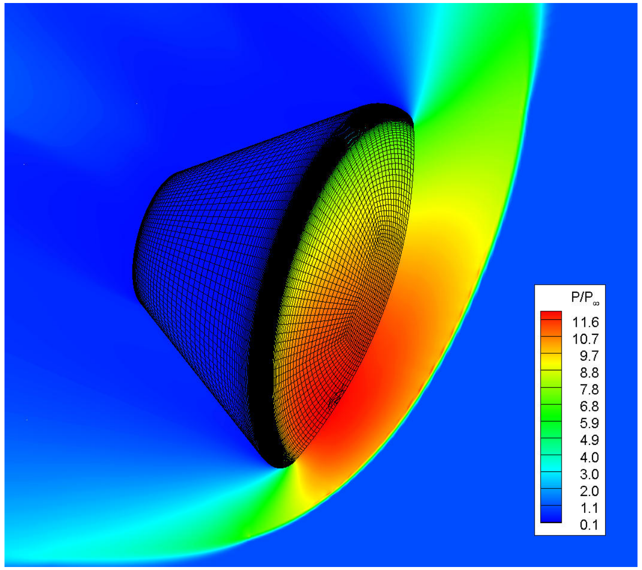
Assessment of Implicit Implementation of the AUSM$^+$ Method and the SST Model for Viscous High Speed Flow



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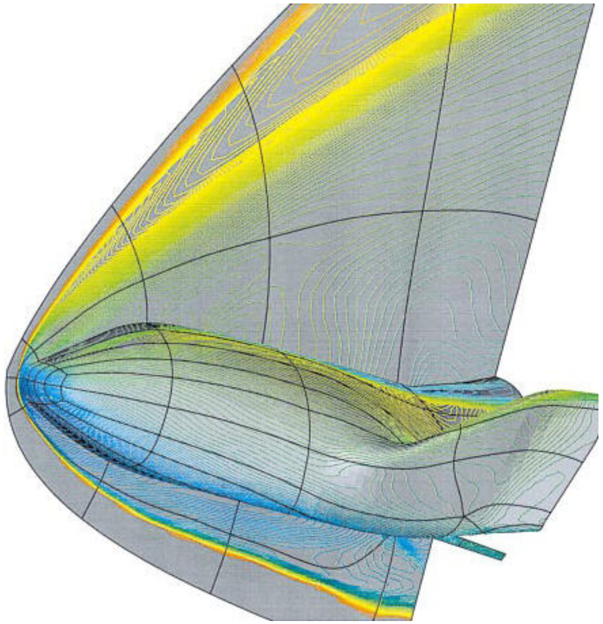
Introduction

- Aerodynamic and thermal environment predictions for high speed vehicles are essential in design and development
- CFD methods have gained significant prominence in recent years



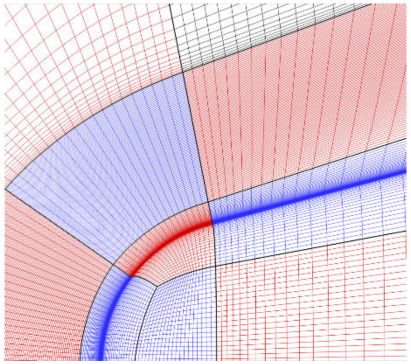
Muylaert J. and W. Berry, 1998. ESA bulletin 96

- A number of challenges remains, including:

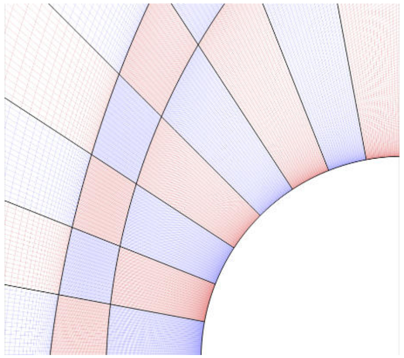
1. devising accurate and robust numerical schemes for the convective flux computation of Navier-Stokes solvers
2. turbulent flow modelling

Current work

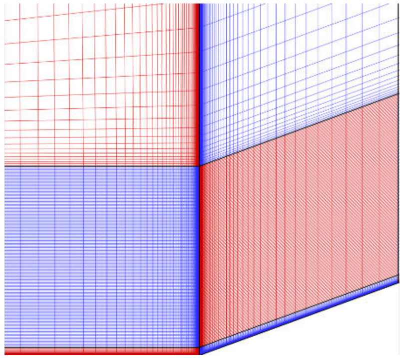
- An analytical Jacobian for the AUSM$^+$ scheme has been implemented into a fully implicit solver
- A description of the derivation procedure will be given along with a brief evaluation of the performance of the formulation
- As examples of aerospace interest:



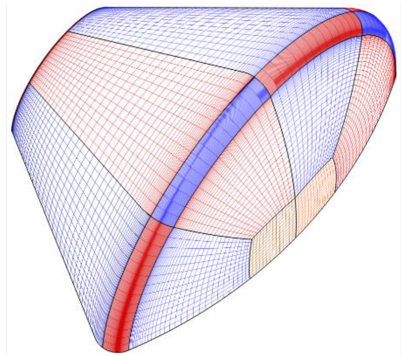
Single Cone



Infinite Cylinder



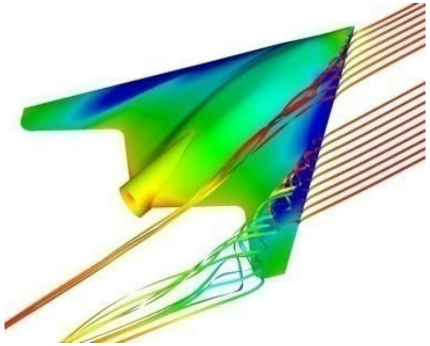
Ramp SWBLI



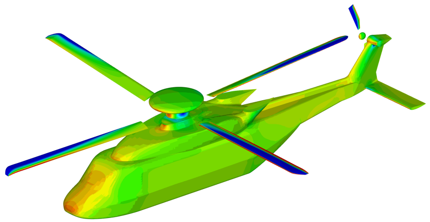
Orion CEV

- For the turbulent cases the SST turbulence model of Menter has been used

HMBv2



- Finite volume spatial discretisation
- Fully un-factored implicit time discretisation
- GCG/ILU(0) linear system solver
- Used successfully for a wide range of aerospace applications including low Mach, subsonic and transonic flows [1], [2]
- Generally employs the Roe or Osher schemes
- In the present work, the $AUSM^{+}$ scheme has been implemented for high Mach number flows



