+1)\}^{-1} , \quad \nu \equiv \{(N-1)(N+2)\}^{-1} . \tag{4.1}$$

In this notation the leading-order (LO) contributions [1, 6, 7] to Eq. (2.7), see also Eq. (2.9), read

$$
\begin{aligned}
\Delta P_{\text{ns}}^{(0)}(N) &= C_F\left(-4S_1 + 2\eta + 3\right) , \\
\Delta P_{\text{ps}}^{(0)}(N) &= 0 , \\
\Delta P_{\text{qg}}^{(0)}(N) &= 2n_f \Delta p_{\text{qg}} , \\
\Delta P_{\text{gq}}^{(0)}(N) &= 2C_F \Delta p_{\text{gq}} , \\
\Delta P_{\text{gg}}^{(0)}(N) &= C_A\left(-4S_1 + 8\eta + 11/3\right) - 2/3\, n_f ,
\end{aligned} \tag{4.2}
$$

and their next-to-leading order (NLO) counterparts of Refs. [23–25] are given by

$$
\begin{aligned}
\Delta P_{\text{ns}}^{+(1)}(N) &= 4C_F^2\Big(-4\left(S_{-3} - 2S_{1,-2} - S_{1,2} - S_{2,1}\right) - 3S_2 + 3/8 - 4\eta S_{-2} \\
&\quad - 2\eta S_2 + 2\left(2\eta + \eta^2 - 2D_0^2\right)S_1 - \eta - 11\eta^2 - 5\eta^3 + D_0^2 + 2D_0^3\Big) \\
&\quad + 4C_A C_F\Big(2\left(S_{-3} - S_3\right) - 4S_{1,-2} + 11/3\, S_2 - 67/9\, S_1 + 17/24 \\
&\quad + 2\eta S_{-2} + 217/18\,\eta + 35/6\,\eta^2 + 2\eta^3 - 11/3\, D_0^2\Big) \\
&\quad + 4/9\, C_F n_f\Big(-6S_2 + 10S_1 - 3/4 - 17\eta - 3\eta^2 + 6D_0^2\Big) ,
\end{aligned} \tag{4.3}
$$

$$
\Delta P_{\text{ps}}^{(1)}(N) = 4C_F n_f\Big(-5\eta + 3\eta^2 + 2\eta^3 + 4D_0^2 - 4D_0^3\Big) , \tag{4.4}
$$

$$
\begin{aligned}
\Delta P_{\text{qg}}^{(1)}(N) &= 4C_F n_f\Big(2\Delta p_{\text{qg}}\left(S_{1,1} - S_2\right) - 2\left(2D_0 - D_0^2 - 2D_1\right)S_1 \\
&\quad - 11D_0 + 9/2\, D_0^2 - D_0^3 + 27/2\, D_1 + 4D_1^2 - 2D_1^3\Big) \\
&\quad + 4C_A n_f\Big(-2\Delta p_{\text{qg}}\left(S_{-2} + S_{1,1}\right) + 4\left(D_0 - D_1 - D_1^2\right)S_1 \\
&\quad + 12D_0 - D_0^2 - 2D_0^3 - 11D_1 - 12D_1^2 - 12D_1^3\Big) ,
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
\Delta P_{\text{gq}}^{(1)}(N) &= 4C_F^2\Big(-\Delta p_{\text{gq}}\left(2S_{1,1} - S_1\right) + 2\left(D_1 + D_1^2\right)S_1 \\
&\quad - 17/2\, D_0 + 2D_0^2 + 2D_0^3 + 4D_1 + 1/2\, D_1^2 + D_1^3\Big) \\
&\quad + 4C_A C_F\Big(2\Delta p_{\text{gq}}\left(S_{1,1} - S_{-2} - S_2\right) - \left(10/3\, D_0 + 4D_0^2 + 1/3\, D_1\right)S_1 \\
&\quad + 41/9\, D_0 - 4D_0^2 + 4D_0^3 + 35/9\, D_1 + 38/3\, D_1^2 + 6D_1^3\Big) \\
&\quad + 8/9\, C_F n_f\Big(3\Delta p_{\text{gq}} S_1 - 4D_0 - D_1 - 3D_1^2\Big) ,
\end{aligned} \tag{4.6}
$$

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$$\Delta P_{\text{gg}}^{(1)}(N) = 4C_A^2 \Big(4\,(S_{1,-2}+S_{1,2}+S_{2,1}) - 2\,(S_3+S_{-3}) - 67/9\,S_1 + 8/3$$
$$- 8\eta\,(S_2+S_{-2}) + 8\,(2\eta+\eta^2-2D_0^2)\,S_1$$
$$+ 901/18\,\eta - 149/3\,\eta^2 - 24\,\eta^3 - 32D_0^2 + 32D_0^3 \Big)$$
$$+ 4/3\,C_A n_f \Big(10/3\,S_1 - 2 - 26/3\,\eta + 2\eta^2 \Big)$$
$$+ 4C_F n_f \Big(-1/2 - 7\eta + 5\eta^2 + 2\eta^3 + 6D_0^2 - 4D_0^3 \Big). \tag{4.7}$$

For completeness also including the non-singlet contribution, which is identical to the function $P_{\text{ns}}^{-(2)}(N)$ given (in a different notation) already in Eq. (3.8) of Ref. [12], the polarized next-to-next-to-leading (NNLO) quark-quark splitting function $\Delta P_{\text{qq}}^{(2)}(N)$ is the sum of

$$\Delta P_{\text{ns}}^{+(2)}(N) = 16C_F^3 \Big(-12S_{-5} + 24S_{-4,1} + 4S_{-3,-2} + 4S_{-3,2} + 12S_{-2,-3}$$
$$- 24S_{-2,1,-2} + 44S_{1,-4} - 64S_{1,-3,1} + 16S_{1,-2,-2} - 8S_{1,-2,2} - 80S_{1,1,-3}$$
$$+ 96S_{1,1,-2,1} - 16S_{1,2,-2} - 8S_{1,2,2} - 16S_{1,3,1} - 8S_{1,4} + 52S_{2,-3} - 56S_{2,-2,1}$$
$$- 16S_{2,1,-2} - 8S_{2,1,2} - 8S_{2,2,1} + 4S_{2,3} + 12S_{3,-2} - 8S_{3,1,1} + 8S_{3,2} + 4S_{4,1}$$
$$+ 4S_5 - S_{-4}\,(9+22\eta) - S_{-3,1}\,(6-32\eta) + 4\eta\,(S_{-2,2} - 2S_{-2,-2} + 2S_{3,1} + S_4)$$
$$+ 2S_{1,-3}\,(3+20\eta) + 4S_{1,-2,1}\,(3-12\eta) + 12S_{1,3} + 2S_{2,-2}\,(3+4\eta)$$
$$+ 2S_{2,2}\,(3+2\eta) + S_{-3}\,(+3\eta - 4\eta^2 - 12D_0^2) + 2S_{-2,1}\,(5\eta + 10\eta^2 + 4D_0^2)$$
$$- 4S_{1,-2}\,(7\eta + 5\eta^2 - 4D_0^2) - 4\,(S_{1,2}+S_{2,1})\,(2\eta + \eta^2 - 2D_0^2)$$
$$- S_3\,(13/4 - 4\eta - 5\eta^2 + 4D_0^2) - S_2\,(3/8 - 2\eta - 17/2\,\eta^2 - 4\eta^3 + 2D_0^2 + 4D_0^3)$$
$$- S_{-2}\,(3 - 12\zeta_3 + 2\eta - 14\eta^2 - 6\eta^3 - 2D_0^2 + 8D_0^3) + 4S_{1,1}\,(3\eta^2 + \eta^3)$$
$$- S_1\,(47/2\,\eta + 53/4\,\eta^2 + 48\eta^3 + 13\eta^4 - 18D_0^2 + 18D_0^3 - 24D_0^4)$$
$$+ \zeta_3\,(15/2 + 6\eta + 6\eta^2) + 29/32 - 215/8\,\eta + 26\eta^2 + 45\eta^3 + 49\eta^4 + 11\eta^5$$
$$+ 175/8\,D_0^2 - 43/2\,D_0^3 + 15/2\,D_0^4 - 16D_0^5 \Big)$$

$$+ 8C_F^2 C_A \Big(20S_{-5} - 40S_{-4,1} + 4S_{-3,-2} - 4S_{-3,2} - 20S_{-2,-3} - 16S_{-2,-2,1}$$
$$+ 56S_{-2,1,-2} - 68S_{1,-4} + 128S_{1,-3,1} - 64S_{1,-2,-2} + 8S_{1,-2,2} + 144S_{1,1,-3}$$
$$- 224S_{1,1,-2,1} - 32S_{1,1,3} + 16S_{1,2,-2} + 32S_{1,3,1} + 44S_{1,4} - 84S_{2,-3} + 120S_{2,-2,1}$$
$$+ 16S_{2,1,-2} + 20S_{2,3} - 20S_{3,-2} + 4S_{3,2} + 4S_{4,1} - 20S_5 + (89/3 + 34\eta)\,S_{-4}$$
$$+ 268/9\,(S_{1,2} + S_{2,1} + 2S_{1,-2} - S_{-3}) + 2S_{-3,1}\,(31/3 - 32\eta) + 4S_{-2,-2}\,(3+8\eta)$$
$$- 4\eta\,S_{-2,2} - 2S_{1,-3}\,(31/3 + 36\eta) - 4S_{1,-2,1}\,(31/3 - 28\eta) - 4S_{1,3}\,(31/3 - 4\eta)$$
$$- 2S_{2,-2}\,(31/3 + 4\eta) - 44/3\,S_{2,2} - 8S_{3,1}\,(1/3 + 2\eta) - S_4\,(23/3 + 22\eta)$$
$$+ S_{-3}\,(37/3\,\eta + 14\eta^2 + 12D_0^2) - 2S_{-2,1}\,(53/3\,\eta + 30\eta^2 + 4D_0^2)$$
$$+ 4S_{1,-2}\,(13\eta + 10\eta^2 - 4D_0^2) + S_3\,(13 + 26/3\,\eta - 10\eta^2 - 4D_0^2)$$
$$+ S_{-2}\,(9 - 36\zeta_3 - 586/9\,\eta - 34/3\,\eta^2 + 38/3\,D_0^2 + 8D_0^3)$$
$$- S_2\,(151/12 + 350/9\,\eta + 46/3\,\eta^2 + 4\eta^3 - 44/3\,D_0^2)$$

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$$+ S_1 \left(715/9\,\eta + 494/9\,\eta^2 + 137/3\,\eta^3 + 8\,\eta^4 - 580/9\,D_0^2 + 16\,D_0^3 - 24\,D_0^4 \right)$$
$$- \zeta_3 \left(45/2 + 18\,\eta + 18\,\eta^2 \right) + 151/32 - 4\,\eta^5 - 341/6\,\eta^4 - 1805/9\,\eta^3$$
$$\left. - 3691/18\,\eta^2 - 5/18\,\eta + 217/36\,D_0^2 + 185/9\,D_0^3 + 38\,D_0^4 + 16\,D_0^5 \right)$$

$$+ 8\,C_F\,C_A^2 \left(-4\,S_{-5} + 8\,S_{-4,1} - 4\,S_{-3,-2} + 4\,S_{-2,-3} + 8\,S_{-2,-2,1} - 16\,S_{-2,1,-2} \right.$$
$$+ 12\,S_{1,-4} - 32\,S_{1,-3,1} + 24\,S_{1,-2,-2} - 32\,S_{1,1,-3} + 64\,S_{1,1,-2,1} + 16\,S_{1,1,3} - 16\,S_{1,3,1}$$
$$- 12\,S_{1,4} + 16\,S_{2,-3} - 32\,S_{2,-2,1} - 8\,S_{2,3} + 4\,S_{3,-2} + 4\,S_{4,1} + 4\,S_5 - S_{-4}\,(31/3 + 6\,\eta)$$
$$- (11/3 - 8\,\eta)\,(2\,S_{-3,1} + S_{3,1} - 4\,S_{1,-2,1}) - 6\,S_{-2,-2}\,(1 + 2\,\eta) + 2\,S_{1,-3}\,(11/3 + 8\,\eta)$$
$$+ S_{1,3}\,(11 - 8\,\eta) + 22/3\,S_{2,-2} + S_4\,(31/3 + 6\,\eta) + S_{-3}\,(134/9 - 23/3\,\eta - 5\,\eta^2)$$
$$- 4\,S_{1,-2}\,(67/9 + 3\,\eta + 5/2\,\eta^2) - 1/2\,S_3\,(389/9 + \eta - 9\,\eta^2) + 1043/54\,S_2$$
$$+ S_{-2,1}\,(38/3\,\eta + 20\,\eta^2) - S_{-2}\,(3 - 12\,\zeta_3 - 302/9\,\eta + 4/3\,\eta^2 + 3\,\eta^3 + 22/3\,D_0^2)$$
$$- S_1\,(245/12 + 6\,\eta + 7/6\,\eta^2 + 11/6\,\eta^3 - 1/2\,\eta^4) + \zeta_3\,(15/2 + 6\,\eta + 6\,\eta^2)$$
$$- 1657/288 + 20521/216\,\eta + 4819/54\,\eta^2 + 261/4\,\eta^3 + 11/3\,\eta^4 - 3\,\eta^5$$
$$\left. - 2759/54\,D_0^2 + 44/3\,D_0^3 - 22\,D_0^4 \right)$$

$$+ 8/3\,C_F^2\,n_f \left(-8\,S_{-4} - 8\,S_{-3,1} + 8\,S_{1,-3} + 16\,S_{1,-2,1} + 16\,S_{1,3} + 8\,S_{2,-2} + 8\,S_{2,2} \right.$$
$$+ 8\,S_{3,1} - 4\,S_4 - 80/3\,S_{1,-2} - 40/3\,S_{1,2} - 40/3\,S_{2,1} + 4\,S_{-3}\,(10/3 - \eta) - 8\,\eta\,S_{-2,1}$$
$$- S_3\,(6 + 8\,\eta) + 4\,S_{-2}\,(22/3\,\eta + \eta^2 - 2\,D_0^2) + S_2\,(5/2 + 56/3\,\eta + 4\,\eta^2 - 8\,D_0^2)$$
$$+ S_1\,(55/4 - 64/3\,\eta - 92/3\,\eta^2 - 8\,\eta^3 - 12\,\zeta_3 + 64/3\,D_0^2) + \zeta_3\,(9 + +6\,\eta)$$
$$\left. - 69/8 + 83/24\,\eta + 457/6\,\eta^2 + 278/3\,\eta^3 + 19\,\eta^4 - 71/6\,D_0^2 + 10/3\,D_0^3 - 24\,D_0^4 \right)$$

$$+ 8/3\,C_F\,C_A\,n_f \left(4\,S_{-4} + 4\,S_{-3,1} - 4\,S_{1,-3} - 8\,S_{1,-2,1} - 6\,S_{1,3} - 4\,S_{2,-2} + 2\,S_{3,1} - 4\,S_4 \right.$$
$$+ 4\,\eta\,S_{-2,1} + 40/3\,S_{1,-2} - 2\,S_{-3}\,(10/3 - \eta) + S_3\,(14 + 3\,\eta) - 167/9\,S_2$$
$$- 2\,S_{-2}\,(22/3\,\eta + \eta^2 - 2\,D_0^2) + S_1\,(209/18 + 2\,\eta^2 + \eta^3 + 12\,\zeta_3) - \zeta_3\,(9 + 6\,\eta)$$
$$\left. + 15/2 - 943/12\,\eta - 953/18\,\eta^2 - 121/3\,\eta^3 - 8\,\eta^4 + 389/9\,D_0^2 - 8\,D_0^3 + 12\,D_0^4 \right)$$
$$+ 8/9\,C_F\,n_f^2 \left(2/3\,S_1 + 10/3\,S_2 - 2\,S_3 - 17/8 + 34/3\,\eta + 20/3\,\eta^2 + \eta^3 - 22/3\,D_0^2 \right) \tag{4.8}$$

and

$$\Delta P_{\text{ps}}^{(2)}(N) = 8\,C_A\,C_F\,n_f \left(-S_{-3}\,(5\,\eta - 6\,\eta^2) - (S_{1,-2} - S_{1,1,1} + S_{1,2} - 3\,\zeta_3)\,(2\,\eta - 4\,\eta^2) \right.$$
$$+ 4\,\eta\,S_{-2,1} - S_3\,(5/2\,\eta - 7\,\eta^2) + S_{-2}\,(21\,\eta - 13\,\eta^2 - 14\,\eta^3 - 20\,D_0^2 + 16\,D_0^3)$$
$$- S_{1,1}\,(11/6\,\eta + 1/3\,\eta^2 - 2\,\eta^3) + S_2\,(5\,\eta - \eta^2 - 4\,\eta^3 - 5\,D_0^2 + 4\,D_0^3) + S_1\,(203/9\,\eta$$
$$- 115/9\,\eta^2 - 3/2\,\eta^3 - \eta^4 - 41/3\,D_0^2 + 34/3\,D_0^3 + 2\,D_0^4) + 1268/27\,\eta - 107/54\,\eta^2$$
$$\left. + 93\,\eta^3 - 283/3\,\eta^4 - 38\,\eta^5 - 575/9\,D_0^2 + 1367/18\,D_0^3 - 83\,D_0^4 + 32\,D_0^5 \right)$$

$$+ 8\,C_F^2\,n_f \left(-(2\,S_{1,1,1} - 2\,S_{1,2} - S_3 + 6\,\zeta_3)\,(\eta - 2\,\eta^2) + S_{1,1}\,(3/2\,\eta - 2\,\eta^2 - 2\,\eta^3) \right.$$
$$+ 2\,S_2\,(7\,\eta - 4\,\eta^2 - 2\,\eta^3 - 6\,D_0^2 + 6\,D_0^3) - 2\,S_1\,(45/4\,\eta - 3\,\eta^2 - 21\,\eta^3 - 7\,\eta^4 - 6\,D_0^2$$
$$\left. + 3\,D_0^3 + D_0^4) + 5\,\eta + 3\,\eta^2 - 75/2\,\eta^3 - 39\,\eta^4 - 8\,\eta^5 + 7\,D_0^2 - 29/2\,D_0^3 + 9\,D_0^4 \right)$$

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$$+ 8/3\, C_F\, n_f^2 \left(S_{1,1}\,(\eta - 2\eta^2) - S_1\,(44/3\,\eta - 31/3\,\eta^2 - 6\eta^3 - 11\,D_0^2 + 10\,D_0^3) \right.$$
$$\left. + 160/9\,\eta - 53/9\,\eta^2 - 30\,\eta^3 - 8\,\eta^4 - 34/3\,D_0^2 + 17/3\,D_0^3 + 6\,D_0^4 \right). \qquad (4.9)$$

In N-space the off-diagonal NNLO entries of the matrix (2.7) are given by

$$\Delta P_{qg}^{(2)}(N) \;=\; 8\,C_F^2\, n_f \left(2\Delta p_{qg}\,(-S_{-4} + 2S_{-2,-2} + 4S_{1,-3} + 2S_{1,1,1,1} - S_{1,1,2} - 5S_{1,2,1} \right.$$
$$+ 4S_{1,3} + 2S_{2,-2} - 6S_{2,1,1} + 6S_{2,2} + 7S_{3,1} - 3S_4) - 4S_{-3}\,(2D_0 - D_0^2 - 2D_1)$$
$$- 8S_{1,-2}\,(D_0 - D_1 - 2D_1^2) - S_{1,1,1}\,(21/2\,D_0 - 5D_0^2 - 12D_1 + 2D_1^2)$$
$$+ 2S_{1,2}\,(5D_0 - 2D_0^2 - 5D_1 + 2D_1^2) + 2S_{2,1}\,(11D_0 - 4D_0^2 - 11D_1 - 2D_1^2)$$
$$- 2S_3\,(11D_0 - 3D_0^2 - 11D_1 - 6D_1^2) - 6D_2\,(S_{-2} + 1)$$
$$+ 2S_{-2}\,(10D_0 - 5D_0^2 - 9D_1 - 6D_1^2 + 8D_1^3) - S_{1,1}\,(29D_0 - 35/2\,D_0^2 + 10D_0^3$$
$$- 36D_1 - 5D_1^2 + 6D_1^3) + 2S_2\,(17D_0 - 10D_0^2 + 4D_0^3 - 22D_1 - 4D_1^2 + 6D_1^3)$$
$$+ S_1\,(-78D_0 + 99/2\,D_0^2 - 43/2\,D_0^3 + 7D_0^4 + 329/4\,D_1 + 18D_1^2 - 15D_1^3 + 4D_1^4)$$
$$+ 3\zeta_3\,(9D_0 - 2D_0^2 - 12D_1 - 4D_1^2) - 1357/32\,D_0 + 77/8\,D_0^2 + 59/8\,D_0^3$$
$$\left. - 15/2\,D_0^4 + 777/16\,D_1 + 213/8\,D_1^2 + 53/4\,D_1^3 - 3D_1^4 + 32D_1^5 \right)$$

$$+ 8\,C_A\,C_F\, n_f \left(2\Delta p_{qg}\,(-11/2\,S_{-4} + 6S_{-3,1} - 3S_{-2,-2} - 2S_{-2,1,1} + 2S_{-2,2} + 6S_{1,-3} \right.$$
$$- 6S_{1,-2,1} - 6S_{1,1,-2} - 4S_{1,1,1,1} - 3S_{1,1,2} + 3S_{1,2,1} + S_{1,3} + 3S_{2,-2} + 6S_{2,1,1}$$
$$- 6S_{3,1} + 3/2\,S_4 + 3\zeta_3\,S_1) - 3D_2\,(2S_{-3} - 2S_{-2,1} - 2S_1 - 2S_{1,-2} - S_{-2} - 1)$$
$$- S_{-3}\,(15D_0 - 6D_0^2 - 18D_1 - 8D_1^2) + 2S_{-2,1}\,(5D_0 - 2D_0^2 - 8D_1)$$
$$+ S_{1,1,1}\,(37/3\,D_0 - D_0^2 - 47/3\,D_1 - 2D_1^2) + 8S_{1,-2}\,(13/4\,D_0 - D_0^2 - 4D_1 - 2D_1^2)$$
$$+ 3S_{1,2}\,(11/2\,D_0 - 4D_0^2 - 5D_1 - 4D_1^2) - 3S_{2,1}\,(11/2\,D_0 - 5D_1 - 4D_1^2)$$
$$+ S_3\,(61/3\,D_0 - D_0^2 - 59/3\,D_1 - 18D_1^2) + S_{-2}\,(8D_0 - 2D_0^2 + 2D_0^3 - 11D_1 - 4D_1^3)$$
$$+ S_{1,1}\,(317/9\,D_0 - 41/6\,D_0^2 - 6D_0^3 - 313/9\,D_1 - 31D_1^2 - 2D_1^3)$$
$$+ 2S_2\,(17/18\,D_0 - 5D_0^2 + 6D_0^3 - 23/9\,D_1 + 10D_1^2 + 6D_1^3) + S_1\,(1195/27\,D_0$$
$$- 29/9\,D_0^2 - 11D_0^3 + 8D_0^4 - 1595/27\,D_1 - 67/2\,D_1^2 + 3D_1^3 + 34D_1^4)$$
$$- 6\zeta_3\,(18D_0 - 5D_0^2 - 21D_1 - 10D_1^2) + 69407/288\,D_0 - 15259/216\,D_0^2$$
$$- 701/72\,D_0^3 + 89/6\,D_0^4 - 4D_0^5 - 34927/144\,D_1 - 36461/216\,D_1^2 - 3359/36\,D_1^3$$
$$\left. - 1/3\,D_1^4 + 8D_1^5 \right)$$