[1] Lawson, S.-J., R. Steijl, M. Woodgate, and G.N. Barakos. 2012. High Performance Computing for Challenging Problems in Computational Fluid Dynamics. *Progress in Aerospace Science*, 52:19–29
[2] Barakos, G.N., R. Steijl, A. Brocklehurst, and K. Badcock. 2005. Development of CFD Capability for Full Helicopter Engineering Analysis. *31st European Rotorcraft Forum*

The AUSM+ scheme

Introduced by M.-S. Liou ref. [3]:

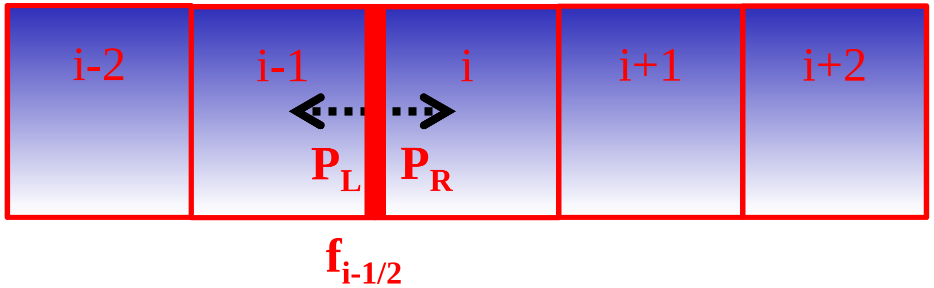
- Formulated to guarantee the enthalpy preservation and resolve contact discontinuities
- Capable of solving flow fields at a wide range of Mach numbers; perform reasonably well for high speed flows

Compared to the Roe scheme is shown to be less prone to shock anomalies [4], and in better agreement with experimental data [5]

[3] Liou, M.-S. 1996. A Sequel to AUSM: AUSM+. *Journal of Computational Physics*, 129:364–382.

[4] Kitamura, K., E. Shima, and P.L. Roe. 2012. Carbuncle Phenomena and Other Shock Anomalies in Three Dimensions. *AIAA Journal*, 50(12):2655–2669.

[5] Darracq, D., S. Campagneux, and A. Corjon. 1999. Computation of Unsteady Turbulent Airfoils Flows with an Aeroelastic AUSM+ Implicit Solver. *16th AIAA Applied Aerodynamics Conference*.



The inviscid flux is split into mass flow and pressure terms

$$\mathbf{f}_{i-1/2} = \dot{m}_{1/2}\mathbf{P} + \mathbf{p}_{1/2} \quad (1)$$

where with an up-winding approach

$$\mathbf{P} = \{\ \mathbf{P}_L \quad \text{if } \dot{m}_{1/2} > 0; \quad \mathbf{P}_R \quad \text{otherwise}\ \} = (1, u, H)^{\mathrm{T}} \quad (2)$$

and

$$\mathbf{p}_{1/2} = (0, p_{1/2}, 0)^{\mathrm{T}} \quad (3)$$

$$\dot{m}_{1/2} = a_{1/2} M_{1/2} \{\ \rho_L \quad \text{if } M_{1/2} > 0; \quad \rho_R \quad \text{otherwise}\ \} \quad (4)$$

Following the flux-splitting idea

$$M_{1/2} = M_{(4)}^{+}(M_L) + M_{(4)}^{-}(M_R) \tag{5}$$

$$p_{1/2} = P_{(5)}^{+}(M_L)p_L + P_{(5)}^{-}(M_R)p_R \tag{6}$$

the superscripts "+" and "-" are associated with the right and left waves and

$$M_L = \frac{u_{n,L}}{a_{1/2}} \qquad M_R = \frac{u_{n,R}}{a_{1/2}} \tag{7}$$

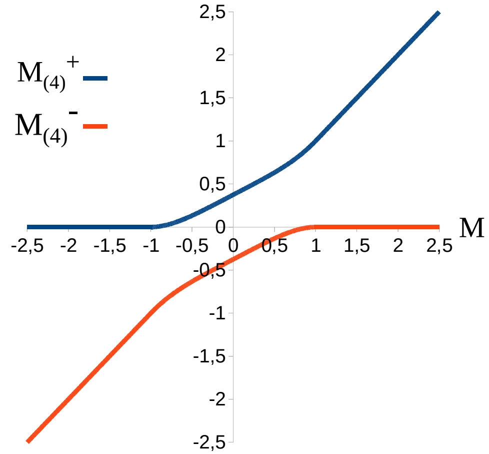
a common speed of sound has to be defined to evaluate the Mach numbers

$$a_{1/2} = min(\widehat{a}_L, \widehat{a}_R); \quad \widehat{a}_L = a_L^{*2}/max(a_L^{*}, u_{n,L}), \quad \widehat{a}_R = a_R^{*2}/max(a_R^{*}, -u_{n,R}) \tag{8}$$

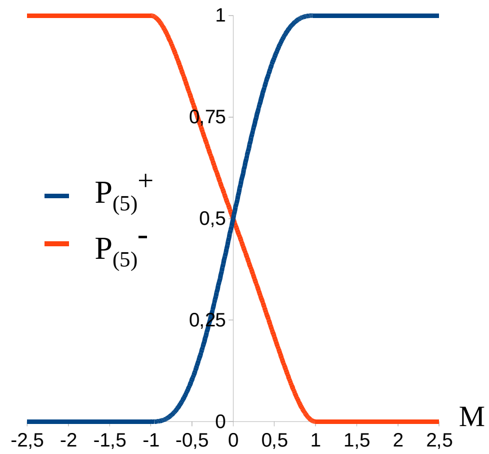
where

$$a_{L/R}^{*2} = \frac{2(\gamma - 1)}{\gamma + 1} H_{L/R} \tag{9}$$

The split formulas are given by the following polynomials



$$M_{(4)}^{\pm}(M) = \begin{cases} M_{(1)}^{\pm}(M) & \text{if } |M| \ge 1 \\ M_{(2)}^{\pm}(M)(1 \mp 16\beta M_{(2)}^{\mp}(M)) & \text{otherwise} \end{cases} \qquad (10)$$



$$P_{(5)}^{\pm}(M) = \begin{cases} \frac{1}{M} M_{(1)}^{\pm}(M) & \text{if } |M| \ge 1 \\ M_{(2)}^{\pm}(M)(\pm 2 - M \mp 16\alpha M M_{(2)}^{\mp}(M)) & \text{otherwise} \end{cases} \qquad (11)$$

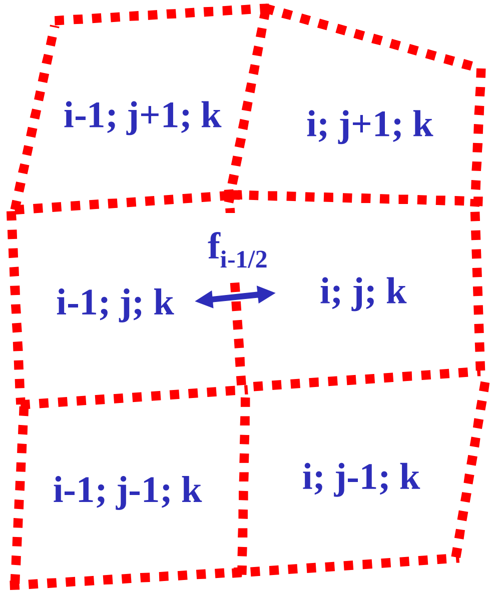
$$M_{(1)}^{\pm}(M) = \tfrac{1}{2}(M \pm |M|); \quad M_{(2)}^{\pm}(M) = \pm\tfrac{1}{4}(M \pm 1)^2 \qquad (12)$$

Fully un-factored implicit solver

In a fully un-factored approach the new time level is introduced simultaneously for all the cells

$$\frac{d}{dt}(\mathbf{W}_{i,j,k}V_{i,j,k}) = -\mathbf{R}_{i,j,k}(\mathbf{W}) \longrightarrow \left[\frac{V_{i,j,k}}{\Delta t^*}\frac{\partial \mathbf{W}_{i,j,k}}{\partial \mathbf{P}_{i,j,k}} + \frac{\partial \mathbf{R}_{i,j,k}}{\partial \mathbf{P}_{i,j,k}}\right]\Delta \mathbf{P}_{i,j,k} = -\mathbf{R}_{i,j,k}(\mathbf{W}^m)$$

The system of linear equation can be solved, by mean of a Krylov subspace method, as example the GCG method



The efficiency of Krylov-subspace methods depends strongly on the preconditioning which has the purpose to cluster the eigenvalues of the system matrix around unity $\longrightarrow$ ILU(0)

A Jacobian matrix for the AUSM+ scheme

Derivative of equation (1)

$$\mathbf{f}_{i-1/2} = \dot{m}_{1/2}\mathbf{P} + \mathbf{p}_{1/2} \quad (1)$$

$$\frac{\partial \mathbf{f}_{i-\frac{1}{2}}}{\partial \mathbf{P}_{L/R}} = \frac{\partial \dot{m}_{1/2}}{\partial \mathbf{P}_{L/R}}\mathbf{P} + \dot{m}_{1/2}\frac{\partial \mathbf{P}}{\partial \mathbf{P}_{L/R}} + \frac{\partial \mathbf{p}_{1/2}}{\partial \mathbf{P}_{L/R}}$$

From equations (3) and (4) it is clear that, if

$$\mathbf{p}_{1/2} = (0, p_{1/2}, 0) \quad (3)$$

$$\dot{m}_{1/2} = a_{1/2} M_{1/2} \begin{cases} \rho_L & \text{if } M_{1/2} > 0; \\ \rho_R & \text{otherwise} \end{cases} \quad (4)$$

$$\frac{\partial a_{1/2}}{\partial \mathbf{P}_{L/R}}; \quad \frac{\partial M_{1/2}}{\partial \mathbf{P}_{L/R}}; \text{ and } \quad \frac{\partial p_{1/2}}{\partial \mathbf{P}_{L/R}}$$

are known, the evaluation of the inviscid flux contributions to the full Jacobian matrix is straighforward

Derivative of the interface speed of sound

The interface speed of sound definition is critical in the AUSM$^+$

$$a_{1/2} = min(\widehat{a}_L, \widehat{a}_R); \quad \widehat{a}_L = a_L^{*2}/max(a_L^*, u_{n,L}), \quad \widehat{a}_R = a_R^{*2}/max(a_R^*, -u_{n,R}) \tag{8}$$