$$+ 8\,C_A^2\, n_f \left(2\Delta p_{qg}\,(-3/2\,S_{-4} + 2S_{-3,1} + 3S_{-2,-2} + 2S_{-2,1,1} + 2S_{1,-3} - 2S_{1,-2,1} - S_{1,3} \right.$$
$$+ 6S_{1,1,-2} + 2S_{1,1,1,1} + 4S_{1,1,2} + 2S_{1,2,1} - S_{2,-2} - 2S_{2,2} + 3S_{3,1} - 5/2\,S_4 - 3\zeta_3\,S_1)$$
$$- S_{-3}\,(104/3\,D_0 - 13D_0^2 - 115/3\,D_1 - 14D_1^2) + 4S_{-2,1}\,(2D_0 - D_0^2 - 2D_1)$$
$$- 6S_{1,-2}\,(7D_0 - 5/3\,D_0^2 - 7D_1 - 6D_1^2) - S_{1,1,1}\,(11/6\,D_0 + 4D_0^2 - 11/3\,D_1 - 4D_1^2)$$
$$- S_{2,1}\,(35/6\,D_0 - 23/3\,D_1 - 8D_1^2) - S_3\,(106/3\,D_0 - 25/2\,D_0^2 - 223/6\,D_1 - 17D_1^2)$$
$$- S_{1,2}\,(157/6\,D_0 - 8D_0^2 - 73/3\,D_1 - 24D_1^2) + 3\zeta_3\,(27D_0 - 8D_0^2 - 30D_1 - 16D_1^2)$$

$$
\begin{aligned}
&- 3\,D_2\,(S_{-2}+1) - S_{-2}\,(776/9\,D_0 - 21\,D_0^2 - D_0^3 - 709/9\,D_1 - 69\,D_1^2 - 62\,D_1^3) \\
&- 2\,S_{1,1}\,(1/9\,D_0 + 7\,D_0^2 - 4\,D_0^3 + 65/18\,D_1 - 71/6\,D_1^2 - 12\,D_1^3) \\
&- S_2\,(36\,D_0 - 12\,D_0^2 - 35\,D_1 - 61/3\,D_1^2 - 16\,D_1^3) + S_1\,(2515/54\,D_0 - 91/2\,D_0^2 \\
&+ 35/2\,D_0^3 + 9/2\,D_0^4 - 4555/108\,D_1 - 59/9\,D_1^2 + 233/6\,D_1^3 + 49\,D_1^4) \\
&- 16099/36\,D_0 + 2867/27\,D_0^2 - 75/2\,D_0^3 + 82/3\,D_0^4 - 15\,D_0^5 + 8227/18\,D_1 \\
&+ 8941/27\,D_1^2 + 2143/9\,D_1^3 + 691/3\,D_1^4 + 158\,D_1^5\Big) \\[4pt]
&+ 8/9\,C_F\,n_f^2\,\Big(3\,\Delta p_{\mathrm{qg}}\,(S_{1,1,1} - 2\,S_3) + S_{1,1}\,(4\,D_0 + 3\,D_0^2 - 14\,D_1) - S_2\,(11\,D_0 - 16\,D_1) \\
&+ S_1\,(14/3\,D_0 - 4\,D_0^2 + 19/6\,D_1) + 4193/16\,D_0 - 3217/12\,D_0^2 + 901/4\,D_0^3 \\
&- 129\,D_0^4 + 36\,D_0^5 - 2113/8\,D_1 + 97/12\,D_1^2 + 151/2\,D_1^3 - 42\,D_1^4 - 72\,D_1^5\Big) \\[4pt]
&+ 8/9\,C_A\,n_f^2\,\Big(3\,\Delta p_{\mathrm{qg}}\,(-2\,S_{-3} - S_{1,1,1} + S_{1,2} - S_{2,1} - S_3) - 2\,S_{-2}\,(2\,D_0 - 7\,D_1) \qquad\qquad (4.10)\\
&- 2\,S_{1,1}\,(2\,D_0 - 7\,D_1 + 3\,D_1^2) + 6\,D_1^2\,S_2 - S_1\,(23/3\,D_0 - 4/3\,D_1 - 17\,D_1^2 + 12\,D_1^3) \\
&+ 118\,D_0 - 1067/12\,D_0^2 + 99/2\,D_0^3 - 527/4\,D_1 - 46/3\,D_1^2 + 65\,D_1^3 - 12\,(D_0^4 + D_1^4)\Big)\;,
\end{aligned}
$$

$$
\begin{aligned}
\Delta P_{\mathrm{gq}}^{(2)}(N) \;=\; & 8\,C_F^3\,\Big(2\,\Delta p_{\mathrm{gq}}\,(-S_{-4} + 6\,S_{-2,-2} + 4\,S_{1,-3} + 2\,S_{1,1,1,1} + S_{1,1,2} + 3\,S_{1,2,1} \\
& - 3\,S_{1,3} + 2\,S_{2,-2} + 2\,S_{2,1,1} - 2\,S_{2,2} - 9\,\zeta_3 + 6\,\zeta_3\,S_1) + 4\,S_{-3}\,(D_0 - 2\,D_0^2 - D_1) \\
& + 8\,S_{1,-2}\,(2\,D_0 - 2\,D_1 - D_1^2) - S_{1,1,1}\,(6\,D_0 - 2\,D_0^2 + 3/2\,D_1 + 5\,D_1^2) \\
& - 2\,S_{1,2}\,(4\,D_0 - D_1 + D_1^2) + 4\,S_{2,1}\,(D_0 - D_0^2 - 7/4\,D_1 - D_1^2) - 6\,D_{-1}\,(S_{-2}+1) \\
& + S_3\,(6\,D_0 + 2\,D_0^2 - 3/2\,D_1 + D_1^2) + S_{-2}\,(26\,D_0 - 4\,D_0^2 - 24\,D_1 - 18\,D_1^2 - 8\,D_1^3) \\
& - S_{1,1}\,(2\,D_0 + 4\,D_0^2 + 6\,D_0^3 - 7/4\,D_1 + 5\,D_1^2 + 6\,D_1^3) + 2\,S_2\,(10\,D_0 - 4\,D_1 + 2\,D_1^2 \\
& + D_1^3) + S_1\,(8\,D_0 + 27/2\,D_0^2 - 4\,D_0^3 - 6\,D_0^4 - 23/4\,D_1 - 39/4\,D_1^2 - 23/2\,D_1^3 - 7\,D_1^4) \\
& + 2427/16\,D_0 - 341/8\,D_0^2 - 39/4\,D_0^3 + 23\,D_0^4 - 8\,D_0^5 - 4547/32\,D_1 - 767/8\,D_1^2 \\
& - 427/8\,D_1^3 - 28\,D_1^4 - 12\,D_1^5\Big) \\[4pt]
+ & 8\,C_A\,C_F^2\,\Big(\Delta p_{\mathrm{gq}}\,(-3\,S_{-4} - 10\,S_{-2,-2} + 4\,S_{-2,1,1} - 8\,S_{1,-3} + 4\,S_{1,-2,1} + 12\,S_{1,1,-2} \\
& - 8\,S_{1,1,1,1} + 6\,S_{1,1,2} + 2\,S_{1,2,1} + 10\,S_{1,3} - 6\,S_{2,-2} + 4\,S_{2,1,1} - 5\,S_4 - 18\,\zeta_3\,S_1 + 27\,\zeta_3) \\
& + 9\,D_{-1}\,(S_{-2}+1) + 2\,S_{-3}\,(6\,D_0 - 3\,D_1 + D_1^2) - 2\,S_{-2,1}\,(4\,D_0 - D_1 + 2\,D_1^2) \\
& - 4\,S_{1,-2}\,(7\,D_0 - 4\,D_1) + S_{1,1,1}\,(73/3\,D_0 + 2\,D_0^2 - 23/3\,D_1 + D_1^2) \\
& + S_{1,2}\,(35/3\,D_0 + 4\,D_0^2 - 71/6\,D_1 - 8\,D_1^2) - S_{2,1}\,(5/3\,D_0 - 8\,D_0^2 + 13/6\,D_1 + 6\,D_1^2) \\
& - S_3\,(10\,D_0 + 16\,D_0^2 - 3\,D_1 - 4\,D_1^2) + S_{-2}\,(6\,D_0 - 6\,D_0^2 - 4\,D_0^3 - D_1 + 7\,D_1^2 + 2\,D_1^3) \\
& - S_{1,1}\,(31/18\,D_0 - 8/3\,D_0^2 + 2\,D_0^3 + 137/9\,D_1 + 22\,D_1^2 + 14\,D_1^3) \\
& - 4/3\,S_2\,(10\,D_0 + 5/2\,D_0^2 + 6\,D_0^3 - 5/4\,D_1 + 67/8\,D_1^2 + 3\,D_1^3) \\
& + S_1\,(293/54\,D_0 - 64/9\,D_0^2 + 8/3\,D_0^3 - 8\,D_0^4 + 613/108\,D_1 + D_1^2 - 39/2\,D_1^3 \\
& - 24\,D_1^4) - 3343/48\,D_0 + 11093/216\,D_0^2 + 365/36\,D_0^3 - 89/3\,D_0^4 + 16\,D_0^5 \\
& + 11273/288\,D_1 - 3197/216\,D_1^2 - 701/72\,D_1^3 + 8/3\,D_1^4 - 8\,D_1^5\Big)
\end{aligned}
$$

$$
\begin{aligned}
+\,8\,C_A^2\,C_F\,\Big(&\Delta p_{\mathrm{gq}}\,(-11\,S_{-4}+16\,S_{-3,1}+2\,S_{-2,-2}-4\,S_{-2,1,1}+4\,S_{-2,2}+24\,S_{1,-3}\\
&+4\,S_{1,3}-20\,S_{1,-2,1}-12\,S_{1,1,-2}+4\,S_{1,1,1,1}-8\,S_{1,1,2}-8\,S_{1,2,1}+10\,S_{2,-2}\\
&-8\,S_{2,1,1}+12\,S_{2,2}+8\,S_{3,1}-3\,S_4-9\,\zeta_3+6\,\zeta_3\,S_1)-6\,D_{-1}\,(S_{-3}-S_{-2,1}-S_1\\
&-S_{1,-2}+3/2\,S_{-2}+3/2)+1/3\,S_{-3}\,(133\,D_0-114\,D_0^2-137\,D_1-39\,D_1^2)\\
&-4/3\,S_{-2,1}\,(10\,D_0-12\,D_0^2-11\,D_1-3\,D_1^2)+2/3\,S_{1,-2}\,(53\,D_0-6\,D_0^2-40\,D_1\\
&-15\,D_1^2)-S_{1,1,1}\,(55/3\,D_0+4\,D_0^2-55/6\,D_1-4\,D_1^2)+7/6\,S_3\,(35\,D_0-18\,D_0^2\\
&-223/7\,D_1-9\,D_1^2)+(S_{1,2}+S_{2,1})\,(7/3\,D_0+12\,D_0^2+41/6\,D_1+2\,D_1^2)\\
&+S_{-2}\,(124/3\,D_0-3\,D_0^2-2\,D_0^3-173/3\,D_1-202/3\,D_1^2-31\,D_1^3)\\
&+S_{1,1}\,(25/18\,D_0+7\,D_0^2+24\,D_0^3+581/36\,D_1+80/3\,D_1^2+12\,D_1^3)\\
&+S_2\,(5/9\,D_0+38/3\,D_0^2-32\,D_0^3-148/9\,D_1-79/2\,D_1^2-18\,D_1^3)\\
&-1/3\,S_1\,(883/9\,D_0+152/3\,D_0^2-29/2\,D_0^3+75\,D_0^4-403/18\,D_1+1/4\,D_1^2\\
&+65\,D_1^3+75/2\,D_1^4)+1913/6\,D_0-5513/54\,D_0^2+776/9\,D_0^3-47\,D_0^4\\
&+30\,D_0^5-3349/12\,D_1-17843/108\,D_1^2-7373/36\,D_1^3-629/3\,D_1^4-79\,D_1^5\Big)\\[4pt]
+\,8/3\,C_F^2\,n_f\,\Big(&\Delta p_{\mathrm{gq}}\,(-5\,S_{1,1,1}-4\,S_{1,2}-2\,S_{2,1}+3\,S_3+12\,\zeta_3)-6\,D_{-1}\,(S_{-2}+1)\\
&-6\,S_{-2}\,(4\,D_0-4\,D_0^2-5\,D_1-2\,D_1^2)+S_{1,1}\,(41/3\,D_0-2\,D_0^2-4/3\,D_1+6\,D_1^2)\\
&+S_2\,(4\,D_0+4\,D_0^2+D_1+2\,D_1^2)-1/9\,S_1\,(31\,(D_0+D_1)-48\,D_0^2-36\,D_0^3-54\,D_1^3)\\
&-1685/8\,D_0+3371/36\,D_0^2-337/6\,D_0^3+50\,D_0^4-24\,D_0^5+10043/48\,D_1\\
&+3769/36\,D_1^2+829/12\,D_1^3+46\,D_1^4+12\,D_1^5\Big)\\[4pt]
+\,8/3\,C_A\,C_F\,n_f\,\Big(&\Delta p_{\mathrm{gq}}\,(4\,S_{-3}-4\,S_{-2,1}-8\,S_{1,-2}+5\,S_{1,1,1}-5\,S_{1,2}-5\,S_{2,1}-2\,S_3)\\
&-12\,\zeta_3\,\Delta p_{\mathrm{gq}}+6\,D_{-1}\,(S_{-2}+1)+2\,S_{-2}\,(10\,D_0-6\,D_0^2-8\,D_1-D_1^2)\\
&-S_{1,1}\,(80/3\,D_0+6\,D_0^2-37/3\,D_1-4\,D_1^2)+2/3\,S_2\,(5\,D_0+6\,D_0^2+5\,D_1+9/2\,D_1^2)\\
&+S_1\,(91/9\,D_0+2/3\,D_0^2+8\,D_0^3+118/9\,D_1+55/2\,D_1^2+17\,D_1^3)+345/4\,D_0\\
&-248/9\,D_0^2-41/3\,D_0^3-643/6\,D_1-2671/36\,D_1^2-59/6\,D_1^3+14\,D_1^4\Big)\\[4pt]
+\,8/9\,C_F\,n_f^2\,\Big(&3\,\Delta p_{\mathrm{gq}}\,S_{1,1}+S_1\,(-4\,D_0-D_1-3\,D_1^2)-6\,D_0+5\,D_1-D_1^2-3\,D_1^3\Big)\,. \qquad(4.11)
\end{aligned}
$$

Finally the polarized third-order gluon-gluon splitting function reads

$$
\begin{aligned}
\Delta P_{\mathrm{gg}}^{(2)}(N)\;=\;16\,C_A^3\,\Big(&-4\,S_{-5}+8\,S_{-4,1}+4\,S_{-3,-2}+2\,S_{-3,2}+4\,S_{-2,-3}-4\,S_{-2,-2,1}\\
&-4\,S_{-2,1,-2}+16\,S_{1,-4}-16\,S_{1,-3,1}-4\,S_{1,-2,-2}-4\,S_{1,-2,2}-24\,S_{1,1,-3}\\
&+16\,S_{1,1,-2,1}-8\,S_{1,1,3}-8\,S_{1,2,-2}-8\,S_{1,2,2}-8\,S_{1,3,1}+8\,S_{1,4}+18\,S_{2,-3}-12\,S_{2,-2,1}\\
&-8\,S_{2,1,-2}-8\,S_{2,1,2}-8\,S_{2,2,1}+10\,S_{2,3}+4\,S_{3,-2}-8\,S_{3,1,1}+10\,S_{3,2}+8\,S_{4,1}-4\,S_5\\
&+11/6\,(2\,S_{-2,-2}-S_{1,3}-S_{3,1})-67/9\,(S_{-3}+S_3-2\,S_{1,-2}-2\,S_{1,2}-2\,S_{2,1})\\
&+1/6\,S_2-245/24\,S_1+79/32+8\,\eta\,(-4\,S_{-4}+4\,S_{-3,1}+S_{-2,-2}+S_{-2,2}+6\,S_{1,-3}\\
&-4\,S_{1,-2,1}+2\,S_{1,3}+2\,S_{2,-2}+2\,S_{2,2}+2\,S_{3,1}-2\,S_4)-11\,\nu\,(S_{-3}-S_{-2,1}
\end{aligned}
$$

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$$
\begin{aligned}
&-S_{1,-2}+S_{-2}-S_1+1)+S_{-3}\,(33\eta-16\eta^2-24D_0^2)+S_3\,(86/3\,\eta-6\eta^2-24D_0^2)\\
&-S_{1,-2}\,(43\eta+32\eta^2-32D_0^2)-(S_{1,2}+S_{2,1})\,(32\eta+16\eta^2-32D_0^2)\\
&-S_{-2,1}\,(23\eta-16D_0^2)-S_{-2}\,(802/9\,\eta-338/3\,\eta^2-60\eta^3-64D_0^2+64D_0^3)\\
&+S_{1,1}\,(48\eta^2+16\eta^3)+S_2\,(-1745/18\,\eta+173/3\,\eta^2+32\eta^3+64D_0^2-64D_0^3)\\
&+S_1\,(487/18\,\eta-17/3\,\eta^2-761/3\,\eta^3-74\eta^4-365/9\,D_0^2-76/3\,D_0^3+48D_0^4)\\
&-1571/54\,\eta-32503/216\,\eta^2+1493/36\,\eta^3+1666/3\,\eta^4+156\eta^5+638/9\,D_0^2\\
&-644/9\,D_0^3+172D_0^4-128D_0^5\Big)\\[4pt]
&+8C_A^2\,n_f\,\Big(2/3\,(S_{1,3}+S_{3,1}-2S_{-2,-2})+20/9\,(S_{-3}-2S_{1,-2}-2S_{1,2}-2S_{2,1}+S_3)\\
&-1/3\,S_2+209/54\,S_1-233/144+4\zeta_3\,(S_1-2\eta+3\eta^2)-\nu\,(4S_{-3}-4S_{-2,1}\\
&-4S_{1,-2}-2S_{-2}-4S_1-2)+S_{-3}\,(3\eta+6\eta^2)-4S_{-2,1}\,\eta-2S_{1,-2}\,(\eta+6\eta^2)\\
&-S_3\,(11/6\,\eta+3\eta^2)+S_{-2}\,(77/9\,\eta-13/3\,\eta^2+2\eta^3)+4/3\,S_2\,(23/3\,\eta-\eta^2)\\
&-S_1\,(901/36\,\eta+166/9\,\eta^2+43/6\,\eta^3+3\eta^4-232/9\,D_0^2+16/3\,D_0^3)-2662/27\,\eta\\
&+4375/54\,\eta^2+169/9\,\eta^3-17/3\,\eta^4+2\eta^5+716/9\,D_0^2-704/9\,D_0^3+16D_0^4\Big)\\[4pt]
&+8C_A\,C_F\,n_f\,\Big(55/12\,S_1-241/144-2\zeta_3\,(2S_1-\eta+12\eta^2)-S_{-3}\,(10\eta+8\eta^2)\\
&+8\nu\,(S_{-3}-S_{-2,1}-S_{1,-2}-S_1-5/4\,(S_{-2}+1))+8S_{-2,1}\,\eta+S_{1,-2}\,(4\eta+32\eta^2)\\
&-(S_{1,1,1}-S_{1,2})\,(2\eta-4\eta^2)-S_3\,(\eta-14\eta^2)+S_{1,1}\,(11/6\,\eta+1/3\,\eta^2-2\eta^3)\\
&+S_{-2}\,(33\eta-20\,(\eta^2+D_0^2)-16\,(\eta^3-D_0^3))+S_2\,(40/3\,\eta-29/3\,\eta^2-4\eta^3-15D_0^2\\
&+12D_0^3)+S_1\,(89/18\,\eta+202/9\,\eta^2+130/3\,\eta^3+14\eta^4-3D_0^2+2D_0^3-2D_0^4)\\
&-1483/54\,\eta+3845/54\,\eta^2+169/9\,\eta^3-554/3\,\eta^4-56\eta^5+30D_0^2-95/6\,D_0^3\\
&-35D_0^4+32D_0^5\Big)\\[4pt]
&+8C_F^2\,n_f\,\Big(1/8+6\zeta_3\,(\eta+2\eta^2)+8\,(S_{-3}-2S_{1,-2})\,\eta^2+(S_{1,1,1}-S_{1,2})\,(2\eta-4\eta^2)\\
&-S_3\,(\eta+6\eta^2)+12\nu\,(S_{-2}+1)-10S_{-2}\,\eta+S_2\,(11\eta-10\eta^2-4\eta^3-8D_0^2+4D_0^3)\\
&-S_{1,1}\,(3/2\,\eta-2\eta^2-2\eta^3)-S_1\,(23/2\,\eta+6\eta^2-10\eta^3-2\eta^4-14D_0^2+10D_0^3\\
&-2D_0^4)-55\eta+12\eta^2-19/2\,\eta^3-21\eta^4-4\eta^5+38D_0^2-75/2\,D_0^3+15D_0^4\Big)\\[4pt]
&+2/27\,C_A\,n_f^2\,\Big(87/4+S_1\,(8-27\eta+48\eta^2)-3\eta-16\eta^2-24\eta^3\Big)\\[4pt]
&+8/27\,C_F\,n_f^2\,\Big(33/8+(S_{1,1}-2S_2)\,(-9\eta+18\eta^2)-S_1\,(84\eta-51\eta^2-18\eta^3\\
&-81D_0^2+54D_0^3)-16\eta+65\eta^2-120\eta^3-36\eta^4-45D_0^3+54D_0^4\Big)\,. \qquad (4.12)
\end{aligned}
$$

All these results refer to the standard transformation to the $\overline{\text{MS}}$ scheme of Ref. [56], see Eq. (2.17). With the exception of the $C_A n_f^2$ part of Eq. (4.12), which was derived in Ref. [73] (see also Ref. [74]), Eqs. (4.9) – (4.12) are new results of the present article.

The last two equations include the denominator ν defined in Eq. (4.1), and are therefore only valid at $N \geq 3$. The first moment of the NNLO quark-gluon splitting function is

$$\Delta P_{\text{gq}}^{(0)}(N\!=\!1) \;=\; 3 C_F \,, \tag{4.13}$$

$$\Delta P_{\text{gq}}^{(1)}(N\!=\!1) \;=\; \frac{71}{3} C_F C_A - 9 C_F^2 - \frac{2}{3} C_F n_f \,, \tag{4.14}$$

$$\Delta P_{\text{gq}}^{(2)}(N\!=\!1) \;=\; \frac{1607}{12} C_F C_A^2 - \frac{461}{4} C_F^2 C_A + \frac{63}{2} C_F^3 + \left(\frac{41}{3} - 72\zeta_3\right) C_F C_A n_f$$
$$- \left(\frac{107}{2} - 72\zeta_3\right) C_F^2 n_f - \frac{13}{3} C_F n_f^2 \,. \tag{4.15}$$

The corresponding results for the gluon-gluon splitting function are identical to the coefficients of the beta function recalled in Eq. (2.16). The NLO and NNLO pure-singlet results are related to Eqs. (4.13) and (4.14) by

$$\Delta P_{\text{ps}}^{(n)}(N\!=\!1) \;=\; -2 n_f \, \Delta P_{\text{gq}}^{(n-1)}(N\!=\!1) \,. \tag{4.16}$$

In the OPE, this relation for the anomalous dimension of the pure-singlet axial current together with Eq. (3.10) for the first moment of $\Delta P_{\text{gg}}^{(2)}$ has been shown in Ref. [49] to be a direct consequence of the requirement that the axial anomaly [82, 83] should preserve the one-loop character of the operator relation [84]

$$\partial^\mu j_\mu^5 \;=\; -2 n_f \, a_s \, \widetilde{G}_a^{\mu\nu} G_{a,\mu\nu} \tag{4.17}$$

in dimensional regularization, where $j_\mu^5 = \overline{\psi}\gamma_5\gamma_\mu\psi$ and $G_a^{\mu\nu}$ ($\widetilde{G}_a^{\mu\nu} = 1/2\,\varepsilon^{\mu\nu\alpha\beta} G_{a,\alpha\beta}$) denote the renormalized axial current and the (dual) gluon field-strength tensor. In this context Eqs. (3.10) and (4.16) are thus consistency requirements ensuring the correct renormalization of the pure-singlet axial current with the chosen finite renormalization constants Z_{ik}, see Eq. (2.17). Consequently Eq. (4.16) for $n=3$, together with Eq. (4.15) and $\Delta P_{\text{ns}}^{+(n)}(N\!=\!1) = \Delta P_{\text{qg}}^{(n)}(N\!=\!1) = 0$, fixes the first moments of the upper-row splitting functions at order α_{s}^4.