Differentiating expression (8), the presence of the min/max operators leads to a dual formulation when:

$$\widehat{a}_L = \widehat{a}_R$$

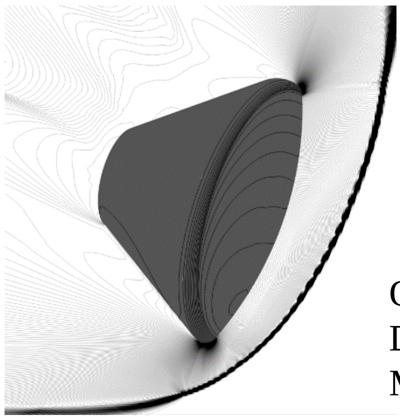
$$a^*_{L/R} = \pm u_{n,L/R}$$

Indeed, the interface speed of sound is a discountinuous function

At the points of discontinuity, we consider the interface speed of sound derivative as the average of the left and right limits

$$\frac{\partial a_{1/2}}{\partial \mathbf{P}_{L/R}} = \begin{cases} \text{for } \widehat{a}_L > \widehat{a}_R & \begin{cases} \dfrac{\partial a_R^*}{\partial \mathbf{P}_{L/R}} & \text{if } a_R^* > -u_{n,R} \\ \dfrac{\partial}{\partial \mathbf{P}_{L/R}} \dfrac{a_R^{*2}}{-u_{n,R}} & \text{if } a_R^* < -u_{n,R} \\ \dfrac{\partial}{\partial \mathbf{P}_{L/R}} \left[\dfrac{1}{2} \left(a_R^* + \dfrac{a_R^{*2}}{-u_{n,R}} \right) \right] & \text{if } a_R^* = -u_{n,R} \end{cases} \\ \text{for } \widehat{a}_L < \widehat{a}_R & \begin{cases} \dfrac{\partial a_L^*}{\partial \mathbf{P}_{L/R}} & \text{if } a_L^* > u_{n,L} \\ \dfrac{\partial}{\partial \mathbf{P}_{L/R}} \dfrac{a_L^{*2}}{u_{n,L}} & \text{if } a_L^* < u_{n,L} \\ \dfrac{\partial}{\partial \mathbf{P}_{L/R}} \left[\dfrac{1}{2} \left(a_L^* + \dfrac{a_L^{*2}}{u_{n,L}} \right) \right] & \text{if } a_L^* = u_{n,L} \end{cases} \end{cases}$$

$$\frac{\partial a_{1/2}}{\partial \mathbf{P}_{L/R}} = \longrightarrow \text{for } \widehat{a}_L = \widehat{a}_R \begin{cases} \frac{1}{2}\frac{\partial}{\partial \mathbf{P}_{L/R}}\left(a_L^* + a_R^*\right) & \text{if } a_L^* > u_{n,L} \ \& \ a_R^* > -u_{n,R} \\ \frac{1}{2}\frac{\partial}{\partial \mathbf{P}_{L/R}}\left(\frac{a_L^{*2}}{u_{n,L}} + \frac{a_R^{*2}}{-u_{nR}}\right) & \text{if } \frac{a_L^{*2}}{u_{n,L}} = \frac{a_R^{*2}}{-u_{n,R}} \\ \frac{1}{2}\frac{\partial}{\partial \mathbf{P}_{L/R}}\left(a_L^* + \frac{a_R^{*2}}{-u_{n,R}}\right) & \text{if } a_L^* = \frac{a_R^{*2}}{-u_{n,R}} \\ \frac{1}{2}\frac{\partial}{\partial \mathbf{P}_{L/R}}\left(\frac{a_L^{*2}}{u_{n,L}} + a_R^*\right) & \text{if } \frac{a_L^{*2}}{u_{L,n}} = a_R^* \end{cases}$$



Orion CEV
Density Countours
M=3 and α=160º

20° Ramp
Density Countours
M=3

Derivative of the interface Mach number

From equation (5)

$$\frac{\partial M_{1/2}}{\partial \mathbf{P}_{L/R}} = \frac{\partial M_{(4)}^{+}(M_L)}{\partial \mathbf{P}_{L/R}} + \frac{\partial M_{(4)}^{-}(M_R)}{\partial \mathbf{P}_{L/R}}$$

The derivative of the interface Mach number polynomials can be directly obtained from equation (10)

$$\frac{\partial M_{(4)}^{+}(M_L)}{\partial \mathbf{P}_{L/R}} = \begin{cases} \dfrac{\partial M_{(1)}^{+}(M_L)}{\partial \mathbf{P}_{L/R}} & \text{if } |M_L| \geq 1 \\ \dfrac{\partial M_{(2)}^{+}(M_L)}{\partial \mathbf{P}_{L/R}}(1 + 16\beta M_{(2)}^{-}(M_L)) - 16\beta M_{(2)}^{+}(M_L)\dfrac{\partial M_{(2)}^{-}(M_L)}{\partial \mathbf{P}_{L/R}} & \text{otherwise} \end{cases}$$

$$\frac{\partial M_{(4)}^{-}(M_R)}{\partial \mathbf{P}_{L/R}} = \begin{cases} \dfrac{\partial M_{(1)}^{-}(M_R)}{\partial \mathbf{P}_{L/R}} & \text{if } |M_R| \geq 1 \\ \dfrac{\partial M_{(2)}^{-}(M_R)}{\partial \mathbf{P}_{L/R}}(1 + 16\beta M_{(2)}^{+}(M_R)) + 16\beta M_{(2)}^{-}(M_R)\dfrac{\partial M_{(2)}^{+}(M_R)}{\partial \mathbf{P}_{L/R}} & \text{otherwise} \end{cases}$$

and (12)

$$\frac{\partial M^{\pm}_{(1)}(M_{L/R})}{\partial \mathbf{P}_{L/R}} = \pm\frac{1}{2}\left(\frac{\partial M_{L/R}}{\partial \mathbf{P}_{L/R}} \pm \frac{\partial |M_{L/R}|}{\partial \mathbf{P}_{L/R}}\right); \qquad \frac{\partial M^{\pm}_{(2)}(M_{L/R})}{\partial \mathbf{P}_{L/R}} = \pm\frac{1}{2}(M_{L/R} \pm 1)\frac{\partial M_{L/R}}{\partial \mathbf{P}_{L/R}}$$

where

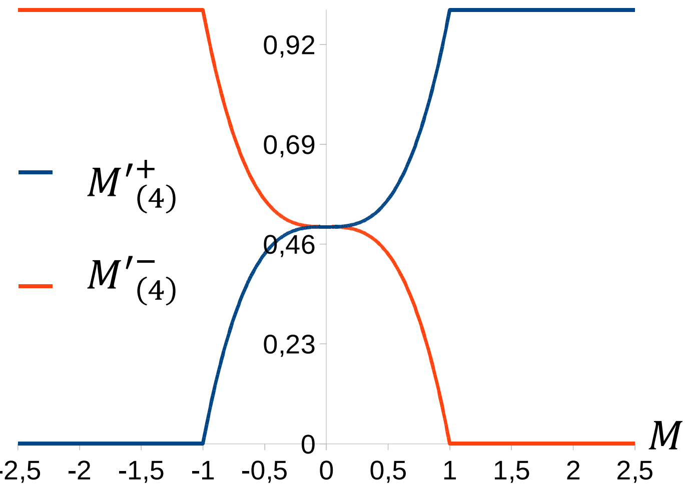
$$\frac{\partial M_L}{\partial \mathbf{P}_{L/R}}, \ \frac{\partial |M_L|}{\partial \mathbf{P}_{L/R}}, \ \frac{\partial M_R}{\partial \mathbf{P}_{L/R}} \text{ and } \frac{\partial |M_R|}{\partial \mathbf{P}_{L/R}}$$

can be easily obtained from equation (7) knowing the interface speed of sound and the cell-face normal velocity derivatives

Note:

$$\frac{\partial M^{\pm}_{(4)}}{\partial P_{L/R}} = M'^{\pm}_{(4)}(M)\frac{\partial M}{\partial P_{L/R}}$$

the interface Mach polynomials and the relative derivatives are continuous functions



Derivative of the pressure flux

From equation (6)

$$\frac{\partial p_{1/2}}{\partial \mathbf{P}_{L/R}} = \frac{\partial P_{(5)}^{+}(M_L)}{\partial \mathbf{P}_{L/R}} p_L + \frac{\partial P_{(5)}^{-}(M_R)}{\partial \mathbf{P}_{L/R}} p_R + P_{(5)}^{+}(M_L)\frac{\partial p_L}{\partial \mathbf{P}_{L/R}} + P_{(5)}^{-}(M_R)\frac{\partial p_R}{\partial \mathbf{P}_{L/R}}$$

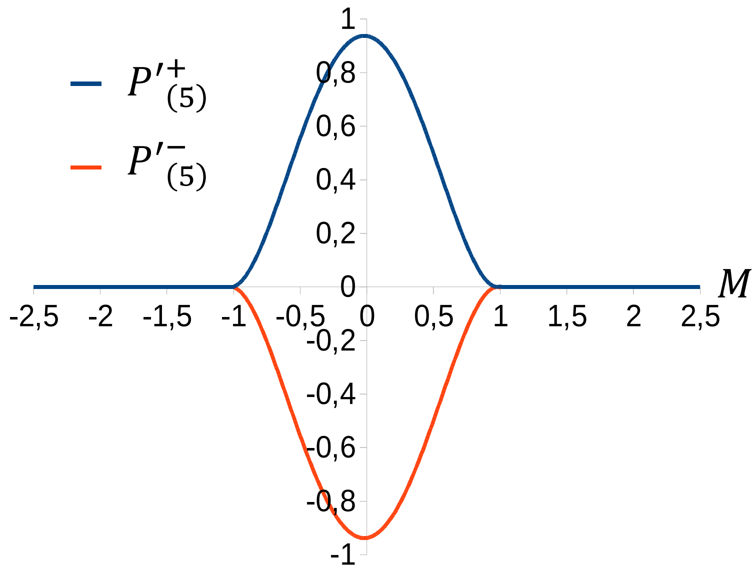
and eq. (12)

$$\frac{\partial P_{(5)}^{+}(M_L)}{\partial \mathbf{P}_{L/R}} = \begin{cases} \dfrac{1}{M_L}\dfrac{\partial M_{(1)}^{+}(M_L)}{\partial \mathbf{P}_{L/R}} - \dfrac{M_{(1)}^{+}(M_L)}{M^2_L}\dfrac{\partial M_L}{\partial \mathbf{P}_{L/R}} & \text{if } |M_L| \geq 1 \\ \dfrac{\partial M_{(2)}^{+}(M_L)}{\partial \mathbf{P}_{L/R}}(2 - M_L - 16\alpha M_L M_{(2)}^{-}(M_L)) - \\ M_{(2)}^{+}(M_L)\left(\dfrac{\partial M_L}{\partial \mathbf{P}_{L/R}} + 16\alpha(M_L\dfrac{\partial M_{(2)}^{-}(M_L)}{\partial \mathbf{P}_{L/R}} + M_{(2)}^{-}(M_L)\dfrac{\partial M_L}{\partial \mathbf{P}_{L/R}})\right) & \text{otherwise} \end{cases}$$

$$\frac{\partial P^{-}_{(5)}(M_R)}{\partial \mathbf{P}_{L/R}} = \begin{cases} \dfrac{1}{M_R}\dfrac{\partial M^{-}_{(1)}(M_R)}{\partial \mathbf{P}_{L/R}} - \dfrac{M^{-}_{(1)}(M_R)}{M^2_R}\dfrac{\partial M_R}{\partial \mathbf{P}_{L/R}} & if\ |M_R| \geq 1 \\ \dfrac{\partial M^{-}_{(2)}(M_R)}{\partial \mathbf{P}_{L/R}}(-2 - M_R + 16\alpha M_R M^{+}_{(2)}(M_R)) - & \\ \quad M^{-}_{(2)}(M_L)\left(\dfrac{\partial M_R}{\partial \mathbf{P}_{L/R}} - 16\alpha(M_R \dfrac{\partial M^{+}_{(2)}(M_R)}{\partial \mathbf{P}_{L/R}} + M^{+}_{(2)}(M_R)\dfrac{\partial M_R}{\partial \mathbf{P}_{L/R}})\right) & \text{otherwise} \end{cases}$$