The quantities given above do not provide the complete set of third-order helicity-difference splitting functions. Additional even-N functions $\Delta P_{\text{ns}}^{-,\nu}$ exist for the quark-antiquark differences

$$\Delta f_{ik}^- \;=\; \Delta f_{q_i} - \Delta f_{\bar{q}_i} - \left(\Delta f_{q_k} - \Delta f_{\bar{q}_k}\right) \,, \tag{4.18}$$

$$\Delta f^\nu \;=\; \sum_{i=1}^{n_f} \left\{\Delta f_{q_i} - \Delta f_{\bar{q}_i}\right\} \tag{4.19}$$

that occur in the (so far practically irrelevant) structure functions g_3 and g_4 in polarized charged-current DIS which has been analyzed at NLO in Ref. [75]. The corresponding NNLO corrections may be addressed in a future publication together with the generalization of Refs. [76, 77] to all N. It appears safe to assume $\Delta P_{\text{ns}}^{-(2)} = P_{\text{ns}}^{+(2)}$ as given in Eq. (3.7) of Ref. [12], $\Delta P_{\text{ns}}^{\nu(2)}$ is unknown though at this point.

5 The NNLO splitting functions in x-space

The expressions for the x-space splitting functions in Eq. (2.5) in terms of harmonic polyloga-rithms [71] can be obtained from their N-space counterparts in terms of harmonic sums [61] by a completely algebraic procedure [71, 78] based on the fact that latter functions occur as coefficients of the Taylor expansion of the former. Our notation for the harmonic polylogarithms follows Ref. [71], with the lowest-weight ($w = 1$) functions $H_m(x)$ given by

$$H_0(x) = \ln x, \qquad H_{\pm 1}(x) = \mp \ln(1 \mp x) \tag{5.1}$$

and the higher-weight ($w \geq 2$) functions recursively defined as

$$H_{m_1,\dots,m_w}(x) = \begin{cases} \dfrac{1}{w!} \ln^w x, & \text{if } m_1, \dots, m_w = 0, \dots, 0 \\[2mm] \displaystyle\int_0^x dz\, f_{m_1}(z)\, H_{m_2,\dots,m_w}(z), & \text{else} \end{cases} \tag{5.2}$$

with

$$f_0(x) = \frac{1}{x}, \qquad f_{\pm 1}(x) = \frac{1}{1 \mp x}. \tag{5.3}$$

For chains of indices zero we employ the abbreviated notation

$$H_{\underbrace{0,\dots,0}_{m},\pm 1,\underbrace{0,\dots,0}_{n},\pm 1,\dots}(x) = H_{\pm(m+1),\pm(n+1),\dots}(x). \tag{5.4}$$

Also here we recall, for completeness, the LO and NLO contributions

$$\begin{aligned} \Delta P_{\text{ns}}^{(0)}(x) &= 2C_F\left(\Delta p_{qq}(x) + 3/2\,\delta(1-x)\right), \\ \Delta P_{\text{ps}}^{(0)}(x) &= 0, \\ \Delta P_{\text{qg}}^{(0)}(x) &= 2n_f\left(-1 + 2x\right), \\ \Delta P_{\text{gq}}^{(0)}(x) &= 2C_F\left(2 - x\right), \\ \Delta P_{\text{gg}}^{(0)}(x) &= 4C_A\left(\Delta p_{gg}(x) + 11/12\,\delta(1-x)\right) - 2/3\,n_f\,\delta(1-x), \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \Delta P_{\text{ns}}^{+(1)}(x) &= 4C_F^2\left(2\Delta p_{qq}(-x)\left(\zeta_2 + 2H_{-1,0} - H_{0,0}\right) + 2\Delta p_{qq}(x)\left(H_{1,0} + H_2 - 3/4\,H_0\right)\right. \\ &\quad \left. - 9\left(1-x\right) - \left(1+x\right)H_{0,0} - 1/2\left(7+11x\right)H_0 + \delta(1-x)\left(3/8 + 6\zeta_3 - 3\zeta_2\right)\right) \\ &\quad + 4C_A C_F\left(-\Delta p_{qq}(-x)\left(\zeta_2 + 2H_{-1,0} - H_{0,0}\right) + \Delta p_{qq}(x)\left(H_{0,0} + 11/6\,H_0 - \zeta_2\right.\right. \\ &\quad \left.\left. + 67/18\right) + 26/3\left(1-x\right) + 2\left(1+x\right)H_0 + \delta(1-x)\left(17/24 - 3\zeta_3 + 11/3\,\zeta_2\right)\right) \\ &\quad + 4/3\,C_F\,n_f\left(-\Delta p_{qq}(x)\left(5/3 + H_0\right) - 2\left(1-x\right) - \delta(1-x)\left(1/4 + 2\zeta_2\right)\right), \end{aligned} \tag{5.6}$$

$$\Delta P_{\text{ps}}^{(1)}(x) = 4 C_F n_f \left(-(1-3x) H_0 + 1 - x - 2(1+x) H_{0,0} \right), \tag{5.7}$$

$$\Delta P_{\text{qg}}^{(1)}(x) = 4 C_A n_f \left(2(1-2x) H_{1,1} + 4(1-x) H_1 - 2(1+2x)(H_{-1,0} + H_{0,0}) \right.$$
$$+ (1+8x) H_0 - 2\zeta_2 + 12 - 11x \Big)$$
$$+ 2 C_F n_f \left(4(1-2x)(\zeta_2 - 1/2 H_{0,0} - H_{1,0} - H_{1,1} - H_2) - 8(1-x) H_1 \right.$$
$$- 9 H_0 - 22 + 27x \Big), \tag{5.8}$$

$$\Delta P_{\text{gq}}^{(1)}(x) = 4 C_F C_A \left(2(2-x)(H_{1,0} + H_{1,1} + H_2) + 2(2+x)(H_{-1,0} + H_{0,0}) \right.$$
$$+ (4 - 13x) H_0 - 1/3(10+x) H_1 + 41/9 + 35/9 x + 2x \zeta_2 \Big)$$
$$+ 2 C_F^2 \left(2(2-x)(H_{0,0} - 2 H_{1,1}) + 2(2+x) H_1 - (4-x) H_0 - 17 + 8x \right)$$
$$+ 8/3 C_F n_f \left((2-x) H_1 - 4/3 - 1/3 x \right), \tag{5.9}$$

$$\Delta P_{\text{gg}}^{(1)}(x) = 4 C_A^2 \left(2 \Delta p_{\text{gg}}(-x)(\zeta_2 + 2 H_{-1,0} - H_{0,0}) - 2 \Delta p_{\text{gg}}(x)(\zeta_2 - H_{0,0} - 2 H_{1,0}) \right.$$
$$- 2 H_2 - 67/18) - 19/2(1-x) + 8(1+x) H_{0,0} + 1/3(29 - 67x) H_0$$
$$+ \delta(1-x)(8/3 + 3\zeta_3) \Big)$$
$$+ 8/3 C_A n_f \left(-5/3 \Delta p_{\text{gg}}(x) - 3(1-x) - (1+x) H_0 - \delta(1-x) \right)$$
$$+ 2 C_F n_f \left(-10(1-x) - 4(1+x) H_{0,0} - 2(5-x) H_0 - \delta(1-x) \right). \tag{5.10}$$

Here and in Eqs. (5.12) – (5.16) we have suppressed the argument x of the polylogarithms and used

$$\Delta p_{\text{qq}}(x) = 2(1-x)^{-1} - 1 - x,$$
$$\Delta p_{\text{gg}}(x) = (1-x)^{-1} + 1 - 2x. \tag{5.11}$$

Divergences for $x \to 1$ are to be understood as plus-distributions.

The polarized NNLO non-singlet and pure singlet quark-quark splitting functions, obtained by Mellin-inverting Eqs. (4.8) and (4.9) are given by

$$\Delta P_{\text{ns}}^{+(2)}(x) = 16 C_F^3 \left(2 \Delta p_{\text{qq}}(-x)(9/4\zeta_3 - 7/4\zeta_2^2 + 3 H_{-3,0} - 16 H_{-2}\zeta_2 - 4 H_{-2,-1,0} \right.$$
$$+ 13 H_{-2,0,0} + 14 H_{-2,2} - 4 H_{-1,-2,0} + 24 H_{-1,-1}\zeta_2 - 20 H_{-1,-1,0,0} - 24 H_{-1,-1,2}$$
$$- 20 H_{-1,0}\zeta_2 + 11 H_{-1,0,0,0} + 2 H_{-1,2,0} + 16 H_{-1,3} + 7 H_{0,0}\zeta_2 - 3 H_{0,0,0,0} - 3 H_2\zeta_2$$
$$- H_{3,0} - 6 H_4 - 3/2 H_{-2,0} - 3 H_{-1}\zeta_2 - 3/2 H_{-1,0,0} + 3 H_{-1,2} + 3/4 H_0\zeta_2 - 3/2 H_3$$
$$+ 9/4 H_{0,0,0} - 18 H_{-1}\zeta_3 + 3/4 H_0 + 13/2 H_0\zeta_3) + 2 \Delta p_{\text{qq}}(x)(9/20\zeta_2^2 - H_{-3,0}$$
$$+ 3 H_{-2}\zeta_2 + 6 H_{-2,-1,0} - 3 H_{-2,0,0} - H_{0,0,0,0} + 4 H_{1,-2,0} - 2 H_{1,0,0,0} + 2 H_{1,2,0}$$
$$+ 4 H_{1,3} + H_{2,0,0} + 2 H_{2,1,0} + 2 H_{2,2} + 2 H_{3,0} + 2 H_{3,1} + H_4 - 3/4 H_0\zeta_2 - 3 H_{1,0,0}$$
$$- 3/2 H_{2,0} - 3/32 H_0 + 1/2 H_0\zeta_3 + 13/16 H_{0,0} + 6 H_1\zeta_3) - (11 + 31x) H_3$$
$$+ (1-x)(-25 H_1 - 151/8 - 4 H_{-2,0,0} - H_{0,0}\zeta_2 + 3 H_{0,0,0,0} - 6 H_1\zeta_2 - 9 H_{1,0})$$

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$$+ (1+x) \left(37/10\, \zeta_2^2 - 18\, H_{-1}\, \zeta_2 + 24\, H_{-1,2} + 14\, H_{-1,0,0} + 12\, H_{-1,-1,0} - 3\, H_{2,0,0} \right.$$
$$\left. - 2\, H_{3,0} - H_4 - 6\, H_{-1,0} \right) + 1/16\, (-307 + 437\, x)\, H_0 + (1 - 5x)\, H_{-2,0} + 6x\, H_0\, \zeta_3$$
$$- 2\, (1 - 3x)\, H_{-3,0} - 3\, (2 + 5x)\, H_{0,0,0} + 3/2\, (5 + 11x)\, \zeta_3 - 1/2\, (5 + 13x)\, H_{2,0}$$
$$+ (12 + 31x)\, H_0\, \zeta_2 + 3/4\, (17 + x)\, \zeta_2 - 3/4\, (25 + x)\, H_2 - 1/8\, (73 - 15x)\, H_{0,0}$$
$$+ \delta(1-x)\, \left(29/32 + 9/8\, \zeta_2 + 17/4\, \zeta_3 + 18/5\, \zeta_2^2 - 15\, \zeta_5 - 2\, \zeta_2\, \zeta_3 \right) \Big)$$

$$+ 8\, C_F^2\, C_A\, \Big(2\, \Delta p_{\mathrm{qq}}(-x)\, (-31/4\, \zeta_3 - 1/4\, \zeta_2^2 + 67/9\, \zeta_2 + 134/9\, H_{-1,0} - 67/9\, H_{0,0} $$
$$- 5\, H_{-3,0} + 32\, H_{-2}\, \zeta_2 + 4\, H_{-2,-1,0} - 21\, H_{-2,0,0} - 30\, H_{-2,2} + 36\, H_{-1,-1,0,0}$$
$$+ 4\, H_{-1,-2,0} - 56\, H_{-1,-1}\, \zeta_2 + 56\, H_{-1,-1,2} + 42\, H_{-1,0}\, \zeta_2 - 17\, H_{-1,0,0,0} - 2\, H_{-1,2,0}$$
$$- 32\, H_{-1,3} - 13\, H_{0,0}\, \zeta_2 + 5\, H_{0,0,0,0} + 7\, H_2\, \zeta_2 + H_{3,0} + 10\, H_4 + 31/6\, H_{-2,0}$$
$$+ 31/3\, H_{-1}\, \zeta_2 + 31/6\, H_{-1,0,0} - 31/3\, H_{-1,2} - 13/12\, H_0\, \zeta_2 - 89/12\, H_{0,0,0} + 31/6\, H_3$$
$$+ 42\, H_{-1}\, \zeta_3 - 9/4\, H_0 - 29/2\, H_0\, \zeta_3) + 2\, \Delta p_{\mathrm{qq}}(x)\, (5/6\, \zeta_3 - 69/20\, \zeta_2^2 - H_{-3,0}$$
$$- 3\, H_{-2}\, \zeta_2 - 14\, H_{-2,-1,0} + 5\, H_{-2,0,0} - 4\, H_{-2,2} - 4\, H_{0,0}\, \zeta_2 + 5\, H_{0,0,0,0} - 16\, H_{1,-2,0}$$
$$- 2\, H_{1,0}\, \zeta_2 + 11\, H_{1,0,0,0} + 8\, H_{1,1,0,0} - 8\, H_{1,3} - 2\, H_2\, \zeta_2 + 5\, H_{2,0,0} + H_{3,0} + H_4$$
$$+ 3\, H_{-2,0} + 41/12\, H_0\, \zeta_2 - 23/12\, H_{0,0,0} + 31/3\, H_{1,0,0} + 11/3\, H_{2,0} + 2/3\, H_3$$
$$- 13/4\, H_{0,0} + 67/9\, H_{1,0} + 67/9\, H_2 - 151/48\, H_0 - 17/2\, H_0\, \zeta_3 - 24\, H_1\, \zeta_3)$$
$$+ 4\, (1 - 2x)\, H_{-3,0} + 2\, (1 - x)\, (379/12 - H_{-2}\, \zeta_2 - 2\, H_{-2,-1,0} + 3\, H_{-2,0,0} + 7\, H_1\, \zeta_2$$
$$+ 4\, H_{1,0,0} + 26/3\, H_{1,0} + 251/6\, H_1) + 2\, (1 + x)\, (25\, H_{-1}\, \zeta_2 - 14\, H_{-1,-1,0} - 32\, H_{-1,2}$$
$$- 13\, H_{-1,0,0} + 2\, H_{2,0} + H_2\, \zeta_2 + 2\, H_{2,0,0} - 3\, H_4 + 19/3\, H_{-1,0}) - (6 + 7x)\, \zeta_2^2$$
$$+ 2\, (2 - 3x)\, H_0\, \zeta_3 - 5\, (3 - 7x)\, H_{-2,0} + 2\, (5 + 3x)\, H_{0,0}\, \zeta_2 + 2\, (9 + 31x)\, H_3$$
$$- (33 + 62x)\, H_0\, \zeta_2 + 1/18\, (157 - 557x)\, H_{0,0} - (39 + 17x)\, \zeta_2 - 1/2\, (97 + 39x)\, \zeta_3$$
$$+ 1/2\, (35 + 13x)\, H_{0,0,0} + 1/72\, (2627 - 3869x)\, H_0 + (155/3 + 17x)\, H_2 - 8\, H_{0,0,0,0}$$
$$+ \delta(1-x)\, (151/32 - 205/12\, \zeta_2 + 211/6\, \zeta_3 - 247/30\, \zeta_2^2 + 15\, \zeta_5 + 2\, \zeta_2\, \zeta_3) \Big)$$

$$+ 8\, C_F\, C_A^2\, \Big(2\, \Delta p_{\mathrm{qq}}(-x)\, (11/4\, \zeta_3 + \zeta_2^2 - 67/18\, \zeta_2 - 67/9\, H_{-1,0} + 67/18\, H_{0,0} + H_{-3,0}$$
$$- 8\, H_{-2}\, \zeta_2 + 4\, H_{-2,0,0} + 8\, H_{-2,2} + 16\, H_{-1,-1}\, \zeta_2 - 8\, H_{-1,-1,0,0} - 16\, H_{-1,-1,2}$$
$$- 11\, H_{-1,0}\, \zeta_2 + 3\, H_{-1,0,0,0} + 8\, H_{-1,3} + 3\, H_{0,0}\, \zeta_2 - H_{0,0,0,0} - 2\, H_2\, \zeta_2 - 2\, H_4$$
$$- 11/6\, H_{-2,0} - 11/3\, H_{-1}\, \zeta_2 - 11/6\, H_{-1,0,0} + 11/3\, H_{-1,2} + 1/6\, H_0\, \zeta_2 + 31/12\, H_{0,0,0}$$
$$- 11/6\, H_3 - 12\, H_{-1}\, \zeta_3 + 3/4\, H_0 + 4\, H_0\, \zeta_3) + 2\, \Delta p_{\mathrm{qq}}(x)\, (245/48 + 1/2\, \zeta_3 + 12/5\, \zeta_2^2$$
$$- 67/18\, \zeta_2 + 389/72\, H_{0,0} + H_{-3,0} + 4\, H_{-2,-1,0} - H_{-2,0,0} + 2\, H_{-2,2} - H_{0,0,0,0}$$
$$+ 6\, H_{1,-2,0} - H_{1,0}\, \zeta_2 - 3\, H_{1,0,0,0} - 4\, H_{1,1,0,0} + 4\, H_{1,3} - 2\, H_{2,0,0} + H_4 - 3/2\, H_{-2,0}$$
$$- 31/12\, H_0\, \zeta_2 + 31/12\, H_{0,0,0} - 11/4\, H_{1,0,0} + 11/12\, H_3 + 1043/216\, H_0 + 4\, H_0\, \zeta_3$$
$$+ 9\, H_1\, \zeta_3) - (1 - x)\, (74/3\, H_1 - 391/27 + H_{-3,0} - H_{-2}\, \zeta_2 - 2\, H_{-2,-1,0} + H_{-2,0,0}$$
$$+ 4\, H_1\, \zeta_2 + 4\, H_{1,0,0}) - (1 + x)\, (16\, H_{-1}\, \zeta_2 - 8\, H_{-1,-1,0} - 6\, H_{-1,0,0} - 20\, H_{-1,2}$$
$$+ 10/3\, H_{-1,0} + 28/3\, H_2 + H_2\, \zeta_2 + 1/2\, H_{2,0,0} - 3/2\, H_4) + 1/4\, (3 + 5x)\, \zeta_2^2 - 2\, H_0\, \zeta_3$$
$$+ 9\, (1 + 2x)\, H_0\, \zeta_2 - 2\, (1 + 9x)\, H_3 + 2/3\, (3 + 10x)\, H_{0,0} - 1/2\, (5 + 3x)\, H_{0,0}\, \zeta_2$$

$$
\begin{aligned}
&+ (7 - 15x)\,H_{-2,0} + 2/3\,(9 + 14x)\,\zeta_2 + 1/9\,(43 - 21x)\,H_0 + 1/2\,(41 + 3x)\,\zeta_3 \\
&- 7\,H_{0,0,0} + H_{0,0,0,0} - \delta(1-x)\,(1657/288 - 5\,\zeta_5 + 194/9\,\zeta_3 - 562/27\,\zeta_2 + 1/4\,\zeta_2^2)\Big) \\[4pt]
&+ 8/3\,C_F^2\,n_f\,\Big(4\Delta p_{qq}(-x)\,(3/2\,\zeta_3 - 5/3\,\zeta_2 - 10/3\,H_{-1,0} + 5/3\,H_{0,0} - H_{-2,0} - 2\,H_{-1}\,\zeta_2 \\
&- H_{-1,0,0} + 2\,H_{-1,2} + 1/2\,H_0\,\zeta_2 + H_{0,0,0} - H_3) + 2\Delta p_{qq}(x)\,(-55/16 + 5\,\zeta_3 + H_0\,\zeta_2 \\
&- H_{0,0,0} - 4\,H_{1,0,0} - 2\,H_{2,0} - 2\,H_3 + 3/2\,H_{0,0} - 10/3\,H_{1,0} - 10/3\,H_2 + 5/8\,H_0) \\
&+ (1-x)\,(34 - 8\,H_1 - 4\,H_{1,0}) - (1+x)\,(8\,H_{-1,0} - 3\,H_{0,0,0}) + 1/3\,(31 + 55x)\,H_{0,0} \\
&+ 1/12\,(269 + 253x)\,H_0 - 8\,H_2 - \delta(1-x)\,(69/8 - 5/2\,\zeta_2 + 17\,\zeta_3 - 29/5\,\zeta_2^2)\Big) \\[4pt]
&+ 8/3\,C_A\,C_F\,n_f\,\Big(2\Delta p_{qq}(-x)\,(-3/2\,\zeta_3 + 5/3\,\zeta_2 + 10/3\,H_{-1,0} - 5/3\,H_{0,0} + H_{-2,0} \\
&+ 2\,H_{-1}\,\zeta_2 + H_{-1,0,0} - 2\,H_{-1,2} - 1/2\,H_0\,\zeta_2 - H_{0,0,0} + H_3) + 2\Delta p_{qq}(x)\,(-209/72 \\
&- 9/2\,\zeta_3 + 5/3\,\zeta_2 - 7/2\,H_{0,0} + H_0\,\zeta_2 - H_{0,0,0} + 3/2\,H_{1,0,0} - 1/2\,H_3 - 167/36\,H_0) \\
&- (1-x)\,(440/9 - 2\,H_1) + (1+x)\,(4\,H_{-1,0} + H_2) + (3-x)\,\zeta_2 - (6+5x)\,H_{0,0} \\
&- 2/3\,(33-x)\,H_0 + \delta(1-x)\,(15/2 - 167/9\,\zeta_2 + 25/3\,\zeta_3 + 3/10\,\zeta_2^2)\Big) \\[4pt]
&+ 8/9\,C_F\,n_f^2\,\Big(\Delta p_{qq}(x)\,(-1/3 + 5/3\,H_0 + H_{0,0}) + (1-x)\,(13/3 + 2\,H_0) \\
&- \delta(1-x)\,(17/8 - 10/3\,\zeta_2 + 2\,\zeta_3)\Big) \tag{5.12}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta P_{\mathrm{ps}}^{(2)}(x) \;=\; & 4\,C_A\,C_F\,n_f\,\Big(4(1-x)\,(5/2\,H_1\,\zeta_2 - 33/4\,H_{1,0,0} + 5\,H_{1,1,0} + 5\,H_{1,1,1} - 4439/54 \\
&- H_{-2}\,\zeta_2 - 2\,H_{-2,-1,0} - 3\,H_{-2,0,0} - 1/2\,H_{0,0}\,\zeta_2 + 17/2\,H_{1,0} + 65/12\,H_{1,1} + 266/9\,H_1) \\
&- 2(1+x)\,(H_{-1}\,\zeta_2 + 10\,H_{-1,-1,0} + 17\,H_{-1,0,0} + 4\,H_{-1,2} - 2\,H_2\,\zeta_2 + 7\,H_{2,0,0} - 4\,H_{2,1,0} \\
&- 4\,H_{2,1,1} - 2\,H_{3,1} + H_4 - 37\,H_{-1,0}) + 1/5\,(117 + 107x)\,\zeta_2^2 - 1/9\,(427 - 1151x)\,H_{0,0} \\
&- 1/27\,(2257 + 8899x)\,H_0 - 4(1 - 5x)\,H_{-3,0} - 4(3 - 4x)\,H_{0,0,0,0} + 2(6+x)\,H_{2,0} \\
&+ 2(9 - 19x)\,H_{-2,0} + 4(9 + 13x)\,H_0\,\zeta_3 + 2/3\,(19 - 11x)\,H_{2,1} + 14/3\,(25 - 26x)\,\zeta_3 \\
&- 4/3\,(19 + 37x)\,H_{0,0,0} - 1/3\,(29 + 47x)\,H_0\,\zeta_2 + 1/3\,(83 + 47x)\,H_3 \\
&+ 1/9\,(91 - 134x)\,\zeta_2 + 1/9\,(575 + 134x)\,H_2\Big) \\[4pt]
&+ 4\,C_F^2\,n_f\,\Big(10(1-x)\,(H_{1,0,0} - 2\,H_{1,1,0} - 2\,H_{1,1,1} - 6/5 - 6/5\,H_{1,0} - 13/10\,H_{1,1} \\
&- 25/2\,H_1) - 4(1+x)\,(37/10\,\zeta_2^2 + 7\,H_0\,\zeta_3 - 6\,H_{0,0}\,\zeta_2 + 4\,H_{0,0,0,0} - H_{2,0,0} + 2\,H_{2,1,0} \\
&+ 2\,H_{2,1,1} + 4\,H_{3,0} + H_{3,1} + 6\,H_4) - 4(2 - 3x)\,H_{2,1} + 20(2-x)\,(H_0\,\zeta_2 - H_3) \\
&- 4(4 - 7x)\,H_{2,0} - 4(5 - 6x)\,H_{0,0,0} - 4(11 - 21x)\,\zeta_3 - (25 - 114x)\,H_0 \\
&- (32 + 25x)\,H_{0,0} + (64 + 27x)\,(\zeta_2 - H_2)\Big) \\[4pt]
&+ 2/9\,C_F\,n_f^2\,\Big(4(1-x)\,(86/3 + 2\,H_1 + 15\,H_{1,1}) + 8(5 - 4x)(\zeta_2 - H_2) + 4(23 + 17x)\,H_{0,0} \\
&+ 24(1+x)\,(\zeta_3 + 2\,H_0\,\zeta_2 + H_{0,0,0} + H_{2,1} - 2\,H_3) + 4/3\,(65 - 43x)\,H_0\Big)\,. \tag{5.13}
\end{aligned}
$$