Note:

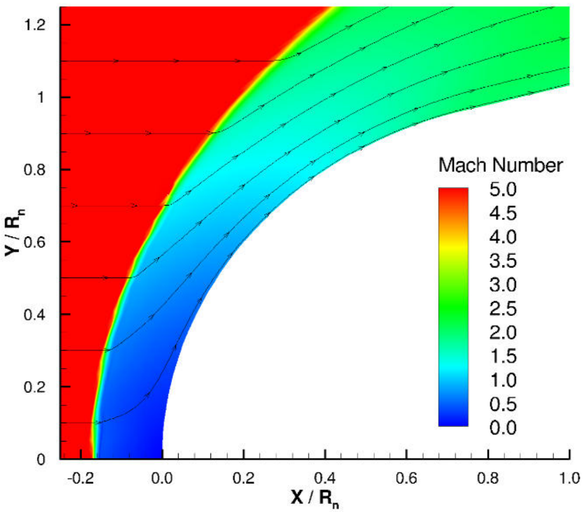
$$\frac{\partial P^{\pm}_{(5)}}{\partial P_{L/R}} = P'^{\pm}_{(5)}(M)\frac{\partial M}{\partial P_{L/R}}$$



also the pressure flux polynomials are continuous functions and so are their derivatives

Comparison of the AUSM+ and Roe schemes

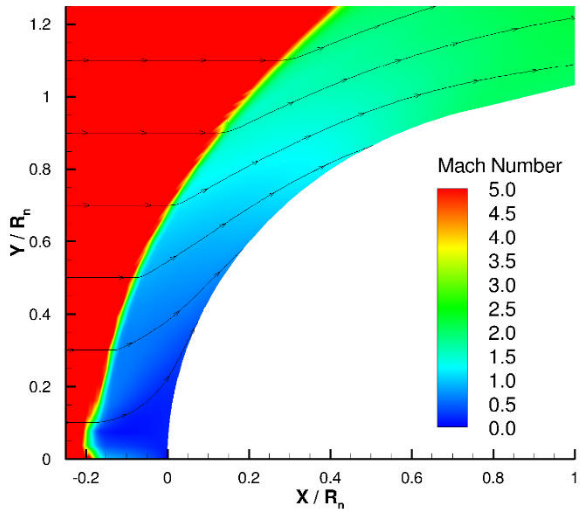
Test case: inviscid flow around a single cone with blunt nose



(a) AUSM +

Non-dimensional quantity	Method	Values
Stagnation	AUSM +	5, 439
point	Roe	5.716
ρ/ρ_∞	THEORY	5, 442
Stagnation	AUSM +	32.593
point	Roe	38.5
p/p_∞	THEORY	32.653
Standoff	AUSM +	0.152
distance	Roe	0.128
δ/R_n^a	[6]	0.163

$^a R_n$ is the nose curvature radius



(b) Roe

- AUSM+ showed the best agreement with the theory and the correlation results
- The shock predicted by the AUSM+ is less spurious unlike the Roe scheme, especially around the stagnation point

[6] Billing, F.S. 1967. Shock-Wave Shapes Around Spherical- and Cylindrical-Nosed Bodies. *Journal of Spacecraft and Rockets*, 4:822–834.

Performance of the implicit scheme

Test cases:

- Inviscid flow around a infinite cylinder
- Laminar (Re=10^5) flow around the ORION spacecraft

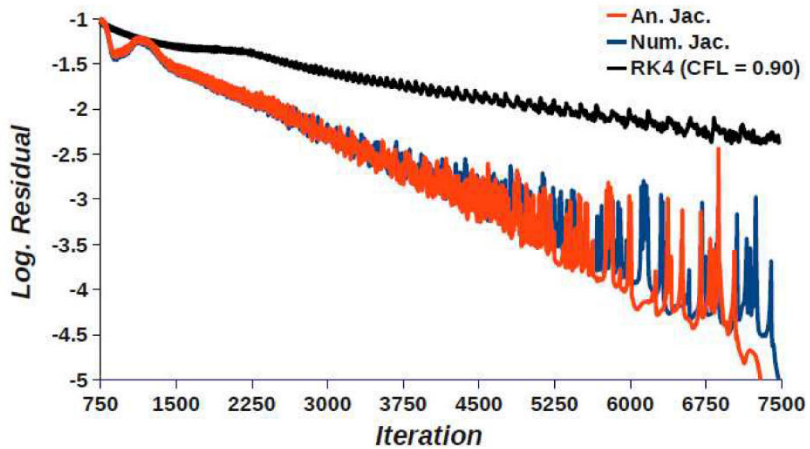
Note: the following norm-of-the-error has been used

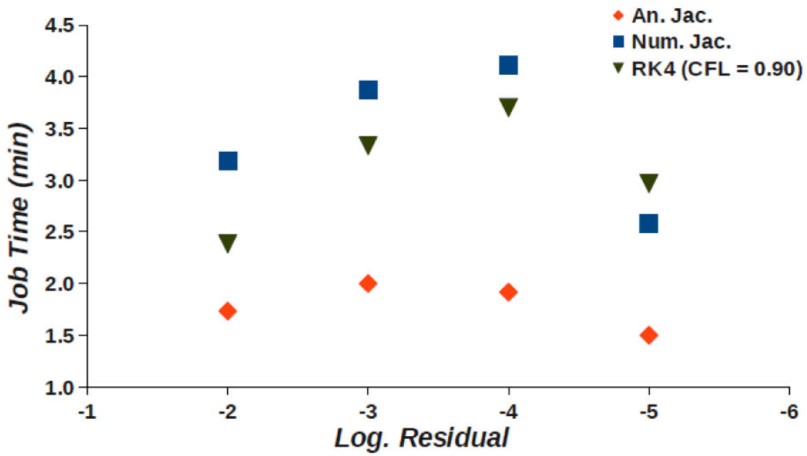
$$log\left(\frac{L_2(Res.\ t>0)}{L_2(Res.\ t=0)}\right)$$

- The analytical Jacobian leads to a solver that is about two times faster than with the numerical Jacobiab
- Compared to the 4-stage Runge Kutta it is 30% and 40% faster
- A comparison of the time needed to obtain a convergent solution with:

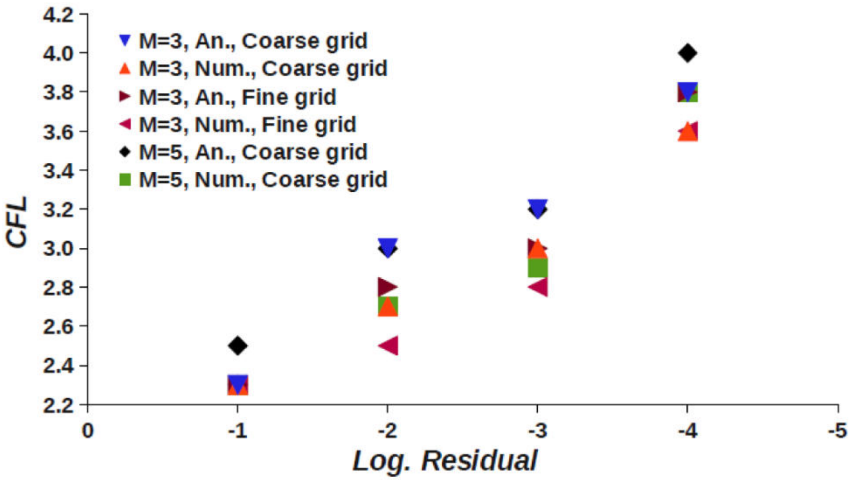
  RK4, CFL=0.9
  RK4 till −1 (-2) and then the implicit scheme, CFL=2.5 (3,0)

confirms that the implicit approach is 30 − 40% faster than the explicit one

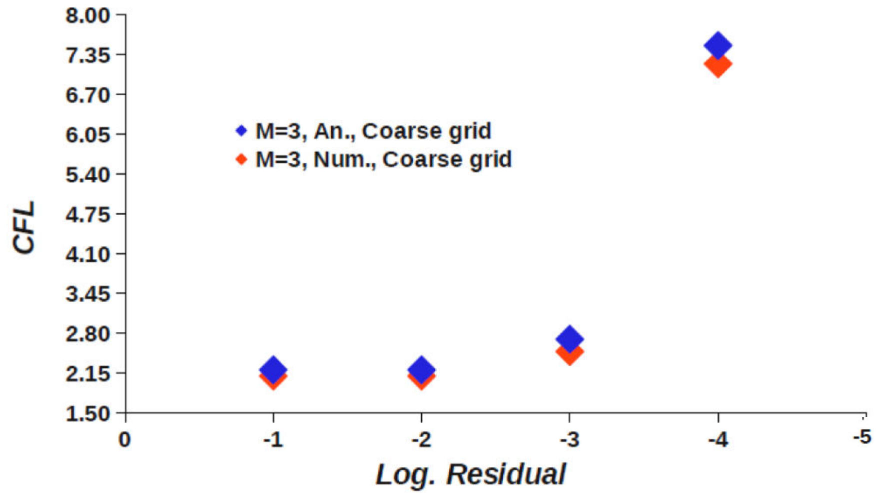




Infinite cylinder, inviscid flow.



(a) Infinite cylinder, inviscid flow.



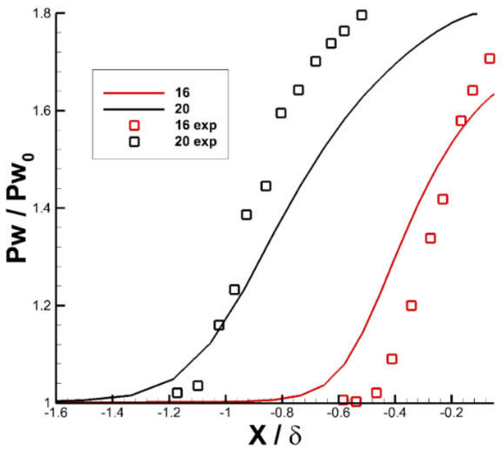
(b) Orion CEV, laminar flow (Re = 1 x 10 ^{5}).

- For both Mach 3 and 5, and grid refinements, the solver can run at CFL numbers equal and often higher than with the numerical Jacobian
- The implicit scheme allows to run at least CFL numbers around 2.5 also in presence of the strong shocks, expansions and interactions characterising the flow field around the Orion.

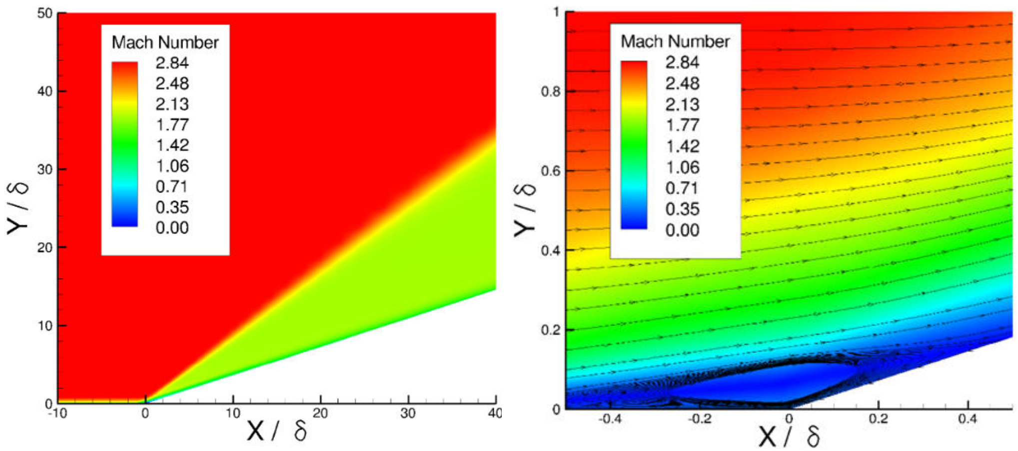
Shock-wave/ turbulent boundary-layer interaction test case