Eqs. (4.10) and (4.11) result in the third-order gluon-quark and quark-gluon splitting functions

$$
\begin{aligned}
\Delta P_{\text{qg}}^{(2)}(x) = {}& 8\,C_A^2\,n_f\Big((1-2x)\,(31\,\mathrm{H}_1\,\zeta_3 + 6\,\mathrm{H}_{1,-2,0} + 10\,\mathrm{H}_{1,0}\,\zeta_2 - 5\,\mathrm{H}_{1,0,0,0} + 2\,\mathrm{H}_{1,1}\,\zeta_2 \\
& + 2\,\mathrm{H}_{1,1,0,0} + 8\,\mathrm{H}_{1,1,1,0} - 4\,\mathrm{H}_{1,1,1,1} + 4\,\mathrm{H}_{1,1,2} + 4\,\mathrm{H}_{1,2,0} - 6\,\mathrm{H}_{1,3} - 11/6\,\mathrm{H}_{1,1,1}) \\
& - 1/36\,(16099 - 16346\,x) + 1/18\,(733 + 12\,x + 54\,x^2)\,\zeta_2 + 1/6\,(273 - 4\,x)\,\mathrm{H}_2 \\
& - 1/18\,(675 - 2356\,x + 54\,x^2)\,\mathrm{H}_{0,0} - (1 - 18\,x)\,\mathrm{H}_{-3,0} + (1 + x)\,(8\,\mathrm{H}_{2,1,0} + 4\,\mathrm{H}_{2,1,1} \\
& - 13\,\mathrm{H}_{-1}\,\zeta_2 - 42\,\mathrm{H}_{-1,-1,0} - 8\,\mathrm{H}_{-1,2}) - (1 - 14\,x)\,\mathrm{H}_{-2}\,\zeta_2 + 1/20\,(495 + 538\,x)\,\zeta_2^2 \\
& - 4\,(1 + 2\,x)\,(\mathrm{H}_{-2,2} + 5/4\,\mathrm{H}_{-1}\,\zeta_3 + 1/2\,\mathrm{H}_{-1,-2,0} - 1/2\,\mathrm{H}_{-1,-1}\,\zeta_2 - 3\,\mathrm{H}_{-1,-1,-1,0} \\
& - \mathrm{H}_{-1,-1,0,0} - \mathrm{H}_{-1,-1,2} + 3/4\,\mathrm{H}_{-1,0,0,0} - \mathrm{H}_{-1,2,1} + \mathrm{H}_{-1,3}) + 4\,(2 + 3\,x)\,\mathrm{H}_{3,1} \\
& - 1/9\,(2 + 65\,x)\,\mathrm{H}_{1,1} - 3/2\,(3 - 2\,x)\,\mathrm{H}_4 + 4\,(3 - x)\,\mathrm{H}_{2,0} + 12\,(3 + 4\,x)\,\mathrm{H}_0\,\zeta_3 \\
& - 2\,(5 - 6\,x)\,\mathrm{H}_{-2,-1,0} + (5 + 6\,x)\,\mathrm{H}_2\,\zeta_2 + 1/2\,(7 - 31\,x)\,\mathrm{H}_0\,\zeta_2 + 2\,(7 - 10\,x)\,\mathrm{H}_{2,1} \\
& + 3\,(7 - 9\,x)\,\mathrm{H}_{-2,0} + 1/2\,(7 - 6\,x)\,\mathrm{H}_{0,0}\,\zeta_2 - (13 - 6\,x)\,\mathrm{H}_{-2,0,0} - (15 - 16\,x)\,\mathrm{H}_{0,0,0,0} \\
& - 1/2\,(25 + 42\,x)\,\mathrm{H}_{2,0,0} + 1/6\,(35 - 46\,x)\,\mathrm{H}_{1,2} + 1/2\,(35 + 31\,x)\,\mathrm{H}_3 + (36 - 35\,x)\,\mathrm{H}_{1,0} \\
& - 2/3\,(41 + 40\,x)\,\mathrm{H}_{0,0,0} + 1/6\,(91 - 80\,x)\,\mathrm{H}_1\,\zeta_2 - 1/3\,(104 + 115\,x)\,\mathrm{H}_{-1,0,0} \\
& + 1/6\,(157 - 146\,x)\,\mathrm{H}_{1,1,0} - 1/6\,(212 - 223\,x)\,\mathrm{H}_{1,0,0} - 1/108\,(11468 + 40643\,x)\,\mathrm{H}_0 \\
& + 1/2\,(315 - 268\,x)\,\zeta_3 + 5/108\,(1006 - 911\,x)\,\mathrm{H}_1 + 1/9\,(776 + 709\,x + 27\,x^2)\,\mathrm{H}_{-1,0}\Big) \\[4pt]
& + 8\,C_A\,C_F\,n_f\Big(-2\,(1 - 2\,x)\,(39/2\,\mathrm{H}_1\,\zeta_3 + 3\,\mathrm{H}_{1,-2,0} + 9\,\mathrm{H}_{1,0}\,\zeta_2 - 3/2\,\mathrm{H}_{1,0,0,0} \\
& + 6\,\mathrm{H}_{1,1}\,\zeta_2 + \mathrm{H}_{1,1,0,0} + 3\,\mathrm{H}_{1,1,1,0} - 4\,\mathrm{H}_{1,1,1,1} - 3\,\mathrm{H}_{1,1,2} - 6\,\mathrm{H}_{1,2,1} - 6\,\mathrm{H}_{1,3}) \\
& - (59/2 - 31\,x + 3\,x^2)\,\mathrm{H}_1\,\zeta_2 - 4\,(1 - x)\,\mathrm{H}_2\,\zeta_2 - 3\,(5 + 6\,x + 2\,x^2)\,\mathrm{H}_{-1,0,0} \\
& - (89/6 - 65/3\,x - 6\,x^2)\,\mathrm{H}_{0,0,0} - (701/72 + 1357/36\,x - 9\,x^2)\,\mathrm{H}_{0,0} - (11 - 35\,x \\
& - 6\,x^2)\,\mathrm{H}_3 + (1 - 14\,x)\,\mathrm{H}_{2,1,1} - 2\,(5 + 8\,x + 3\,x^2)\,\mathrm{H}_{-1,2} - (8 + 17\,x + 9\,x^2)\,\mathrm{H}_{-1,0} \\
& - 2\,(1 + 16\,x + 3\,x^2)\,\mathrm{H}_{-2,0} - 2\,(1 - 6\,x)\,\mathrm{H}_{-3,0} + 1/288\,(69407 - 68990\,x) \\
& + 8\,(1 + x)\,\mathrm{H}_{-2,-1,0} - 1/3\,(370 - 293\,x + 45\,x^2)\,\zeta_3 - (101/9 + 85/18\,x + 9\,x^2)\,\zeta_2 \\
& - 1/10\,(101 + 146\,x)\,\zeta_2^2 + 2\,(1 + 2\,x)\,(17/2\,\mathrm{H}_{-1}\,\zeta_3 + 3\,\mathrm{H}_{-1,-2,0} - 9\,\mathrm{H}_{-1,-1}\,\zeta_2 \\
& - 6\,\mathrm{H}_{-1,-1,-1,0} + 6\,\mathrm{H}_{-1,-1,0,0} + 6\,\mathrm{H}_{-1,-1,2} + 9\,\mathrm{H}_{-1,0}\,\zeta_2 - 11/2\,\mathrm{H}_{-1,0,0,0} \\
& - 2\,\mathrm{H}_{-1,2,0} - 2\,\mathrm{H}_{-1,2,1} - 6\,\mathrm{H}_{-1,3} - 2\,\mathrm{H}_{0,0,0,0}) - 8\,(1 + 3\,x)\,\mathrm{H}_4 + 6\,(1 + 4\,x)\,\mathrm{H}_{0,0}\,\zeta_2 \\
& - 4\,(3 + 4\,x)\,\mathrm{H}_{3,0} + (1 + 22\,x)\,\mathrm{H}_{2,0,0} - 5\,(2 - 7\,x)\,\mathrm{H}_{2,0} + 2\,(13 + 16\,x + 3\,x^2)\,\mathrm{H}_{-1,-1,0} \\
& - 4\,(1 + 6\,x)\,\mathrm{H}_{-2,2} - 2\,(3 + 8\,x)\,(\mathrm{H}_{-2,0,0} + \mathrm{H}_{3,1}) - 3/2\,(11 - 10\,x)\,(\mathrm{H}_{1,1,0} - \mathrm{H}_{1,2}) \\
& + (9 - 35\,x - 12\,x^2)\,\mathrm{H}_0\,\zeta_2 - (17/9 - 46/9\,x)\,\mathrm{H}_{1,0} + (37/3 - 47/3\,x)\,\mathrm{H}_{1,1,1} \\
& + (317/9 - 313/9\,x)\,\mathrm{H}_{1,1} + (29/9 + 85/18\,x)\,\mathrm{H}_2 + (61/3 - 59/3\,x)\,\mathrm{H}_{1,0,0} \\
& + 4\,(2 + 7\,x)\,\mathrm{H}_{-2}\,\zeta_2 - 12\,\mathrm{H}_{2,1,0} + (23 + 32\,x + 9\,x^2)\,\mathrm{H}_{-1}\,\zeta_2 - (41 + 22\,x)\,\mathrm{H}_0\,\zeta_3 \\
& + (41/6 + 46/3\,x)\,\mathrm{H}_{2,1} + 1/27\,(1195 - 1433\,x)\,\mathrm{H}_1 + 1/216\,(15259 + 25645\,x)\,\mathrm{H}_0\Big) \\[4pt]
& + 8\,C_F^2\,n_f\Big(2\,(1 - 2\,x)\,(7/2\,\mathrm{H}_{0,0}\,\zeta_2 + 7\,\mathrm{H}_1\,\zeta_3 + 2\,\mathrm{H}_{1,-2,0} + 7\,\mathrm{H}_{1,0}\,\zeta_2 - 3\,\mathrm{H}_{1,0,0,0} \\
& + 5\,\mathrm{H}_{1,1}\,\zeta_2 - 4\,\mathrm{H}_{1,1,0,0} - \mathrm{H}_{1,1,1,0} - 2\,\mathrm{H}_{1,1,1,1} - 5\,\mathrm{H}_{1,1,2} - 6\,\mathrm{H}_{1,2,0} - 6\,\mathrm{H}_{1,2,1} - 7\,\mathrm{H}_{1,3} \\
& - 2\,\mathrm{H}_{2,1,0} - 5/2\,\mathrm{H}_{2,1,1} - 4\,\mathrm{H}_{2,2} - 4\,\mathrm{H}_{3,0} - 5\,\mathrm{H}_{3,1} - 7/2\,\mathrm{H}_4 - 5\,\mathrm{H}_{-2,0}) + 681/16\,x
\end{aligned}
$$

$$+ 2(1-x)(13 H_1\zeta_2 - 11 H_{1,0,0} - 5 H_{1,1,0} - 11 H_{1,2} + 4 H_2\zeta_2) - 4(3/5 - 2x)\zeta_2^2$$
$$- 1357/32 - 2(10 + 9x - 3x^2) H_{-1,0} - 4(1+x)(H_{-1}\zeta_2 + 2 H_{-1,-1,0} + 2 H_{-1,0,0})$$
$$+ (59/2 - 18x + 6x^2)\zeta_2 + 4(1+2x)(H_{-1,-2,0} + 2 H_{-1,-1,0,0} - 1/2 H_{-1,0,0,0})$$
$$- 4(1+4x) H_{-2,0,0} - 2(3 - 2x) H_{2,0,0} - 7/2(5 - 2x) H_{2,1} - 3/2(7 - 8x) H_{1,1,1}$$
$$- 2(10 - x) H_{2,0} - 9/2(11 - 4x) H_2 + (13 - 14x) H_0\zeta_3 + 1/2(15 - 4x) H_{0,0,0}$$
$$- 2(17 - 22x) H_{1,0} + (23/2 - 2x) H_0\zeta_2 + (25 - 11x)\zeta_3 - (29 - 36x) H_{1,1}$$
$$- (43/2 - 2x) H_3 - 1/8(77 - 397x) H_0 + 1/8(59 + 458x - 48x^2) H_{0,0}$$
$$- (78 - 329/4x) H_1 - 4x(4 H_{-3,0} - H_{0,0,0,0} - 2 H_{-2}\zeta_2 - 4 H_{-2,-1,0}))$$
$$+ 2/9 C_A n_f^2 \Big(12(1 - 2x)(H_1\zeta_2 + H_{1,0,0} + H_{1,1,0} + H_{1,1,1} - H_{1,2}) + 24(1 + 2x) H_{-1,0,0}$$
$$+ 48(1 - x) H_{0,0,0} - 8(2 - 7x) H_{1,1} + 8(2 + 7x) H_{-1,0} + 4(4 + 3x)\zeta_2 + 472 - 527x$$
$$- 4/3(23 - 4x) H_1 + 2(99 + 68x) H_{0,0} - 36\zeta_3 + 1067/3 H_0 + 200/3 x H_0 - 12 x H_2 \Big)$$
$$+ 2/9 C_F n_f^2 \Big(12(1 - 2x)(\zeta_3 + 2 H_{1,0,0} - H_{1,1,1} - H_{2,1} + 12 H_{0,0,0,0}) + 4(11 - 16x) H_{1,0}$$
$$+ 1/4(4193 - 4226x) - 8(2 - 7x)(\zeta_2 - H_{1,1} - H_2) + 2/3(28 + 19x) H_1$$
$$+ 12(43 + 10x) H_{0,0,0} + 17(53 + 14x) H_{0,0} + 1/3(3217 - 59x) H_0 \Big) \tag{5.14}$$

and

$$\Delta P_{\text{gq}}^{(2)}(x) = 8 C_A^2 C_F \Big(4(1 - 2x) H_{2,1,1} + 1/12(3718 - 3349x) - 1/20(366 + 193x)\zeta_2^2$$
$$+ 16(1 + x) H_{-2,2} + 2(2 - 11x) H_{-2,-1,0} + (2 - 9x) H_{-3,0} - (106/3 + 3x^{-1}$$
$$+ 173/3x) H_{-1,0} - 1/54(1442 - 403x) H_1 - (46/3 + 3x^{-1} - 121/6x) H_1\zeta_2$$
$$+ (2 - x)(7 H_1\zeta_3 - 2 H_{1,-2,0} - 4 H_{1,0}\zeta_2 + 3 H_{1,0,0,0} - 2 H_{1,1}\zeta_2 + 4 H_{1,1,0,0} + 8 H_{1,1,2}$$
$$+ 8 H_{1,1,1,0} + 4 H_{1,1,1,1} + 12 H_{1,2,0} + 8 H_{1,2,1} + 8 H_{1,3} + 6 H_{2,1,0} + 6 H_{2,2} - 55/6 H_{1,1,1})$$
$$- 4(2 + x)(23/4 H_{-1}\zeta_3 + 5/2 H_{-1,-2,0} - 13/2 H_{-1,-1}\zeta_2 - 3 H_{-1,-1,-1,0} - 4 H_{-1,3}$$
$$+ 6 H_{-1,-1,0,0} + 5 H_{-1,-1,2} + 5 H_{-1,0}\zeta_2 - 11/4 H_{-1,0,0,0} - H_{-1,2,0} - H_{-1,2,1})$$
$$- 7/2(2 + 5x) H_{2,1} - 1/9(5 - 148x) H_{1,0} + 4(6 - x) H_{3,1} + 4(8 + x) H_{3,0}$$
$$- (3 - 122/3x) H_{-2,0} - 2(10 + 7x) H_0\zeta_3 - (14 + 5x) H_2\zeta_2 - (14 + 27x) H_{-2}\zeta_2$$
$$- 1/6(14 + 41x)(H_{1,1,0} + H_{1,2}) + 2(15 - 4x) H_{0,0,0,0} + 1/36(50 + 581x) H_{1,1}$$
$$+ (13/3 - 9/x - 4/3x) H_{-1}\zeta_2 + 1/3(38 - 139x) H_{2,0} + (38 + 11x) H_{-2,0,0}$$
$$- (23 + 13/2x) H_{0,0}\zeta_2 - 1/6(47 - 419x) H_0\zeta_2 + 1/6(245 - 223x) H_{1,0,0}$$
$$+ (25 + 13/2x) H_4 + 2/9(49 + 73x) H_2 + (21 + 13/2x) H_{2,0,0} + (47 - 5/3x) H_{0,0,0}$$
$$+ (40/3 + 6x^{-1} + 44/3x) H_{-1,2} - 1/3(161 - 194x)\zeta_3 - 2/9(208 + 73x)\zeta_2$$
$$+ (133/3 + 6x^{-1} + 137/3x) H_{-1,0,0} + (106/3 - 6x^{-1} + 80/3x) H_{-1,-1,0}$$
$$+ 1/6(29 - 419x) H_3 + 1/18(1444 - 2351x) H_{0,0} + 1/108(11998 + 18649x) H_0 \Big)$$
$$+ 8 C_A C_F^2 \Big(8/3(1 + x) H_3 + (2 - 7x)(2 H_{-3,0} - H_{2,1,1}) - 1/216(13037 - 4423x) H_0$$
$$- 1/5(46 + 49x)\zeta_2^2 - (6 - 9x^{-1} + x) H_{-1,0} - 1/48(2911 - 11273/6x)$$

$$
\begin{aligned}
&+ (2-x)\,(3\,H_1\,\zeta_3 + 10\,H_{1,-2,0} + 2\,H_{1,0}\,\zeta_2 + 5\,H_{1,0,0,0} - 4\,H_{1,1}\,\zeta_2 + 10\,H_{1,1,0,0} \\
&\quad - 6\,H_{1,1,1,0} - 8\,H_{1,1,1,1} - 2\,H_{1,1,2} - 4\,H_{1,2,1} + 2\,H_{2,1,0} + 4\,H_{2,2} + 4\,H_{3,0}) \\
&+ (2+x)\,(11\,H_{-1}\,\zeta_3 + 6\,H_{-1,-2,0} - 10\,H_{-1,-1}\,\zeta_2 - 12\,H_{-1,-1,-1,0} + 8\,H_{-1,-1,0,0} \\
&\quad + 4\,H_{-1,-1,2} - 2\,H_{-1,0}\,\zeta_2 + 3\,H_{-1,0,0,0} - 4\,H_{-1,2,1} + 6\,H_{-1,0,0}) - 3\,(2-3x)\,H_{-2,0} \\
&- (2+7x)\,H_{3,1} - 2\,(4-5x)\,H_2\,\zeta_2 + 4\,(4-x)\,H_0\,\zeta_3 + 2\,(4+x)\,(H_{-1,2} + 2\,H_{0,0,0,0}) \\
&- 2/3\,(5-x)\,H_{2,0} - 4\,(7+4x)\,H_{-1,-1,0} - 1/3\,(8-43x)\,H_{2,1} + 2\,(8-7x)\,H_{2,0,0} \\
&+ 5/3\,(8-x)\,H_{1,0} - (10-3x)\,H_{1,0,0} + 1/6\,(10+13x)\,H_{1,2} - 2\,(11+5x)\,H_{-1}\,\zeta_2 \\
&- 2/3\,(13+4x)\,H_0\,\zeta_2 - 1/18\,(31+274x)\,H_{1,1} + 2/9\,(32-73x)\,H_2 \\
&+ 7/6\,(32-25x)\,\zeta_3 + 1/3\,(89-88x)\,H_{0,0,0} - 1/6\,(70-71x)\,H_{1,1,0} + 8\,H_4 \\
&+ 1/3\,(73-23x)\,H_{1,1,1} + 1/6\,(74-61x)\,H_1\,\zeta_2 - 6x\,(H_{-2,0,0} - H_{-2}\,\zeta_2 - 2\,H_{-2,-1,0}) \\
&- 2/9\,(59-73x)\,\zeta_2 + 1/108\,(586+613x)\,H_1 + 1/72\,(730-821x)\,H_{0,0} - 4\,H_{0,0}\,\zeta_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 8\,C_F^3\,\Big(8\,(1+x)\,(H_{-1}\,\zeta_2 + 2\,H_{-1,-1,0} + 1/2\,H_{-1,0,0} + H_{-2,0,0}) + 2331/16 - 4547/32\,x \\
&\quad - 12\,(1-x)\,H_0\,\zeta_3 - 2\,(13+3x^{-1}+12x)\,H_{-1,0} + (3+93/10\,x)\,\zeta_2^2 - (2-5x)\,H_{2,0,0} \\
&\quad + 2\,(2-3x)\,H_2\,\zeta_2 - 2\,(2-x)\,(3/2\,H_{0,0}\,\zeta_2 + 8\,H_1\,\zeta_3 + 6\,H_{1,-2,0} + 2\,H_{1,0}\,\zeta_2 - 3\,H_{1,1}\,\zeta_2 \\
&\quad + 3\,H_{1,1,0,0} + H_{1,1,1,0} - 2\,H_{1,1,1,1} + 3\,H_{1,1,2} + 2\,H_{1,2,0} + 2\,H_{1,2,1} + 1/2\,H_{2,1,1} + H_{2,2} \\
&\quad + 3/2\,H_{3,1} - 3/2\,H_4 + H_{-2,0}) - 4\,(2+x)\,(H_{-1,-2,0} + 2\,H_{-1,-1,0,0} - H_{-1,0}\,\zeta_2 \\
&\quad - 1/2\,H_{-1,0,0,0}) - (4-7x)\,H_{1,2} - 4\,(5-2x)\,H_{1,0} - 2\,(4+x)\,(H_{0,0,0,0} + 3/4\,H_{1,1,1}) \\
&\quad - (4-x)\,(H_1\,\zeta_2 - 3/2\,H_{1,0,0} - 2\,H_{1,1,0}) - 1/2\,(8-x)\,H_3 - 1/4\,(8-7x)\,H_{1,1} \\
&\quad + 1/2\,(8+7x)\,H_{2,1} - (23+3x)\,H_{0,0,0} - 1/2\,(25+23x)\,\zeta_2 - 3/8\,(26+31x)\,H_{0,0} \\
&\quad - 1/2\,(27-23x)\,H_2 + 1/4\,(32-23x)\,H_1 - (52-21x)\,\zeta_3 + 1/8\,(389+721x)\,H_0 \\
&\quad - x\,(8\,H_{-2,-1,0} + 4\,H_{-2}\,\zeta_2 + 1/2\,H_0\,\zeta_2 - 2\,H_{2,0} - 16\,H_{-3,0})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 8/3\,C_A\,C_F\,n_f\,\Big(369/4 - 643/6\,x - 2\,(10-3x^{-1}+8x)\,H_{-1,0} - 10/3\,(1+x)\,H_{1,0} \\
&\quad + 3\,(2-3x)\,H_{2,1} - (2-x)\,(H_1\,\zeta_2 + 2\,H_{1,0,0} - 5\,H_{1,1,0} - 5\,H_{1,1,1} - 5\,H_{1,2} - 2\,H_{2,0}) \\
&\quad - 4\,(2+x)\,(2\,H_{-1}\,\zeta_2 + 2\,H_{-1,-1,0} - H_{-1,0,0} - H_{-1,2} - H_3) - 1/6\,(4+91x)\,H_2 \\
&\quad - 4\,(5+x)\,H_0\,\zeta_2 - 2\,(6-5x)\,H_{-2,0} - 2\,(23-14x)\,\zeta_3 - 1/3\,(41+74x)\,H_{0,0} \\
&\quad + (194/9 + 3143/36\,x)\,H_0 - 1/3\,(80-37x)\,H_{1,1} + 1/9\,(91+118x)\,H_1 + 2x\,H_{0,0,0} \\
&\quad - (58/3 - 91/6\,x)\,\zeta_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 8/3\,C_F^2\,n_f\,\Big(2\,(2-x)\,(31/2\,\zeta_3 + 6\,H_{-2,0} - H_1\,\zeta_2 + 3/2\,H_{1,0,0} + 2\,H_{1,1,0} - 5/2\,H_{1,1,1} \\
&\quad + H_{1,2} + H_{2,0} + 1/2\,H_{2,1} + H_3 - 6\,H_{0,0,0,0}) - 1/36\,(3155+3893x)\,H_0 - 1733/8 \\
&\quad + 10043/48\,x - 31/9\,(1+x)\,H_1 - (4+x)\,(H_{1,0} + 4/3\,H_2) + 2\,(10+x)\,H_0\,\zeta_2 \\
&\quad + 4/3\,(22+x)\,\zeta_2 + 1/3\,(41-4x)\,H_{1,1} - (50+29x)\,H_{0,0,0} + 6\,(4-x^{-1}+5x)\,H_{-1,0} \\
&\quad - 1/12\,(674-457x)\,H_{0,0}\Big)
\end{aligned}
$$

$$
+ 8/3\,C_F\,n_f^2\,\Big((2-x)\,H_{1,1} - 1/3\,(4+x)\,H_1 - 2 + 5/3\,x\Big)\,. \tag{5.15}
$$

Finally the *x*-space expression corresponding to Eq. (4.12) for the polarized NNLO gluon-gluon splitting function reads

$$
\begin{aligned}
\Delta P_{gg}^{(2)}(x) \; = \; 16\,C_A^3 &\Big(4\,\Delta p_{gg}(-x)\,(-11/8\,\zeta_2^2 + H_{-3,0} - 4\,H_{-2}\,\zeta_2 - 2\,H_{-2,-1,0} + 3\,H_{-2,2} \\
&+ 9/2\,H_{-2,0,0} - 3\,H_{-1}\,\zeta_3 - 2\,H_{-1,-2,0} + 4\,H_{-1,-1}\,\zeta_2 - 6\,H_{-1,-1,0,0} - 4\,H_{-1,-1,2} \\
&- 9/2\,H_{-1,0}\,\zeta_2 + 4\,H_{-1,0,0,0} + H_{-1,2,0} + 4\,H_{-1,3} + 5/4\,H_0\,\zeta_3 + 2\,H_{0,0}\,\zeta_2 - H_{0,0,0,0} \\
&- 1/2\,H_2\,\zeta_2 - 1/2\,H_{3,0} - 2\,H_4 + 11/24\,H_0\,\zeta_2 + 67/36\,(\zeta_2 + 2\,H_{-1,0} - H_{0,0})) \\
&+ 4\,\Delta p_{gg}(x)\,(245/96 - 3/40\,\zeta_2^2 - H_{-3,0} + 3/2\,H_{-2}\,\zeta_2 + H_{-2,-1,0} - H_{-2,0,0} - H_{-2,2} \\
&- 7/4\,H_0\,\zeta_3 - 2\,H_{0,0}\,\zeta_2 + H_{0,0,0,0} - 3/2\,H_1\,\zeta_3 - H_{1,-2,0} - 3/2\,H_{1,0}\,\zeta_2 + 2\,H_{1,0,0,0} \\
&+ 2\,H_{1,1,0,0} + 2\,H_{1,2,0} + 2\,H_{1,3} - H_2\,\zeta_2 + 5/2\,H_{2,0,0} + 2\,H_{2,1,0} + 2\,H_{2,2} + 5/2\,H_{3,0} \\
&+ 2\,H_{3,1} + 2\,H_4 + 11/12\,\zeta_3 + 11/12\,H_{-2,0} + 11/24\,H_{1,0,0} + 11/24\,H_3 \\
&- 67/36\,(\zeta_2 - H_{0,0} - 2\,H_{1,0} - 2\,H_2) + 1/24\,H_0) - 1/3\,(72 - 185\,x - 22\,x^2)\,H_0\,\zeta_2 \\
&- 1/3\,(32 - 161\,x - 11\,x^2)\,H_{-2,0} + 4\,(1 - 5\,x)\,H_{-3,0} - 1/6\,(312 - 393\,x - 55\,x^2)\,\zeta_3 \\
&+ (1 - x)\,(5579/18 + 4\,H_{-2}\,\zeta_2 + 8\,H_{-2,-1,0} + 12\,H_{-2,0,0} - 21/2\,H_1\,\zeta_2 + 37\,H_{1,0,0} \\
&+ 1/18\,H_1 - 19/2\,H_{1,0}) - 1/5\,(43 + 33\,x)\,\zeta_2^2 - 8\,(1 + 3\,x)\,H_0\,\zeta_3 - 2\,(11 + 13\,x)\,H_{0,0}\,\zeta_2 \\
&+ (1 + x)\,(21\,H_{-1,-1,0} - 25/2\,H_{-1}\,\zeta_2 + 65\,H_{-1,0,0} + 23\,H_{-1,2} - 4\,H_2\,\zeta_2 + 10\,H_{2,0,0} \\
&+ 16\,H_{3,0} + 26\,H_4 - 215/3\,H_{-1,0}) - 1/9\,(74 - 97\,x)\,H_2 + 1/3\,(77 - 115\,x)\,H_{2,0} \\
&+ 1/3\,(40 - 185\,x - 11\,x^2)\,H_3 - 1/9\,(571 + 97\,x)\,\zeta_2 + 1/3\,(158 - 87\,x - 11\,x^2)\,H_{0,0,0} \\
&+ 1/12\,(1019 - 1489\,x)\,H_{0,0} + 1/216\,(24625 + 40069\,x)\,H_0 - 11/6\,(x^{-1} - x^2)\,H_1\,\zeta_2 \\
&+ 28\,H_{0,0,0,0} - 11/2\,(x^{-1} + x^2)\,(H_{-1}\,\zeta_2 + 2/3\,H_{-1,-1,0} - 2/3\,H_{-1,0,0} - 2/3\,H_{-1,2}) \\
&+ \delta(1 - x)\,(79/32 - 5\,\zeta_5 + 67/6\,\zeta_3 + 1/6\,\zeta_2 - \zeta_2\,\zeta_3 + 11/24\,\zeta_2^2)\Big) \\[4pt]
+ 8\,C_A^2\,n_f &\Big(2/3\,\Delta p_{gg}(x)\,(10/3\,\zeta_2 - 10/3\,H_{0,0} - 20/3\,H_{1,0} - 20/3\,H_2 - 209/36 - 8\,\zeta_3 \\
&- 2\,H_{-2,0} - H_{1,0,0} - H_3 - 1/2\,H_0) + 2/9\,\Delta p_{gg}(-x)\,(10\,H_{0,0} - 10\,\zeta_2 - 20\,H_{-1,0} \\
&- 3\,H_0\,\zeta_2) - 1/6\,(51 - 61\,x - 16\,x^2)\,H_0\,\zeta_2 - 1/18\,(146 + 227\,x + 36\,x^2)\,H_{0,0} \\
&- 1/3\,(23 + 43\,x - 4\,x^2)\,H_{-2,0} - 1/3\,(1 - 12\,x + 4\,x^2)\,H_{0,0,0} - 2\,(1 - 5\,x)\,H_{-3,0} \\
&+ 2\,(1 - x)\,(512/9 + 3\,H_{-2}\,\zeta_2 + 6\,H_{-2,-1,0} - 3\,H_{-2,0,0} - 11/2\,H_1\,\zeta_2 + 11/4\,H_{1,0,0} \\
&+ 1087/72\,H_1 - 2\,H_{1,0}) + (1 + x)\,(7\,H_{-1}\,\zeta_2 + 22\,H_{-1,-1,0} - 9\,H_{-1,0,0} + 4\,H_{-1,2} \\
&- 4/3\,H_{2,0} - 6\,H_2\,\zeta_2 + 3\,H_{2,0,0} + 3\,H_4 - 19\,H_{-1,0}) - 2/39\,(507 - 195\,x - 65\,x^2)\,\zeta_3 \\
&- 1/18\,(499 + 301\,x - 36\,x^2)\,\zeta_2 + 3/10\,(13 + 23\,x)\,\zeta_2^2 + 1/6\,(5 - 61\,x - 8\,x^2)\,H_3 \\
&- (5 + 3\,x)\,H_{0,0}\,\zeta_2 + 1/18\,(157 + 301\,x)\,H_2 + 1/108\,(2422 + 7609\,x)\,H_0 - 12\,H_0\,\zeta_3 \\
&- 2/3\,(x^{-1} - x^2)\,H_1\,\zeta_2 - 2\,(x^{-1} + x^2)\,(H_{-1}\,\zeta_2 + 2/3\,H_{-1,-1,0} - 2/3\,H_{-1,0,0} \\
&- 2/3\,H_{-1,2} - H_{-1,0}) + 2\,H_{0,0,0,0} - 1/3\,\delta(1 - x)\,(233/48 + 10\,\zeta_3 + \zeta_2 + 1/2\,\zeta_2^2)\Big) \\[4pt]
+ 8/3\,C_A\,C_F\,n_f &\Big(4\,\Delta p_{gg}(x)\,(3\,\zeta_3 - 55/16) + 3\,(1 - x)\,(8\,H_{-2,0,0} - 7507/27 - 16\,H_{-2}\,\zeta_2 \\
&- 32\,H_{-2,-1,0} + 30\,H_1\,\zeta_2 - 29\,H_{1,0,0} - 10\,H_{1,1,0} - 10\,H_{1,1,1} - 26/3\,H_{1,0} - 65/6\,H_{1,1} \\
&- 1127/18\,H_1) + 6\,(1 + x)\,(61/6\,H_{-1,0} - 11\,H_{-1}\,\zeta_2 - 30\,H_{-1,-1,0} + 3\,H_{-1,0,0}
\end{aligned}
$$

$$
\begin{aligned}
&-4\,\mathrm{H}_{-1,2}+6\,\mathrm{H}_{0,0}\,\zeta_2+8\,\mathrm{H}_2\,\zeta_2-7\,\mathrm{H}_{2,0,0}-2\,\mathrm{H}_{2,1,0}-2\,\mathrm{H}_{2,1,1}-4\,\mathrm{H}_{3,0}-\mathrm{H}_{3,1}-6\,\mathrm{H}_4)\\
&+(125+38\,x-20\,x^2)\,\zeta_3+1/6\,(848+341\,x-108\,x^2)\,\zeta_2-1/18\,(8363+3362\,x)\,\mathrm{H}_0\\
&-(181+88\,x-8\,x^2)\,\mathrm{H}_{0,0,0}-1/6\,(1723-692\,x-108\,x^2)\,\mathrm{H}_{0,0}-3/5\,(43+83\,x)\,\zeta_2^2\\
&-(32-43\,x-8\,x^2)\,\mathrm{H}_3-24\,(3-2\,x)\,\mathrm{H}_{0,0,0,0}+6\,(9-x)\,\mathrm{H}_0\,\zeta_3-(19-11\,x)\,\mathrm{H}_{2,1}\\
&+8\,(3+12\,x-x^2)\,\mathrm{H}_{-2,0}+(56-43\,x-16\,x^2)\,\mathrm{H}_0\,\zeta_2-1/6\,(482+341\,x)\,\mathrm{H}_2\\
&-(38-37\,x)\,\mathrm{H}_{2,0}+4\,(x^{-1}-x^2)\,\mathrm{H}_1\,\zeta_2+4\,(x^{-1}+x^2)\,(3\,\mathrm{H}_{-1}\,\zeta_2+2\,\mathrm{H}_{-1,-1,0}\\
&-2\,\mathrm{H}_{-1,0,0}-2\,\mathrm{H}_{-1,2}-9/2\,\mathrm{H}_{-1,0})-48\,x\,\mathrm{H}_{-3,0}-241/48\,\delta(1-x)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+8\,C_F^2\,n_f\,\Big(8\,(1-x)\,(\mathrm{H}_{-2}\,\zeta_2+1+2\,\mathrm{H}_{-2,-1,0}-\mathrm{H}_{-2,0,0}-2\,\mathrm{H}_1\,\zeta_2+11/8\,\mathrm{H}_{1,0,0}\\
&+5/4\,(\mathrm{H}_{1,1,0}+\mathrm{H}_{1,1,1})-7/8\,\mathrm{H}_{1,0}+13/16\,\mathrm{H}_{1,1}+41/16\,\mathrm{H}_1)+4\,(1+x)\,(4\,\mathrm{H}_{-1}\,\zeta_2\\
&+8\,\mathrm{H}_{-1,-1,0}-4\,\mathrm{H}_{-1,0,0}+\mathrm{H}_{0,0}\,\zeta_2-\mathrm{H}_{0,0,0,0}-2\,\mathrm{H}_2\,\zeta_2+3/2\,\mathrm{H}_{2,0,0}+\mathrm{H}_{2,1,0}+\mathrm{H}_{2,1,1}\\
&+1/2\,\mathrm{H}_{3,1}-\mathrm{H}_4+5/2\,\mathrm{H}_{-1,0})+(8-19/2\,x+4\,x^2)\,\zeta_2-(23+3/2\,x+4\,x^2)\,\mathrm{H}_{0,0}\\
&+(9+13\,x)\,\zeta_2^2-2\,(1-7\,x)\,\mathrm{H}_0\,\zeta_3+2\,(2-3\,x)\,\mathrm{H}_{2,1}+2\,(4-x)\,(\mathrm{H}_0\,\zeta_2-\mathrm{H}_3)\\
&-2\,(3+4\,x)\,\mathrm{H}_{2,0}+(2+19/2\,x)\,\mathrm{H}_2-5/2\,(5-2\,x)\,\mathrm{H}_0-2\,(7-3\,x)\,\mathrm{H}_{0,0,0}\\
&-2\,(5+21\,x)\,\zeta_3+4\,(x^{-1}+x^2)\,\mathrm{H}_{-1,0}-16\,x\,(2\,\mathrm{H}_{-2,0}-\mathrm{H}_{-3,0})+1/8\,\delta(1-x)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+2/27\,C_A\,n_f^2\,\Big(-8\,\Delta p_{\mathrm{gg}}(x)+48\,(1+x)\,(\zeta_2-1/2\,\mathrm{H}_{0,0}-\mathrm{H}_2)-3\,(1-x)\,(33+41\,\mathrm{H}_1)\\
&-(56-67\,x)\,\mathrm{H}_0+87/4\,\delta(1-x)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+2/27\,C_F\,n_f^2\,\Big(-4\,(1-x)\,(146+90\,\mathrm{H}_{1,0}+45\,\mathrm{H}_{1,1}+78\,\mathrm{H}_1)-72\,(1+x)\,(\zeta_3-2\,\mathrm{H}_0\,\zeta_2\\
&+\mathrm{H}_{0,0,0}+2\,\mathrm{H}_{2,0}+\mathrm{H}_{2,1}+2\,\mathrm{H}_3)+24\,(13-8\,x)\,(\zeta_2-\mathrm{H}_2)-12\,(7-23\,x)\,\mathrm{H}_{0,0}\\
&-52\,(5-x)\,\mathrm{H}_0+33/2\,\delta(1-x)\Big)\,.
\end{aligned}
\tag{5.16}
$$

The functions (5.12) – (5.16) are shown in Figs. 5 – 8 for $n_f = 3$ effectively massless quark flavours. For the numerical evaluation of the harmonic polylogarithms we have made use of Ref. [79].

Except for the case of ΔP_{gq}, the respective first two terms in the expansion of the entries of the matrix (2.7) powers of $(1-x)$ are identical to their unpolarized counterparts, i.e., Eq. (2.22) holds also for the differences $\delta_{ik}^{(2)}(x)$ defined in Eq. (2.20). The NNLO counterpart to Eq. (2.23) is

$$
\begin{aligned}
\delta_{\mathrm{gq}}^{(2)}(x) \;=\; & \ln^3(1-x)\,8\,C_F\,(C_A-C_F)^2\\[4pt]
&+\ln^2(1-x)\left[\frac{2}{3}\,C_F\,(C_A-C_F)\,(77\,C_A-45\,C_F)-\frac{28}{3}\,C_F\,(C_A-C_F)\,n_f\right]\\[4pt]
&+\ln(1-x)\left[\frac{1870}{9}\,C_F\,C_A^2-\frac{2260}{9}\,C_F^2\,C_A+54\,C_F^3\right.\\[4pt]
&\qquad\left.-8\,\zeta_2\,C_F\,(C_A-C_F)\,(5\,C_A-2\,C_F)-\frac{424}{9}\,C_F\,C_A\,n_f+\frac{304}{9}\,C_F^2\,n_f+\frac{8}{3}\,C_F\,n_f^2\right]\\[4pt]
&+C_F\,C_A^2\left(\frac{2068}{9}-\frac{154}{3}\,\zeta_2\right)-C_F^2\,C_A\left(\frac{466}{3}-30\,\zeta_2\right)+24\,C_F^3+\frac{52}{3}\,C_F^2\,n_f+\frac{40}{9}\,C_F\,n_f^2\\[4pt]
&+8\,\zeta_3\,C_F\,(C_A-C_F)\,(5\,C_A+4\,C_F)-C_F\,C_A\,n_f\left(\frac{632}{9}-\frac{28}{3}\,\zeta_2\right)
\end{aligned}
$$

$$+ (1-x) \ln^3(1-x) \, 8 C_F \, (C_A - C_F)^2$$

$$+ (1-x) \ln^2(1-x) \left[\frac{2}{3} C_F \, (C_A - C_F) \, (41 C_A + 15 C_F) - \frac{28}{3} C_F \, n_f \, (C_A - C_F) \right]$$

$$+ (1-x) \ln(1-x) \left[\frac{1690}{9} C_F \, C_A^2 - \frac{1504}{9} C_F^2 C_A + 22 C_F^3 + \frac{16}{9} C_F^2 n_f + \frac{8}{3} C_F \, n_f^2 \right.$$

$$\left. - 8 \zeta_2 C_F \, (C_A - C_F) \, (5 C_A - 2 C_F) - \frac{208}{9} C_F \, C_A \, n_f \right] \tag{5.17}$$

$$+ (1-x) \left[C_F \, C_A^2 \left(\frac{104}{3} - \frac{34}{3} \zeta_2 \right) + C_F^2 C_A \left(\frac{574}{9} - 42 \zeta_2 \right) - C_F^3 \, (16 - 32 \zeta_2) \right.$$

$$\left. + 8 \zeta_3 C_F \, (C_A - C_F) \, (5 C_A + 4 C_F) - C_F \, C_A \, n_f \left(\frac{280}{9} - \frac{28}{3} \zeta_2 \right) - \frac{4}{9} C_F^2 n_f - \frac{8}{9} C_F \, n_f^2 \right]$$

$$+ O\left((1-x)^2 \right) .$$

All terms shown in this equations are removed by including the additional contribution (2.19) to the transformation (3.7) from the Larin scheme.