In [13] shock-wave/boundary-layer interactions, generated using two-dimensional compression ramps, were studied experimentally. The characteristics of the incoming boundary layer were δ = 24 mm, M_∞ = 2.84, Re = 6.5 x 10 7 m^{-1}

The numerical solutions fit reasonably the experimental data. Indeed, the SST model and the AUSM + scheme are able to capture the recirculation zones with a reasonable level of reliability.



(a) Pressure curves for θ = 16 ^{o} and θ = 20 ^{o} ramp angles.



(b) Mach contours at θ = 20 ^{o}.

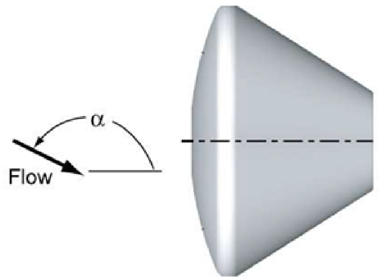
[13] Muck, K.-C., J. Andreopoulos, and J.-P. Dussauge. 1988. Unsteady Nature of Shock-Wave/Turbulent Boundary- Layer Interaction. AIAA Journal, 26(2):179–187.

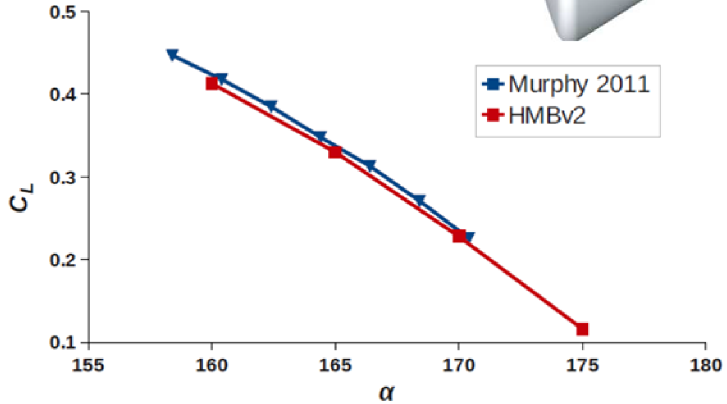
Orion CEV aerodynamic testing

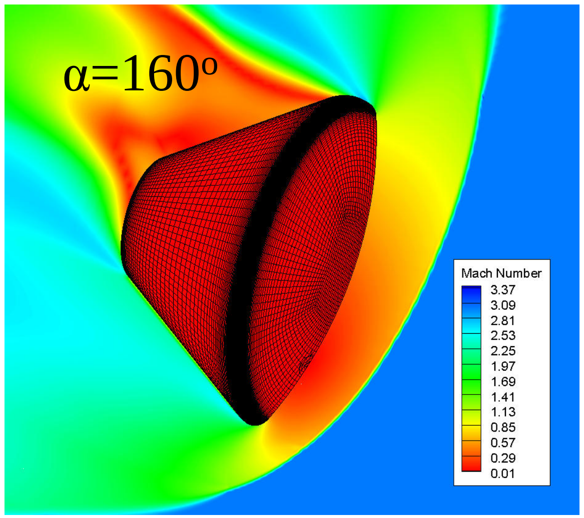
The aerodynamic coefficients of the Orion are compared to the experimental results collected in [14]

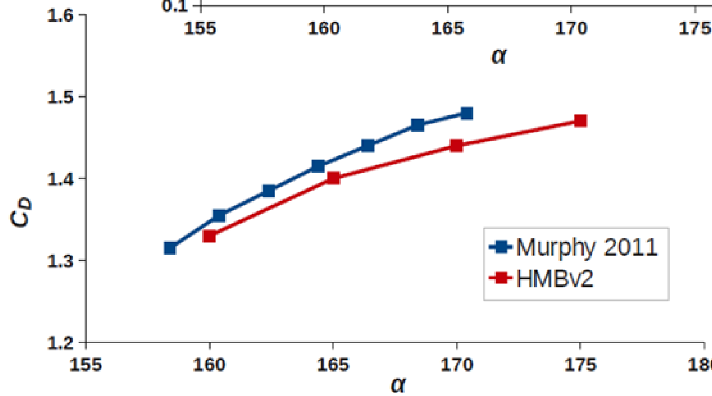
The test case at Mach 3 and Reynolds 1.5 x 10^6 has been considered and solved numerically

Differences between numerical and experimental results are less than 3%









[14] Murphy, K.J., et al. 2011. Orion Crew Module Aerodynamic Testing. In: 29th AIAA Applied Aerodynamics Conference.

Conclusions and Future Works

- The derivation of a fully analytical Jacobian for the AUSM$^+$ has been presented
- The implicit scheme with the analytical Jacobian has been tested resulting to be faster than the same implicit scheme with a numerically approximate Jacobian and the explicit 4-stage Runge-Kutta method
- The results, compared to experimental data, showed that the SST model and the AUSM$^+$ can cope with a wide range of high speed test cases with a good level of reliability

- Additional improvements are still possible and further investigations will be conducted to evaluate possible simplifications that can be made to the analytical Jacobian
- The AUSM$^+$-family fluxes will represent the basis for the continuum part in a hybrid continuum/kinetic Boltzmann method for partially rarefied flow

Thank you