At small-x the polarized splitting functions are double-logarithmically enhanced, i.e., terms up to $\ln^{2n} x$ occur at N^{n}LO. Using the notation

$$\Delta P_{ik}^{(2)}(x) \;=\; D_{ik}^{(0)} \ln^4 x + D_{ik}^{(1)} \ln^3 x + D_{ik}^{(2)} \ln^2 x + D_{ik}^{(3)} \ln x + O(1) \tag{5.18}$$

for the leading logarithmic (LL), next-to-leading logarithmic (NLL) contributions etc at NNLO, the small-x terms of the non-singlet and pure-singlet splitting function are given by

$$D_{\rm ns}^{(0)} \;=\; -C_F \, C_A^2 + 4 C_F^2 C_A - \frac{10}{3} C_F^3$$

$$\cong 1.43210 \, ,$$

$$D_{\rm ns}^{(1)} \;=\; \frac{40}{9} C_F \, C_A^2 - \frac{14}{9} C_F^2 C_A - 4 C_F^3 + \frac{20}{9} C_F^2 n_f - \frac{16}{9} C_F \, C_A \, n_f$$

$$\cong 35.5556 - 3.16049 \, n_f \, ,$$

$$D_{\rm ns}^{(2)} \;=\; \left(81 + 14 \zeta_2 \right) C_F \, C_A^2 - \left(\frac{152}{3} + 96 \zeta_2 \right) C_F^2 C_A - (60 - 104 \zeta_2) C_F^3$$

$$- \frac{196}{9} C_F \, C_A \, n_f + \frac{80}{3} C_F^2 n_f + \frac{4}{9} C_F \, n_f^2$$

$$\cong 399.205 - 39.7037 \, n_f + 0.592592 \, n_f^2 \, ,$$

$$D_{\rm ns}^{(3)} \;=\; \left(\frac{3442}{27} + \frac{100}{3} \zeta_2 + 112 \zeta_3 \right) C_F \, C_A^2 + \left(\frac{1850}{9} - \frac{680}{3} \zeta_2 - 336 \zeta_3 \right) C_F^2 C_A$$

$$- (286 - 192 \zeta_2 - 224 \zeta_3) C_F^3 - \left(\frac{2252}{27} - \frac{8}{3} \zeta_2 \right) C_F \, C_A \, n_f + \left(\frac{568}{9} + \frac{32}{3} \zeta_2 \right) C_F^2 n_f$$

$$+ \frac{88}{27} C_F \, n_f^2$$

$$\cong 1465.93 - 172.693 \, n_f + 4.34568 \, n_f^2 \tag{5.19}$$

31

and

$$D_{\text{ps}}^{(0)} = -2n_f C_F C_A - \frac{8}{3}n_f C_F^2$$

$$\cong -12.7407\,n_f\,,$$

$$D_{\text{ps}}^{(1)} = -\frac{152}{9}n_f C_F C_A - \frac{40}{3}n_f C_F^2 + \frac{8}{9}n_f^2 C_F$$

$$\cong -91.2593\,n_f + 1.18519\,n_f^2\,,$$

$$D_{\text{ps}}^{(2)} = -\left(\frac{854}{9} + 4\,\zeta_2\right)n_f C_F C_A - (64 - 48\,\zeta_2)\,n_f C_F^2 + \frac{92}{9}n_f^2 C_F$$

$$\cong -379.285\,n_f + 13.6296\,n_f^2\,,$$

$$D_{\text{ps}}^{(3)} = -\left(\frac{9028}{27} + \frac{116}{3}\zeta_2 - 144\,\zeta_3\right)n_f C_F C_A - (100 - 160\,\zeta_2 + 112\,\zeta_3)\,n_f C_F^2$$

$$\quad + \left(\frac{520}{27} + \frac{32}{3}\zeta_2\right)n_f^2 C_F$$

$$\cong -848.741\,n_f + 49.0736\,n_f^2 \tag{5.20}$$

where the respective last lines provide the QCD values rounded to six significant figures. The corresponding coefficients for $\Delta P_{\text{qg}}^{(2)}$ and $\Delta P_{\text{gq}}^{(2)}$ read

$$D_{\text{qg}}^{(0)} = -5n_f C_A^2 - \frac{4}{3}n_f C_F C_A + \frac{4}{3}n_f^2 C_F$$

$$\cong -50.3333\,n_f + 1.77778\,n_f^2\,,$$

$$D_{\text{qg}}^{(1)} = -\frac{328}{9}n_f C_A^2 - \frac{178}{9}n_f C_F C_A + 10n_f C_F^2 + \frac{16}{9}n_f^2 C_A + \frac{172}{9}n_f^2 C_F$$

$$\cong -389.334\,n_f + 30.8148\,n_f^2\,,$$

$$D_{\text{qg}}^{(2)} = -(150 - 14\,\zeta_2)\,n_f C_A^2 - \left(\frac{701}{18} - 24\,\zeta_2\right)n_f C_F C_A + \left(\frac{59}{2} + 28\,\zeta_2\right)n_f C_F^2$$

$$\quad + 22n_f^2 C_A + \frac{901}{9}n_f^2 C_F$$

$$\cong -1006.28\,n_f + 199.481\,n_f^2\,,$$

$$D_{\text{qg}}^{(3)} = -\left(\frac{22936}{27} - 28\,\zeta_2 - 288\,\zeta_3\right)n_f C_A^2 + \left(\frac{15259}{27} + 72\,\zeta_2 - 328\,\zeta_3\right)n_f C_F C_A$$

$$\quad - (77 - 92\,\zeta_2 - 104\,\zeta_3)\,n_f C_F^2 + \frac{2134}{27}n_f^2 C_A + \frac{6434}{27}n_f^2 C_F$$

$$\cong -2603.45\,n_f + 554.840\,n_f^2 \tag{5.21}$$

and

$$D_{\text{gq}}^{(0)} = 10C_F C_A^2 + \frac{16}{3}C_F^2 C_A - \frac{8}{3}C_F^3 - \frac{8}{3}n_f C_F^2$$

$$\cong \ 142.123 - 4.74074\, n_f \, ,$$

$$D_{gq}^{(1)} \ = \ \frac{188}{3} C_F C_A^2 + \frac{356}{9} C_F^2 C_A - \frac{92}{3} C_F^3 - \frac{200}{9} n_f C_F^2$$

$$\cong \ 890.272 - 39.5062\, n_f \, ,$$

$$D_{gq}^{(2)} \ = \ \left(\frac{3104}{9} - 92\, \zeta_2 \right) C_F C_A^2 + \left(\frac{365}{9} - 16\, \zeta_2 \right) C_F^2 C_A - (39 + 24\, \zeta_2)\, C_F^3$$
$$- \frac{164}{9} n_f C_F C_A - \frac{674}{9} n_f C_F^2$$

$$\cong \ 2212.57 - 206.025\, n_f \, ,$$

$$D_{gq}^{(3)} \ = \ \left(\frac{22052}{27} - \frac{188}{3} \zeta_2 - 160\, \zeta_3 \right) C_F C_A^2 - \left(\frac{11093}{27} + \frac{208}{3} \zeta_2 - 128\, \zeta_3 \right) C_F^2 C_A$$
$$+ (341 - 96\, \zeta_3)\, C_F^3 + \left(\frac{1984}{27} - \frac{160}{3} \zeta_2 \right) n_f C_F C_A - \left(\frac{6742}{27} - \frac{160}{3} \zeta_2 \right) n_f C_F^2$$

$$\cong \ 4811.85 - 344.947\, n_f \, . \tag{5.22}$$

Finally the small-x coefficients (5.18) of the polarized NNLO gluon-gluon splitting function are

$$D_{gg}^{(0)} \ = \ \frac{56}{3} C_A^3 + \frac{2}{3} n_f C_A^2 - 8\, n_f C_F C_A - \frac{4}{3} n_f C_F^2$$

$$\cong \ 504 - 28.3704\, n_f \, ,$$

$$D_{gg}^{(1)} \ = \ \frac{1264}{9} C_A^3 - \frac{4}{9} n_f C_A^2 - \frac{724}{9} n_f C_F C_A - \frac{56}{3} n_f C_F^2 - \frac{8}{9} n_f^2 C_F$$

$$\cong \ 3792 - 358.963\, n_f - 1.18519\, n_f^2 \, ,$$

$$D_{gg}^{(2)} \ = \ \left(\frac{2126}{3} - 176\, \zeta_2 \right) C_A^3 - \left(\frac{244}{9} + 20\, \zeta_2 \right) n_f C_A^2 - \left(\frac{3542}{9} - 48\, \zeta_2 \right) n_f C_F C_A$$
$$- (92 - 16\, \zeta_2)\, n_f C_F^2 - \frac{8}{9} n_f^2 C_A - \frac{28}{9} n_f^2 C_F$$

$$\cong \ 11317.3 - 1915.25\, n_f - 6.81481\, n_f^2 \, ,$$

$$D_{gg}^{(3)} \ = \ \left(\frac{47810}{27} - \frac{976}{3} \zeta_2 - 192\, \zeta_3 \right) C_A^3 + \left(\frac{4844}{27} - \frac{236}{3} \zeta_2 - 96\, \zeta_3 \right) n_f C_A^2$$
$$- \left(\frac{34172}{27} - \frac{448}{3} \zeta_2 - 144\, \zeta_3 \right) n_f C_F C_A - (68 - 64\, \zeta_2 + 16\, \zeta_3)\, n_f C_F^2$$
$$- \frac{112}{27} n_f^2 C_A - \left(\frac{520}{27} - \frac{32}{3} \zeta_2 \right) n_f^2 C_F$$

$$\cong \ 27129.4 - 3944.01\, n_f - 14.7288\, n_f^2 \, . \tag{5.23}$$

The coefficients $D_{ns}^{(0)}$, $D_{ps}^{(0)}$ and $D_{gg}^{(0)}$, which are identical to the coefficients of the corresponding physical kernels, agree directly with Refs. [64, 67], for $D_{qg}^{(0)}$ and $D_{gq}^{(0)}$ agreement with Ref. [64] is obtained after taking into account Eq. (2.26).
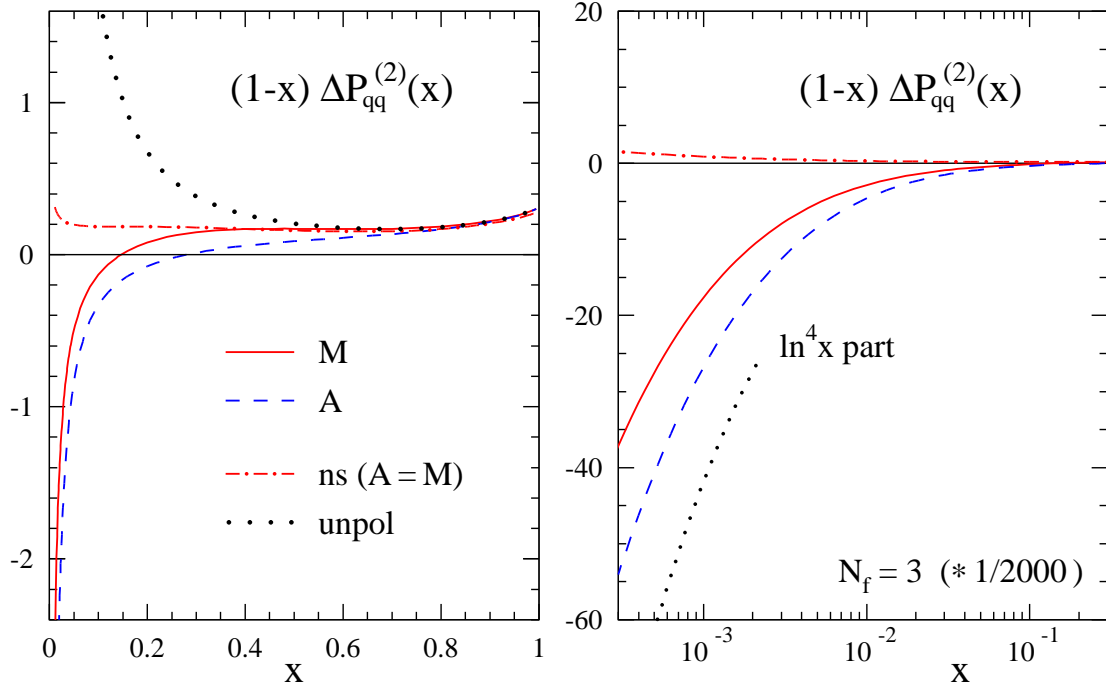
Figure 5: The polarized NNLO quark-quark splitting function in the standard $\overline{\text{MS}}$ scheme (M), as given by the sum of equations (5.12) and (5.13) for three flavours, multiplied by $(1-x)$ for display purposes. Also shown are the non-singlet contribution, the leading-logarithmic small-x part [64], and the splitting function in the alternative scheme (A) with Eq. (3.7), see also Appendix A.
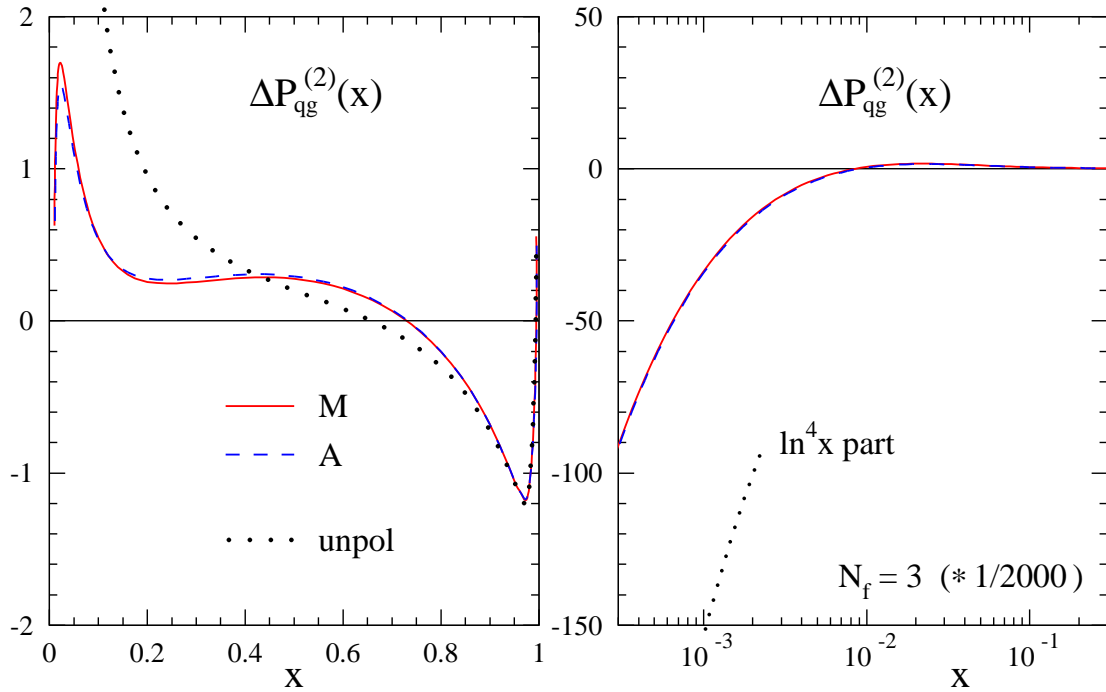
Figure 6: As Figure 5, but for the gluon-quark splitting function (5.14) and its A-scheme analogue. The multiplication with $1/2000 \simeq 1/(4\pi)^3$ approximately converts the results to a series in α_s.
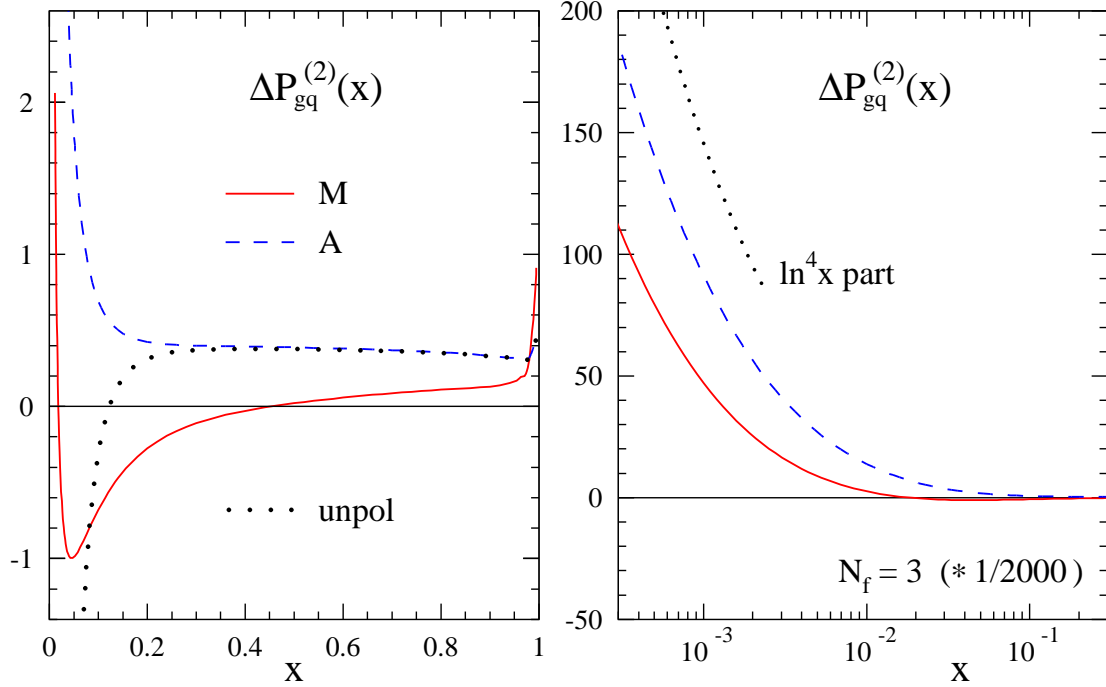
Figure 7: The polarized NNLO quark-gluon splitting function (5.15) for the standard (M) transformation (2.17) [56] from the Larin scheme and an alternative (A) which also includes Eq. (3.7). As for $\Delta P_{qg}^{(2)}(x)$ shown in the previous figure, the leading small-x coefficient is different from Ref. [64], which provides the $\ln^4 x$ terms of the physical kernels $K_{\bar{4}6}$ and $K_{6\bar{4}}$ in these cases.
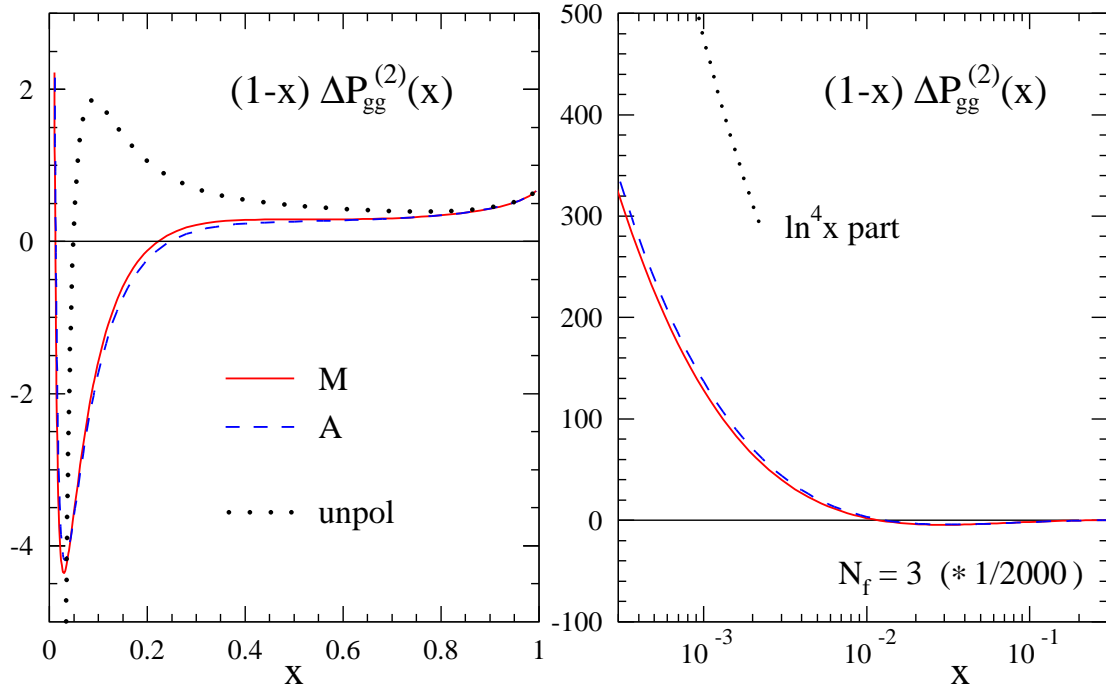
Figure 8: As Figure 5, including the multiplication with $(1-x)$, but for the second diagonal NNLO entry of the splitting-function matrix (2.7) given by Eq. (5.16) in the standard $\overline{\text{MS}}$ scheme.

The small-x behaviour in the right parts of Figs. 5 – 8 is due to the above contributions, which exhibit the usual pattern of alternating LL, NLL etc terms with coefficients strongly increasing towards lower logarithms. Consequently the leading logarithms alone do not provide a good approximation for any practically relevant values of x as illustrated in the figures. Yet it is also clear, from the scale of the ordinates in those right panels and Eq. (5.20) – (5.23), that these logarithms lead to a huge small-x enhancement that can potentially spoil the stability of the expansion in α_s at x-values that would be accessible to an electron-proton collider with polarized beams.

Given the length and complexity of the exact expressions (5.12) – (5.16), it may be useful to also have at one's disposal compact and accurate approximate expressions for the case of QCD, i.e., $C_A = 3$ and $C_F = 4/3$. Such approximations can be build up, besides powers of x, from the non-logarithmic plus distribution and end-point logarithms

$$\mathcal{D}_0 = 1/(1-x)_+ , \quad L_1 = \ln(1-x) , \quad L_0 = \ln x . \tag{5.24}$$

Due to $\Delta P_{\text{ns}}^{+(2)} = P_{\text{ns}}^{-(2)}$, the result (4.23) of Ref. [12] can be used also here; it is given by

$$\begin{aligned}
\Delta P_{\text{ns}}^{+(2)}(x) \cong{} & 1174.898\,\mathcal{D}_0 + 1295.470\,\delta(1-x) + 714.1\,L_1 + 1860.2 - 3505\,x \\
& + 297.0\,x^2 - 433.2\,x^3 + L_0 L_1 (684 + 251.2\,L_0) + 1465.2\,L_0 + 399.2\,L_0^2 \\
& + 320/9\,L_0^3 + 116/81\,L_0^4 \\
& + n_f \left(-183.187\,\mathcal{D}_0 - 173.933\,\delta(1-x) - 5120/81\,L_1 - 216.62 + 406.5\,x + 77.89\,x^2 \right. \\
& \left. + 34.76\,x^3 - 1.136\,x L_0^3 - 65.43\,L_0 L_1 - 172.69\,L_0 - 3216/81\,L_0^2 - 256/81\,L_0^3 \right) \\
& + n_f^2 \left(-\mathcal{D}_0 - (51/16 + 3\zeta_3 - 5\zeta_2)\,\delta(1-x) + x(1-x)^{-1}L_0(3/2\,L_0 + 5) + 1 \right. \\
& \left. + (1-x)(6 + 11/2\,L_0 + 3/4\,L_0^2) \right) 64/81 . \tag{5.25}
\end{aligned}$$

The polarized pure-singlet NNLO splitting function (5.13) can be parametrized as

$$\begin{aligned}
\Delta P_{\text{ps}}^{(2)}(x) \cong{} & n_f (1-x) \left(-344/27\,L_0^4 - (90.9198 + 81.50\,x)L_0^3 \right. \\
& - (368.6 - 349.9\,x)L_0^2 - (739.0 - 232.57\,L_1)L_0 - 1362.6 + 1617.4\,x \\
& \left. - 674.8\,x^2 + 167.41\,x^3 - 204.76\,L_1 - 12.61\,L_1^2 - 6.541\,L_1^3 \right) \\
& + n_f^2 (1-x) \left((1.1741 - 0.8253\,x)L_0^3 + (13.287 + 10.657\,x)L_0^2 + 45.482\,L_0 \right. \\
& \left. + 49.13 - 30.77\,x - 4.307\,x^2 - 0.5094\,x^3 + 9.517\,L_1 + 1.7805\,L_1^2 \right) . \tag{5.26}
\end{aligned}$$

Sufficiently accurate parametrizations of the corresponding off-diagonal quantities in Eqs. (5.14) and (5.15) are given by

$$\begin{aligned}
\Delta P_{\text{qg}}^{(2)}(x) \cong{} & n_f \left(-151/3\,L_0^4 - (385.64 + 73.30\,x)L_0^3 - (894.8 - 1145.3\,x)L_0^2 \right. \\
& - (1461.2 - 825.4\,L_1)L_0 - 2972.4 + 4672\,x - 1221.6\,x^2 - 18.0\,x^3 \\
& \left. + 278.32\,L_1 - 90.26\,L_1^2 - 5.30\,L_1^3 + 3.784\,L_1^4 \right) \\
& + n_f^2 \left(16/9\,L_0^4 + (30.739 + 10.186\,x)L_0^3 + (196.96 + 179.1\,x)L_0^2 \right. \\
& + (526.3 - 47.30\,L_1)L_0 + 499.65 - 432.18\,x - 141.63\,x^2 - 11.34\,x^3 \\
& \left. - 6.256\,L_1 + 7.32\,L_1^2 + 0.7374\,L_1^3 \right) \tag{5.27}
\end{aligned}$$

and

$$\Delta P_{\mathrm{gq}}^{(2)}(x) \cong 11512/81\,L_0^4 + (888.003 + 175.1\,x)L_0^3 + (2140 - 850.7\,x)L_0^2$$
$$+ (4046.6 - 1424.8\,L_1)L_0 + 6159 - 3825.9\,x + 1942\,x^2 - 742.1\,x^3$$
$$+ 1843.7\,L_1 + 451.55\,L_1^2 + 59.3\,L_1^3 + 5.143\,L_1^4$$
$$+ n_f \left(-128/27\,L_0^4 - (39.3872 + 30.023\,x)L_0^3 - (202.46 + 126.53\,x)L_0^2\right.$$
$$- (308.98 + 16.18\,L_1)L_0 - 301.07 - 296.0\,x + 406.13\,x^2 - 101.62\,x^3$$
$$\left.- 171.78\,L_1 - 47.86\,L_1^2 - 4.963\,L_1^3\right)$$
$$+ n_f^2 \left(16/27\,(-12 + 10\,x + (8 + 2\,x)L_1 + (6 - 3\,x)L_1^2)\right)\ . \tag{5.28}$$

Finally the gluon-gluon splitting function (5.16) can be approximately represented by

$$\Delta P_{\mathrm{gg}}^{(2)}(x) \cong 2643.521\,\mathcal{D}_0 + 4427.762\,\delta(1 - x) + 504L_0^4 + (3777.5 + 1167\,x)L_0^3$$
$$+ (10902 - 863\,x)L_0^2 + (23091 - 12292\,L_1)L_0 + 30988 - 39925\,x + 13447\,x^2$$
$$- 4576\,x^3 - 13247\,(1 - x)L_1 + 3801\,L_1$$
$$+ n_f \left(-412.172\,\mathcal{D}_0 - 528.536\,\delta(1 - x) - 766/27\,L_0^4 - (357.798 - 131\,x)L_0^3\right.$$
$$- (1877.2 - 613.1\,x)L_0^2 - (3524 + 7932\,L_1)L_0 - 1173.5 + 2648.6\,x - 2160.8\,x^2$$
$$\left.+ 1251.7\,x^3 - 6746\,(1 - x)L_1 - 295.7\,L_1\right)$$
$$+ n_f^2 \left(-16/9\,\mathcal{D}_0 + 6.4607\,\delta(1 - x) - 1.1809\,L_0^3 - (6.679 - 15.764\,x)L_0^2\right.$$
$$\left.- (13.29 + 16.944\,L_1)L_0 - 16.606 + 32.905\,x - 18.30\,x^2 + 2.637\,x^3 - 0.210\,L_1\right)\ . \tag{5.29}$$

These expressions can be readily transformed to Mellin space for any $N \neq -n$, $n = 0, 1, 2, \ldots$; the most complex objects needed there are the logarithmic derivatives of Euler's Γ-function.

The n_f^2 contributions in Eqs. (5.25) and (5.28) are exact. The same holds for all coefficients of $\ln^4 x$ and, up to the truncation of irrational numbers, those of $1/(1 - x)_+$ in Eqs. (5.25) and (5.29). The other terms at $x < 1$ have fitted to the exact results, evaluated by the FORTRAN code of Ref. [79], at $10^{-6} \leq x \leq 1 - 10^{-6}$ using the MINUIT package [80, 81]. Except for x values very close to zeros of the splitting functions, the above parametrizations deviate from the exact results by less than one part in thousand, which should be sufficient for any foreseeable phenomenological application. As in the unpolarized case [12, 13], the coefficients of $\delta(1 - x)$ have been adjusted in Eq. (5.29) using low integer moments in order to achieve a maximal accuracy of the parametrization and its convolutions with the polarized gluon distribution. For a brief discussion of this slightly subtle point the reader is referred to Ref. [13] (penultimate paragraph of section 4).

The effect of the new results (5.13) – (5.16) on the evolution of polarized parton densities is briefly illustrated in Figs. 9 and 10, where the respective first and second lines of Eq. (2.7) have been evaluated for the schematic, but sufficiently realistic low-scale distributions

$$\Delta f_{\mathrm{q}}(x, \mu_0^2) = 0.8\,x^{0.7}(1 - x)^3(1 + 3\,x + 2.5\,x^2) - 0.25\,x^{0.7}(1 - x)^7\ ,$$
$$\Delta f_{\mathrm{g}}(x, \mu_0^2) = 1.5\,x^{0.5}(1 - x)^5 \tag{5.30}$$

used for the evolution benchmarks in Refs. [85, 86], for $\alpha_{\mathrm{s}}(\mu_0^2) = 0.3$ and $n_f = 3$. After the convolution with the distributions (5.30), the NNLO corrections are fairly small down to small x.
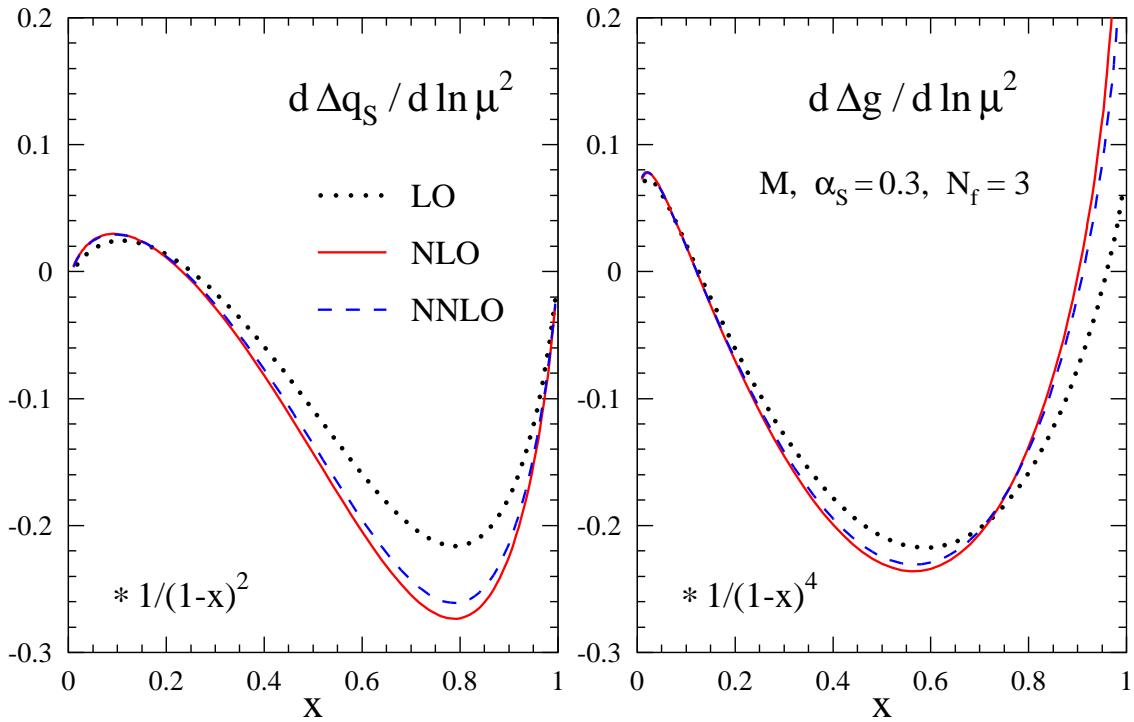
Figure 9: The perturbative expansion of the scale derivatives of the polarized singlet-quark and gluon distributions in the standard $\overline{\text{MS}}$ scheme (M) [56], for the low-scale input distributions in Eq. (5.30) and a rather large value of the strong coupling α_s. The results have been multiplied by powers of $(1-x)$ suitable to clearly display the NLO and NNLO effects up to rather large x.
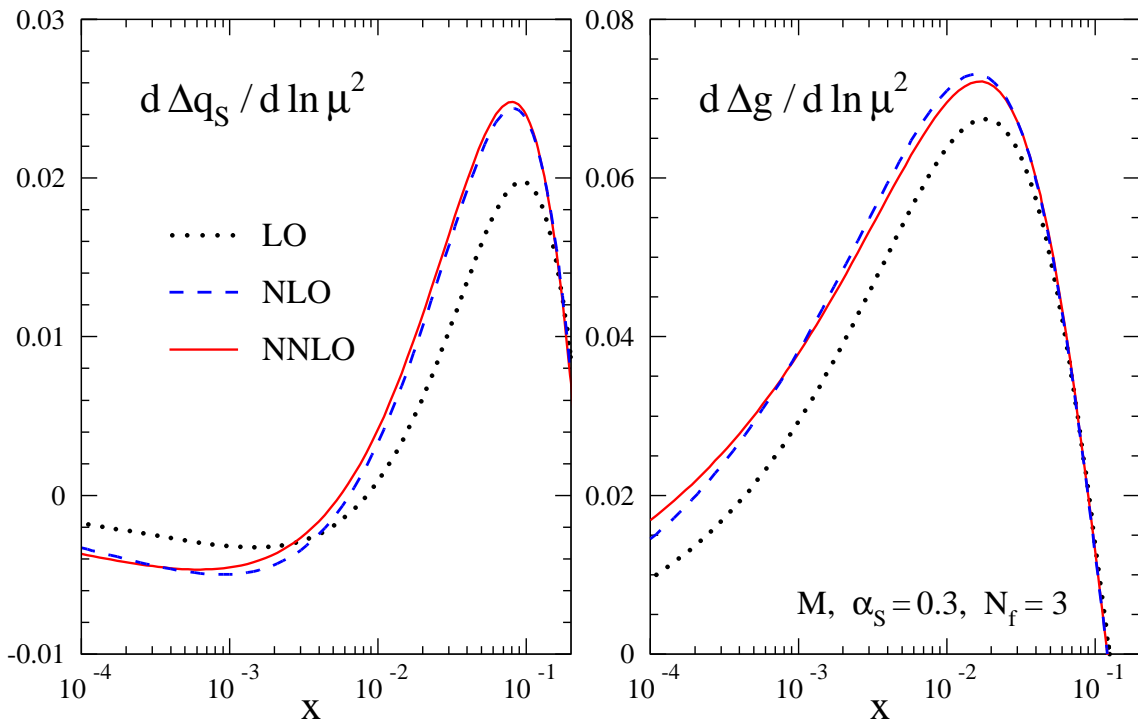

Figure 10: As Figure 10, but using a logarithmic scale in x to show the results down to small x.

6 Summary

We have extended the determination of the helicity-difference (polarized) splitting functions ΔP_{ik}, which were only known at the first [1,6,7] and second [23–25] order in the strong coupling constant α_s so far, to the third order (next-to-next-to-leading order, NNLO) in massless perturbative QCD. These corrections are relevant to the structure function g_1 in polarized deep-inelastic scattering (DIS), for which we also confirm the results of Ref. [44] for the NNLO coefficient functions, and all other observables that are sensitive to the polarized quark and gluon distributions $\Delta f_{q_i} + \Delta f_{\bar{q}_i}$ and Δf_g. The so far practically irrelevant polarized quark-antiquark differences have not been addressed here; the corresponding splitting functions can be calculated, e.g., by extending the analysis of weak-interaction structure functions in Ref. [75] to NNLO accuracy.

The calculation of the upper row of the matrix of NNLO flavour-singlet splitting functions, i.e., of $\Delta P_{qq}^{(2)}(x)$ and $\Delta P_{qg}^{(2)}(x)$, was carried out via the structure function g_1 as a direct extension of our previous calculations of the helicity-averaged (unpolarized) case [12, 13], for an earlier brief account see Ref. [26]. The corresponding lower-row quantities $\Delta P_{gq}^{(2)}(x)$ and $\Delta P_{gg}^{(2)}(x)$ have been determined in a different manner from graviton-exchange DIS, see Ref. [28], which includes structure functions sensitive to the polarized gluon distribution at the Born level.

We have first calculated the relevant structure function at fixed odd moments to $N = 25$, using a large-N optimized version [35] of the MINCER program [33, 34] in (T)FORM [31, 32]. Exploiting in particular the close relation between the polarized and unpolarized splitting functions for the highest-weight harmonic sums [61] and for the threshold limit, cf. Ref. [58] – which includes the so-called supersymmetric relation, see Refs. [3, 87], as far as it can be addressed in $\overline{\text{MS}}$ – we have then been able to determine the all-N expressions of $\Delta P_{gq}^{(2)}$ and $\Delta P_{gg}^{(2)}$. It was crucial for this step that the coefficients of the harmonic sums are integer, up to low powers of 2 and 3 that can be removed by a suitable normalization, which allows the use of advanced tools [39–41] for systems of Diophantine equations; this was observed and exploited before in a comparable but somewhat simpler situation in Ref. [38]. Finally the results have been validated by comparing the next two moments of all-N expressions to additional results calculated using MINCER up to $N = 29$.

Our results have been presented above in N-space and x-space, using the transformation of Ref. [56] from the so-called Larin scheme for γ_5 [48,49] in dimensional regularization to $\overline{\text{MS}}$. This scheme shows an unphysical feature in the threshold limit of the quark-gluon splitting function ΔP_{gq} already at NLO, which can be removed to NNLO by simple additional terms in the scheme transformation. Yet this situation does not appear to necessitate a change of the factorization scheme in practical calculations after almost two decades of NLO analyses in QCD spin physics.

The new functions $\Delta P_{ik}^{(2)}(x)$ are consistent with all known limits and partial results, e.g., for the leading large-n_f terms [73], and expectations. In particular, the first moment of $\Delta P_{gg}^{(2)}(x)$, which is not directly accessible in graviton-exchange DIS [28] but can be determined from the x-space results in terms of harmonic polylogarithms [71], is identical to the NNLO coefficient of the beta function of QCD [54, 55] as theoretically required. We have checked our calculations of graviton-exchange DIS also by re-calculating, and obtaining full agreement for, $\Delta P_{qq}^{(2)}$ and $\Delta P_{qg}^{(2)}$ to fairly

high values odd of N and all unpolarized flavour-singlet NNLO splitting functions at even $N \leq 10$. As those results, the present polarized splitting functions lead to fairly small NNLO corrections, down to low values of x, after the convolution with realistic polarized quark and gluon distributions, despite a double-logarithmic small-x enhancement that dwarfs that of the non-singlet cases.

Our results allow NNLO analyses of spin-dependent hard-scattering observables, provided that the corresponding coefficient functions are known to this accuracy as for the structure function g_1 in DIS [44], for a fixed number of effectively massless flavours n_f. The extension to analyses in the so-called variable flavour-number scheme, where effective theories for different values of n_f are used together, requires non-trivial matching coefficients for the strong coupling [88] and the parton densities at this order. The latter coefficients have been calculated in Ref. [89] for the unpolarized case. As far as we know, the corresponding results for the helicity-difference parton distributions are not yet available in the literature though.

FORM and FORTRAN files of our main analytical results in N-space and x-space, and compact high-accuracy parametrizations of the functions $\Delta P_{ik}^{(2)}(x)$, can be obtained by downloading the source of this article from `http://arxiv.org/` or from the authors upon request.

Acknowledgments

Appendix A   Transformation to the $\overline{\text{MS}}$ scheme

Here we collect, for completeness, the functions entering the transformation of the splitting and coefficient functions from the Larin scheme to $\overline{\text{MS}}$ as discussed in Section 2, Eqs. (2.11) – (2.14) and Eqs. (2.17) – (2.19), and Section 3, see Eq. (3.7).

The NLO and NNLO quark-quark elements (2.17) of the transformation matrix $Z(x, \mu^2)$ read

$$z_{\text{ns}}^{(1)}(x) \;=\; -8\,C_F\,(1-x)\,, \tag{A.1}$$

$$
\begin{aligned}
z_{\text{ns}}^{(2)}(x) \;=\;\; & 8\,C_F^2\Big((1-x)(5-2\,\text{H}_{1,0}-2\,\text{H}_2) - 2\,(1+x)\,(2\,\text{H}_{-1,0}-\text{H}_{0,0}+\zeta_2) + (1+2x)\,\text{H}_0\Big) \\
& + 4\,C_F C_A\Big(4\,(1+x)\,\text{H}_{-1,0} - 4\,(\text{H}_{0,0}-\zeta_2) - (29+7x)/3\,\text{H}_0 - 211/9\,(1-x)\Big) \\
& + 8/9\,C_F n_f\,(1-x)\Big(3\,\text{H}_0+5\Big)\,,
\end{aligned}
\tag{A.2}
$$

$$z_{\text{ps}}^{(2)}(x) \;=\; 4C_F n_f \left((2+x)\,\text{H}_{0,0} + (3-x)\,\text{H}_0 + 2\,(1-x) \right) \tag{A.3}$$

for the standard transformation, above denoted by 'M' where required for clarity, of Ref. [56] where the critical last line has been calculated.

In the alternative ('A') form of the transformation, which restores the $(1-x)^2$ suppression for $x \to 1$ of the difference or the unpolarized and polarized splitting functions for $\Delta P_{\text{gq}}^{(1)}(x)$ and $\Delta P_{\text{gq}}^{(2)}(x)$, there are additional quark-gluon entries (3.7) given by

$$z_{\text{gq},A}^{(1)}(x) \;=\; -2C_F\,(2-x)\,, \tag{A.4}$$

$$z_{\text{gq},A}^{(2)}(x) \;=\; -\tfrac{1}{2}\,\Delta P_{\text{gq}}^{(1)}(x) - 8C_F^2 \left(3\,(1-x) + (2+x)\,\text{H}_0 \right)\,, \tag{A.5}$$

where the last line has been expressed in term of the NLO splitting function (5.8) for brevity. Furthermore Eq. (A.3) is replaced by

$$z_{\text{ps},A}^{(2)}(x) \;=\; z_{\text{ps}}^{(2)}(x) + 12C_F n_f\,(1-x) \tag{A.6}$$

which ensures that Eq. (4.16) holds also in the A-scheme.

B NLO coefficient functions in graviton-exchange DIS

The (un-)polarized graviton DIS structure function H_i of Ref. [28] have been introduced briefly in Section 2. We have defined combinations of those H_i which, at Born level, are either given by the flavour-singlet (un-)polarized quark distribution $(\Delta)f_{\text{q}}$ or by the gluon density $(\Delta)f_{\text{g}}$, cf. Eqs. (2.1), (2.2) and (2.8). Their quark and gluon coefficient functions $C_{i,\text{q}}$ and $C_{i,\text{g}}$ can be expanded in powers of a_{s}, see Eq. (2.6).

In the unpolarized case, using the definitions $H_{\bar{1}} = H_1 - H_3$ and $H_{\bar{2}} = H_2 - 4H_3$, cf. Eq. (2.24), the leading-order results for the corresponding non-vanishing coefficient functions are

$$c_{\bar{1},\text{q}}^{(0)}(x) \;=\; c_{\bar{2},\text{q}}^{(0)}(x) \;=\; c_{3,\text{g}}^{(0)}(x) \;=\; \delta(1-x)\,. \tag{B.1}$$

The normalization of the structure functions is chosen such that all dependence on $D = 4 - 2\varepsilon$ is removed from the structure functions H_i at Born level, i.e., the results in Eq. (B.1) are exact.

The NLO results for the unpolarized graviton-exchange coefficient functions read, at $Q^2 = \mu^2$,

$$c_{\bar{1},\text{q}}^{(1)}(x) \;=\; 2C_F \left(-p_{\text{qq}}(x)(3/4 + \text{H}_0 + \text{H}_1) + 1/4(25-x) - \delta(1-x)(13/2 + 2\zeta_2) \right)\,, $$

$$c_{\bar{1},\text{g}}^{(1)}(x) \;=\; 2/3\,C_A(7p_{\text{qg}}(x) + 9) - 2n_f \left(p_{\text{qg}}(x)(29/6 + \text{H}_0 + \text{H}_1) - 5/2 \right)\,, \tag{B.2}$$

$$c_{\bar{2},\text{q}}^{(1)}(x) \;=\; c_{\bar{1},\text{q}}^{(1)}(x) - 9xC_F\,, $$

$$c_{\bar{2},\text{g}}^{(1)}(x) \;=\; c_{\bar{1},\text{g}}^{(1)}(x) - (2C_A + n_f)\,6x(1-x)\,, \tag{B.3}$$

$$c_{3,\mathrm{q}}^{(1)}(x) \;=\; 2C_F\Big(-p_{\mathrm{gq}}(x)(3/4+\mathrm{H}_0+\mathrm{H}_1)+1/4\,(6+x)\Big)\,,$$

$$c_{3,\mathrm{g}}^{(1)}(x) \;=\; 4C_A\Big(-p_{\mathrm{gg}}(x)(11/12+\mathrm{H}_0+\mathrm{H}_1)+11/12\,(2-x+x^2) \tag{B.4}$$
$$-\,\delta(1-x)(34/9+\zeta_2)\Big) + 2/3\,n_f\Big(p_{\mathrm{gg}}(x)-2+x-x^2+25/6\,\delta(1-x)\Big)\,,$$

where we have used the abbreviations

$$\begin{aligned}
p_{\mathrm{qq}}(x) &= 2\,(1-x)^{-1}-1-x\,, \\
p_{\mathrm{qg}}(x) &= 1-2x+2x^2\,, \\
p_{\mathrm{gq}}(x) &= 2x^{-1}-2+x\,, \\
p_{\mathrm{gg}}(x) &= (1-x)^{-1}+x^{-1}-2+x-x^2\,.
\end{aligned} \tag{B.5}$$

The NLO QCD corrections for unpolarized graviton-exchange DIS at NLO have been presented before in Ref. [29] in terms of the bare structure functions H_1, H_2 and H_3 as a Laurent series in ε, i.e., before mass factorization. The results for the coefficient functions in Eq. (B.2) can be used to construct the corresponding expressions to be compared with Ref. [29]. Accounting, of course, for the different normalization we find agreement except for the result of the coefficient function $c_{3,\mathrm{q}}^{(1)}$ as given in Eq. (3.3) of Ref. [29].

In the polarized case we similarly use $H_{\bar{4}}=2(H_4-H_6)$ and H_6, recall Eq. (2.25) with

$$c_{4,\mathrm{i}}^{(0)}(x) \;=\; \delta_{\mathrm{iq}}\,\delta(1-x)\,, \qquad c_{6,\mathrm{i}}^{(0)}(x) \;=\; \delta_{\mathrm{ig}}\,\delta(1-x)\,. \tag{B.6}$$

Again the structure functions are normalized such that there is no dependence in ε at this order. The NLO results for the polarized graviton DIS coefficient functions in the standard $\overline{\mathrm{MS}}$ scheme, i.e., with the transformation (2.17), are given by

$$c_{4,\mathrm{q}}^{(1)}(x) \;=\; 2C_F\Big(-\Delta p_{\mathrm{qq}}(x)(3/4+\mathrm{H}_0+\mathrm{H}_1)-1/4\,(11-17x)-\delta(1-x)(13/2+2\zeta_2)\Big)\,,$$

$$c_{4,\mathrm{g}}^{(1)}(x) \;=\; 32/3\,C_A(2x-1)-2n_f\Big((2x-1)(\mathrm{H}_0+\mathrm{H}_1)-1/3(13-20x)\Big)\,, \tag{B.7}$$

$$c_{6,\mathrm{q}}^{(1)}(x) \;=\; C_F\Big(-2(2-x)(\mathrm{H}_0+\mathrm{H}_1)-(10-3/x-7x)\Big)\,,$$

$$c_{6,\mathrm{g}}^{(1)}(x) \;=\; 4C_A\Big(-\Delta p_{\mathrm{gg}}(x)(11/12+\mathrm{H}_0+\mathrm{H}_1)-1/12(35-11/x-46x) \tag{B.8}$$
$$-\,\delta(1-x)(34/9+\zeta_2)\Big)+2/3\,n_f\Big(\Delta p_{\mathrm{gg}}(x)+1-1/x-2x+25/6\,\delta(1-x)\Big)\,,$$

in terms of $\Delta p_{\mathrm{qq}}(x)$ and $\Delta p_{\mathrm{gg}}(x)$ defined in Eq. (5.11).

Analogous to our discussion of relations between the unpolarized and polarized splitting functions in N-space in Section 3, is may be interesting to note that all H_0 and H_1 contributions to Eqs. (B.7) and (B.8) are related to those in Eqs. (B.2) – (B.4) by replacing $p_{\mathrm{ik}}(x)$ by their polarized counterparts $\Delta p_{\mathrm{ik}}(x)$ with, cf. Eq. (3.6), $\Delta p_{\mathrm{qg}}(x)=2x-1$ and $\Delta p_{\mathrm{gq}}(x)=2-x$.

C Calculation of graviton-exchange DIS

Here we present some core ingredients of our diagram calculations, starting with the Feynman rules as used for graviton-exchange DIS. They have been taken from various sources [90, 91]. We assume all momenta of the gluons and the graviton to be outgoing, while the momenta of the quarks and ghosts follow the arrows on the lines. The color indices in the fundamental representation are i and j; color indices in the adjoint representation are represented by the letters a,b,c,d,e; the Lorentz indices of the graviton are α and β and those of the gluons are μ, ν, ρ, σ. We also use a gauge parameter which is indicated by ξ.

For completeness we start with the QCD propagators and vertices:

$$-i\delta_{ab}\left(\delta_{\mu\nu} - \xi\frac{Q_\mu Q_\nu}{Q\cdot Q}\right)\Big/Q\cdot Q \qquad \text{(C.1)}$$

$$i\delta_{ij}(\gamma_\mu P^\mu)/P\cdot P \qquad \text{(C.2)}$$

$$i\delta_{ab}/Q\cdot Q \qquad \text{(C.3)}$$

$$ig\,T_{ij}^a\,\gamma_\mu \qquad \text{(C.4)}$$

$$-g f^{abc}(\delta_{\mu\nu}(p_1 - p_2)_\rho \\ + \delta_{\nu\rho}(p_2 - p_3)_\mu + \delta_{\rho\mu}(p_3 - p_1)_\nu) \qquad \text{(C.5)}$$

$$-ig^2\ (\ + f^{abe}f^{cde}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) \\ + f^{ace}f^{dbe}(\delta_{\mu\sigma}\delta_{\rho\nu} - \delta_{\mu\nu}\delta_{\rho\sigma}) \\ + f^{ade}f^{bce}(\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\rho}\delta_{\sigma\nu})\ ) \qquad \text{(C.6)}$$
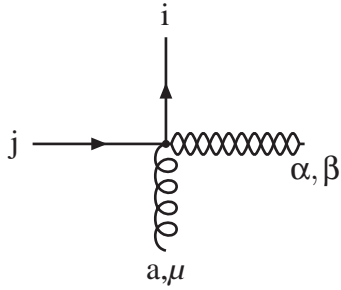
$$-gf^{abc}p_{2\mu} \tag{C.7}$$
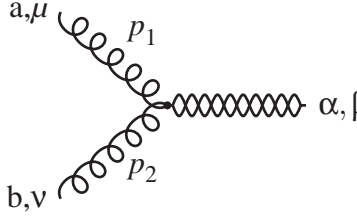
The additional vertices involving the graviton are given by
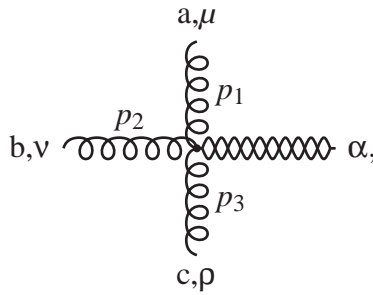
$$-i\frac{\kappa}{8}\delta_{ij}\ (\ \gamma_\alpha(p_1+p_2)_\beta+\gamma_\beta(p_1+p_2)_\alpha$$
$$-2\delta_{\alpha\beta}\gamma_\mu(p_1+p_2)^\mu\ ) \tag{C.8}$$
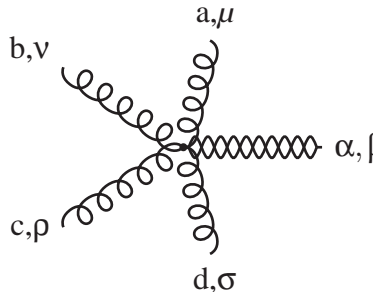

$$ig\frac{\kappa}{4}T^a_{ij}\ (\ \delta_{\alpha\mu}\gamma_\beta+\delta_{\beta\mu}\gamma_\alpha-2\delta_{\alpha\beta}\gamma_\mu\ ) \tag{C.9}$$
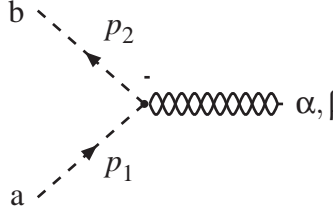

$$-i\frac{\kappa}{2}\delta^{ab}\ (\ p_1\cdot p_2\,C_{\alpha\beta,\mu\nu}+D_{\alpha\beta,\mu\nu}(p_1,p_2)$$
$$+\frac{1}{1-\xi}\,E_{\alpha\beta,\mu\nu}(p_1,p_2)\ ) \tag{C.10}$$
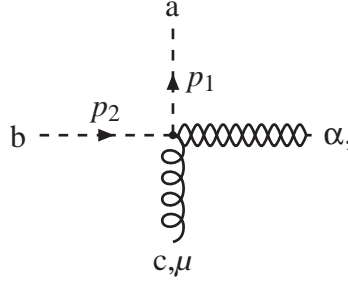
$$-g\frac{\kappa}{2}f^{abc}\ (\ +C_{\alpha\beta,\mu\nu}(p_1-p_2)_\rho$$
$$+C_{\alpha\beta,\mu\rho}(p_3-p_1)_\nu$$
$$+C_{\alpha\beta,\nu\rho}(p_2-p_3)_\mu$$
$$+F_{\alpha\beta,\mu\nu\rho}(p_1,p_2,p_3)\ ) \tag{C.11}$$

$$-ig^2\frac{\kappa}{2}\ (\ +f^{abe}f^{cde}G_{\alpha\beta,\mu\rho\nu\sigma}+f^{ace}f^{bde}G_{\alpha\beta,\mu\nu\rho\sigma}$$
$$+f^{ade}f^{bce}G_{\alpha\beta,\mu\nu\sigma\rho}\ ) \tag{C.12}$$

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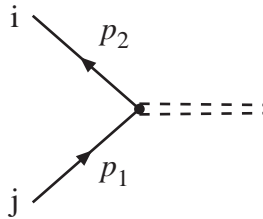$$-i\frac{\kappa}{2}\delta^{ab}C_{\alpha\beta,\mu\nu}\,p_1{}^{\mu}p_2{}^{\nu} \qquad (C.13)$$

$$-g\frac{\kappa}{2}f^{abc}C_{\alpha\beta,\mu\nu}\,p_1{}^{\nu} \qquad (C.14)$$

The tensors C,D,E,F and G in Eqs. (C.10) – (C.14) are defined by

$$
\begin{aligned}
C_{\alpha\beta,\mu\nu} &= \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu} - \delta_{\alpha\beta}\delta_{\mu\nu}\,, \\
D_{\alpha\beta,\mu\nu}(p_1,p_2) &= \delta_{\alpha\beta}p_{1\nu}p_{2\mu} - \delta_{\alpha\nu}p_{1\beta}p_{2\mu} - \delta_{\alpha\mu}p_{1\nu}p_{2\beta} + \delta_{\mu\nu}p_{1\alpha}p_{2\beta} \\
&\quad - \delta_{\beta\nu}p_{1\alpha}p_{2\mu} - \delta_{\beta\mu}p_{1\nu}p_{2\alpha} + \delta_{\mu\nu}p_{1\beta}p_{2\alpha}\,, \\
E_{\alpha\beta,\mu\nu}(p_1,p_2) &= \delta_{\alpha\beta}(p_{1\mu}p_{1\nu} + p_{2\mu}p_{2\nu} + p_{1\mu}p_{2\nu}) \\
&\quad - \delta_{\beta\nu}p_{1\alpha}p_{1\mu} - \delta_{\beta\mu}p_{2\alpha}p_{2\nu} - \delta_{\alpha\nu}p_{1\beta}p_{1\mu} - \delta_{\alpha\mu}p_{2\beta}p_{2\nu}\,, \\
F_{\alpha\beta,\mu\nu\rho}(p_1,p_2,p_3) &= +\delta_{\alpha\mu}\,\delta_{\nu\rho}(p_2 - p_3)_{\beta} + \delta_{\alpha\nu}\,\delta_{\mu\rho}(p_3 - p_1)_{\beta} \\
&\quad + \delta_{\alpha\rho}\,\delta_{\mu\nu}(p_1 - p_2)_{\beta} + \delta_{\beta\mu}\,\delta_{\nu\rho}(p_2 - p_3)_{\alpha} \\
&\quad + \delta_{\beta\nu}\,\delta_{\mu\rho}(p_3 - p_1)_{\alpha} + \delta_{\beta\rho}\,\delta_{\mu\nu}(p_1 - p_2)_{\alpha}\,, \\
G_{\alpha\beta,\mu\nu\rho\sigma} &= \delta_{\alpha\beta}\,(\delta_{\mu\nu}\,\delta_{\rho\sigma} - \delta_{\mu\sigma}\,\delta_{\nu\rho}) \\
&\quad + \delta_{\alpha\mu}\,\delta_{\beta\sigma}\,\delta_{\nu\rho} + \delta_{\alpha\rho}\,\delta_{\beta\nu}\,\delta_{\mu\sigma} - \delta_{\alpha\mu}\,\delta_{\beta\nu}\,\delta_{\rho\sigma} - \delta_{\alpha\rho}\,\delta_{\beta\sigma}\,\delta_{\mu\nu} \\
&\quad + \delta_{\beta\mu}\,\delta_{\alpha\sigma}\,\delta_{\nu\rho} + \delta_{\beta\rho}\,\delta_{\alpha\nu}\,\delta_{\mu\sigma} - \delta_{\beta\mu}\,\delta_{\alpha\nu}\,\delta_{\rho\sigma} - \delta_{\beta\rho}\,\delta_{\alpha\sigma}\,\delta_{\mu\nu}\,. \qquad (C.15)
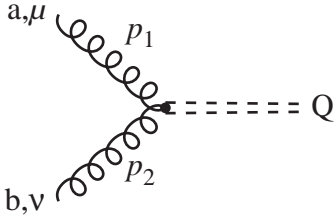\end{aligned}
$$

In addition we need a ghost contribution in the graviton for the unpolarized calculations. We call this particle the g-ghost and we need the vertices ($\omega = \sqrt{\frac{2}{3(D-2)}}$):
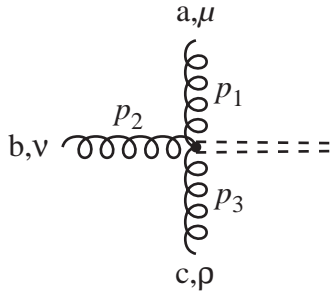

$$i\omega\kappa\delta_{ij}\frac{3}{4}\gamma_{\mu}(p_1 + p_2)^{\mu} \qquad (C.16)$$
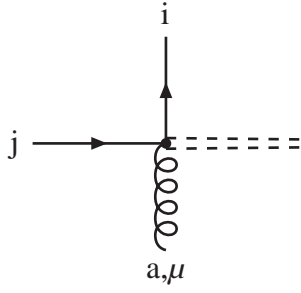
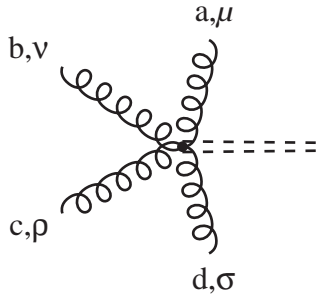$$-i\frac{3}{2}\omega g\kappa T_{ij}^a\gamma_\mu \qquad (C.17)$$

$$i\omega\kappa\delta^{ab}\frac{1}{1-\xi}\left(p_{1\mu}Q_\nu + p_{2\nu}Q_\mu\right) \qquad (C.18)$$

$$0 \qquad (C.19)$$

$$0 \qquad (C.20)$$

Vertices involving both the standard ghost and the g-ghost were not required in our calculation.

We now turn to the projection operators for which we sometimes have more than one choice. The physical operator for the unpolarized gluon is given by [1]

$$\Pi_{\kappa\lambda}(Q,P) = \delta_{\kappa\lambda} - Q_\kappa P_\lambda/Q\cdot P - Q_\lambda P_\kappa/Q\cdot P + P_\kappa P_\lambda Q\cdot Q/Q\cdot P^2 \qquad (C.21)$$

in which $P\cdot P = 0$. One can replace this by $\delta_{\kappa\lambda}$ and a ghost contribution in the regular way. This gives more diagrams, but they are easier to compute. For the polarized gluon we use

$$\Pi_{\kappa\lambda}(Q,P) = \varepsilon_{PQ\kappa\lambda}/Q\cdot P. \qquad (C.22)$$

For the unpolarized and polarized quark the projection operators are

$$\Pi(P) = \gamma^\mu P_\mu \qquad (C.23)$$

---

[1]Here we use Q for the momentum of the probe. Often q is used after which $Q^2 = -q\cdot q$. In the following part $Q\cdot Q$ is just the square of the 4-vector Q, which keeps the notation in line with the computer programs.

and

$$\Pi(P) \;=\; \gamma_5\gamma^\mu P_\mu \;=\; \frac{1}{6}\varepsilon_{\kappa\lambda\nu P}\,\gamma^\kappa\gamma^\lambda\gamma^\nu \,. \tag{C.24}$$

The last form of the operator is necessary to deal with the issue of γ_5 in D dimensions. At a later stage we then contract the Levi-Civita tensors in terms of the D-dimensional metric.

For the graviton the situation is more complicated as there are several possible currents. We follow Ref. [28], assuming a target mass of zero, and add the D-dimensional effects as given in Ref. [29]. Then for unpolarized scattering we have

$$W_{\alpha_1\beta_1,\alpha_2\beta_2} \;=\; F_1 A^{(1)}_{\alpha_1\beta_1\alpha_2\beta_2} + F_2 A^{(2)}_{\alpha_1\beta_1\alpha_2\beta_2} + F_3 A^{(3)}_{\alpha_1\beta_1\alpha_2\beta_2} \,, \tag{C.25}$$

and for polarized scattering

$$W_{\alpha_1\beta_1,\alpha_2\beta_2} \;=\; F_4 A^{(4)}_{\alpha_1\beta_1\alpha_2\beta_2} + F_6 A^{(6)}_{\alpha_1\beta_1\alpha_2\beta_2} \tag{C.26}$$

with

$$
\begin{aligned}
A^{(1)}_{\alpha_1\beta_1\alpha_2\beta_2} &= \overline{\pi}_{\alpha_1\beta_1}\overline{\pi}_{\alpha_2\beta_2} \,, \\
A^{(2)}_{\alpha_1\beta_1\alpha_2\beta_2} &= \overline{P}_{\alpha_1}\overline{P}_{\alpha_2}\overline{G}_{\beta_1\beta_2} + \overline{P}_{\alpha_1}\overline{P}_{\beta_2}\overline{G}_{\beta_1\alpha_2} + \overline{P}_{\beta_1}\overline{P}_{\alpha_2}\overline{G}_{\alpha_1\beta_2} + \overline{P}_{\beta_1}\overline{P}_{\beta_2}\overline{G}_{\alpha_1\alpha_2} \\
&\quad - \frac{4}{D-1}\left(\overline{P}_{\alpha_1}\overline{P}_{\beta_1}\overline{G}_{\beta_2\alpha_2} + \overline{P}_{\alpha_2}\overline{P}_{\beta_2}\overline{G}_{\beta_1\alpha_1}\right) + \frac{4}{(D-1)^2}\overline{G}_{\alpha_1\beta_1}\overline{G}_{\alpha_2\beta_2}\overline{P}\cdot\overline{P} \\
A^{(3)}_{\alpha_1\beta_1\alpha_2\beta_2} &= \overline{G}_{\alpha_1\alpha_2}\overline{G}_{\beta_1\beta_2} + \overline{G}_{\alpha_1\beta_2}\overline{G}_{\alpha_2\beta_1} - \frac{2}{D-1}\overline{G}_{\alpha_1\beta_1}\overline{G}_{\alpha_2\beta_2} \,, \\
A^{(4)}_{\alpha_1\beta_1\alpha_2\beta_2} &= \varepsilon_{\alpha_1\alpha_2 QP}\overline{P}_{\beta_1}\overline{P}_{\beta_2} + \varepsilon_{\alpha_1\beta_2 QP}\overline{P}_{\beta_1}\overline{P}_{\alpha_2} + \varepsilon_{\beta_1\alpha_2 QP}\overline{P}_{\alpha_1}\overline{P}_{\beta_2} + \varepsilon_{\beta_1\beta_2 QP}\overline{P}_{\alpha_1}\overline{P}_{\alpha_2} \,, \\
A^{(6)}_{\alpha_1\beta_1\alpha_2\beta_2} &= \varepsilon_{\alpha_1\alpha_2 QP}\overline{G}_{\beta_1\beta_2} + \varepsilon_{\alpha_1\beta_2 QP}\overline{G}_{\beta_1\alpha_2} + \varepsilon_{\beta_1\alpha_2 QP}\overline{G}_{\alpha_1\beta_2} + \varepsilon_{\beta_1\beta_2 QP}\overline{G}_{\alpha_1\alpha_2} \,. \tag{C.27}
\end{aligned}
$$

Here we have used

$$
\begin{aligned}
\overline{\pi}_{\alpha\beta} &= \overline{P}_\alpha\overline{P}_\beta - \frac{1}{D-1}\overline{G}_{\alpha\beta}\overline{P}\cdot\overline{P} \,, \\
\overline{P}_\alpha &= P_\alpha - Q_\alpha\frac{Q\cdot P}{Q\cdot Q} \,, \\
\overline{G}_{\alpha\beta} &= \delta_{\alpha\beta} - \frac{Q_\alpha Q_\beta}{Q\cdot Q} \,. \tag{C.28}
\end{aligned}
$$

When we construct the projection operators we demand $\Pi_i A_j = \delta_{ij}$, and after also using the symmetry in the graviton indices we have for the unpolarized operators

$$
\begin{aligned}
\Pi_1 \;=\; & 256\frac{(D+1)(D+3)}{D(D-2)}P_{\alpha_1}P_{\beta_1}P_{\alpha_2}P_{\beta_2}\frac{1}{Q\cdot Q^3} \\
& + 1024\frac{D+1}{D(D-2)}P_{\alpha_1}P_{\alpha_2}\delta_{\beta_1\beta_2}\frac{Q\cdot P^2}{Q\cdot Q^4} + 512\,\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}\frac{Q\cdot P^4}{Q\cdot Q^5} \,,
\end{aligned}
$$

$$\Pi_2 = 64\frac{(D+1)}{D(D-2)}P_{\alpha_1}P_{\beta_1}P_{\alpha_2}P_{\beta_2}\frac{1}{Q\cdot Q^3}+32\frac{1}{D(D-3)}\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}\frac{Q\cdot P^4}{Q\cdot Q^5}$$
$$+64\frac{D^2-D-4}{D(D-2)(D-3)}P_{\alpha_1}P_{\alpha_2}\delta_{\beta_1\beta_2}\frac{Q\cdot P^2}{Q\cdot Q^4}\ ,$$

$$\Pi_3 = 16\frac{1}{D(D-2)}P_{\alpha_1}P_{\beta_1}P_{\alpha_2}P_{\beta_2}\frac{1}{Q\cdot Q^3}+32\frac{1}{D(D-3)}P_{\alpha_1}P_{\alpha_2}\delta_{\beta_1\beta_2}\frac{Q\cdot P^2}{Q\cdot Q^4}$$
$$+16\frac{1}{D(D-3)}\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}\frac{Q\cdot P^4}{Q\cdot Q^5}\ . \tag{C.29}$$

For the polarized projection operators the situation is slightly more complicated. In principle we could work with Π_4 and Π_6 but we notice that, if both projections are needed, it is easier to work with the linear combinations Π_D and Π_F. These are defined by

$$\Pi_4 = \frac{D+1}{D(D-2)(D-3)}\Pi_D + \frac{1}{D(D-2)(D-3)}\Pi_F\ ,$$

$$\Pi_6 = \frac{1}{D(D-2)(D-3)}(\Pi_D+\Pi_F)\ . \tag{C.30}$$

In any case we have a Levi-Civita tensor in the operator, and we contract this with the Levi-Civita tensor of the quark or the gluon. For the quark we obtain

$$\Pi_D^q = 4\frac{P_{\alpha_1}P_{\alpha_2}}{Q\cdot Q^3}R_{\beta_1\beta_2}^q\ ,\qquad \Pi_F^q = 4\frac{\delta_{\alpha_1\alpha_2}Q\cdot P^2}{Q\cdot Q^4}R_{\beta_1\beta_2}^q \tag{C.31}$$

with

$$R_{\beta_1\beta_2}^q = \gamma^\mu\gamma^\nu P_\mu Q_\nu(\gamma_{\beta_1}P_{\beta_2}-\gamma_{\beta_2}P_{\beta_1})+\gamma^\mu P_\mu(\gamma_{\beta_1}\gamma_{\beta_2}-\delta_{\beta_1\beta_2})Q\cdot P$$
$$+\gamma^\mu P_\mu(P_{\beta_1}Q_{\beta_2}-Q_{\beta_1}P_{\beta_2})\ , \tag{C.32}$$

and for the gluon we find

$$\Pi_D^g = 4\frac{P_{\alpha_1}P_{\alpha_2}}{Q\cdot Q^3}R_{\beta_1\beta_2}^g\ ,\qquad \Pi_F^g = 4\frac{\delta_{\alpha_1\alpha_2}Q\cdot P^2}{Q\cdot Q^4}R_{\beta_1\beta_2}^g \tag{C.33}$$

with

$$R_{\beta_1\beta_2}^g = (\delta_{\kappa\beta_2}\delta_{\lambda\beta_1}-\delta_{\kappa\beta_1}\delta_{\lambda\beta_2})\frac{Q\cdot P^2}{Q\cdot Q}+(\delta_{\kappa\beta_2}P_{\beta_1}\overline{P}_\lambda-\delta_{\kappa\beta_1}P_{\beta_2}\overline{P}_\lambda)$$
$$-(\delta_{\lambda\beta_2}P_{\beta_1}\overline{P}_\kappa-\delta_{\lambda\beta_1}P_{\beta_2}\overline{P}_\kappa)\ ,$$

$$\overline{P}_\kappa = P_\kappa-Q_\kappa\frac{Q\cdot P}{Q\cdot Q}\ , \tag{C.34}$$

where we have again used the symmetry in the graviton indices to simplify the expressions.

In the polarized case we do not need a ghost contribution, neither for the graviton nor for the gluon. Propagators for the graviton and the corresponding ghost are not required since we do not consider internal gravitons.

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