Zigzags

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Look at the four diagrams in the figure. Each of them is produced by the same procedure, which we describe shortly. In the upper left is a family of lines (360 of them, in fact) which has a very

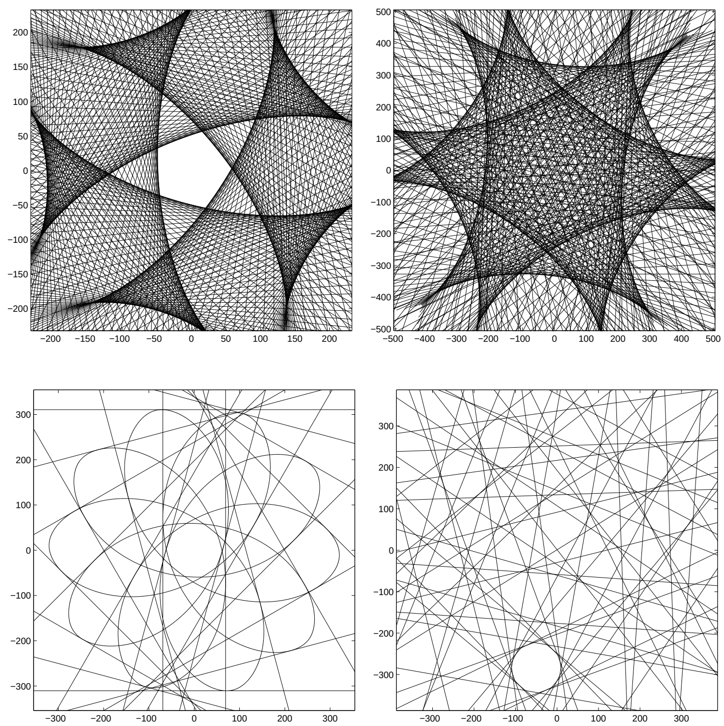

Figure 1: Some families of lines created by the zigzag construction, and, lower left, one incorrect attempt to draw their envelope.

clearly defined *envelope*, that is a curve tangent to all of the lines. Though not explicitly drawn in the figure, the envelope, which has 10 cusps or sharp points and 10 self-crossings, is immediately evident to the eye. In the upper right the lines produce not one connected envelope but three, each one of which has four cusps and no self-crossings though the components cross each other. The lower left figure is rather curious: some lines are drawn and also a very loopy curve which, you can verify, is tangent to all of the lines. However it's a very poor excuse for an envelope of the lines—this should evidently have four cusps and two crossings. Finally in the lower right the lines appear to have a number of circles for their envelope—how many do you see?

In this article I shall describe the way in which these finite sets of lines are generated: the *zigzag construction*. It is very striking that the lines often form visually evident envelopes; indeed that is what prompted this investigation in the first place—the challenge is to find a curve, or several curves, which form precisely this visually evident envelope of the lines. As the lower left example in the figure illustrates, an arbitrary curve tangent to all the lines may well be 'wrong'.

If we are given a family of lines, say $a(t)x+b(t)y = c(t)$, parametrized by a continuous parameter t, then there is a standard method for finding the envelope curve: solve for x and y between this

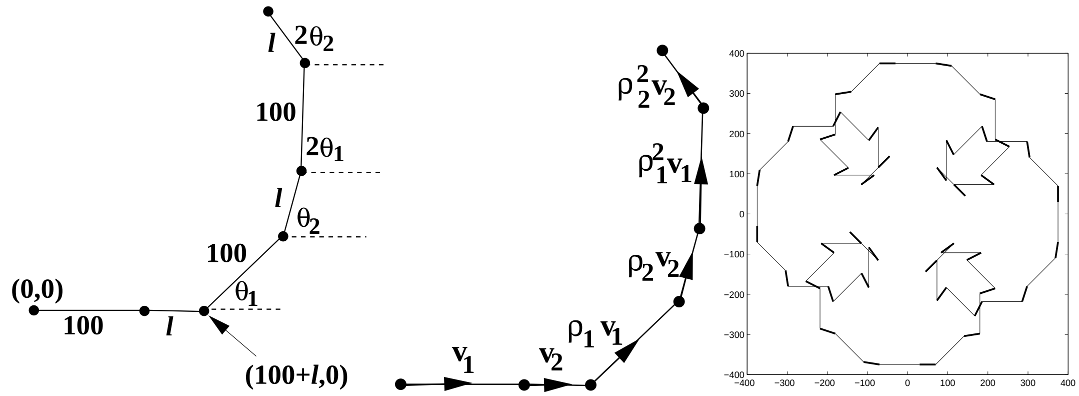

Figure 2: The basic zigzag, defined by lengths of 100 and l, and angles θ_1, θ_2. In the middle figure, ρ_1 and ρ_2 are rotations through θ_1 and θ_2 respectively. Right: a simple completed zigzag with $l = 40,\ \theta_1 = 45°, \theta_2 = 9°$, with the zags drawn heavily. Note that the origin here has been moved to the 'center' of the zigzag (see §1.1). In all the remaining figures in this article, *only* the zags are drawn, and they are extended right across the viewing area.

equation and its derivative with respect to t, namely $a'(t)x + b'(t)y = c'(t)$. (See for example [2, p.57].) But a *finite* family of lines such as those being considered here will have a very wide choice for an envelope curve tangent to all of them—how do we choose the 'right' one? I shall present one method which works quite often—the *whirligig construction*—but I do not know the complete answer.

Two Java applets demonstrating these constructions are at
http://www.liv.ac.uk/~tobyhall/Zigzag/
for the background leading up to the topic of this article, and
http://www.liv.ac.uk/~pjgiblin/Zigzag/
for the specific envelopes considered here.

In §1 I shall describe the zizgag construction and in §2 the whirligig construction. In §3 and §5 I give two ways of marrying the two. In between, in §4, there are several more examples. In §6 there is a discussion of the special case (such as Figure 1, lower right) where the envelope consists of a number of circles.

1 Zigzags

The basic idea of a zigzag is illustrated in Figure 2; the original idea comes from [1, p.114]. A straight horizontal line—the zeroth *zig*—is drawn to the right from the origin, of length 100. At the end of this another straight horizontal line—the zeroth *zag*—is drawn, of length l. If $l < 0$ then the line is drawn to the left and otherwise to the right; in either case it terminates at $(100 + l, 0)$. The vector $(100, 0)$ is denoted $\mathbf{v}_1$ and $(l, 0)$ by $\mathbf{v}_2$ in the figure. At this stage we say that *step zero*—a zig and a zag—has been completed. So far there is not much zigzagging in evidence.

But now the true zigzagging begins. We have two angles θ_1, θ_2 given to us (usually they will be whole numbers of *degrees*). We draw a straight line—the first zig—of length 100 from $(100 + l, 0)$, at an angle θ_1 with the positive x-axis (so this angle is measured anticlockwise from this axis). The termination of this line is at $(100 + l + 100 \cos \theta_1, 100 \sin \theta_1)$. From this point we draw a line—the first zag—of length l at an angle θ_2 with the horizontal, thereby arriving at the point

$$(100 + l + 100 \cos \theta_1 + l \cos \theta_2, 100 \sin \theta_1 + l \sin \theta_2).$$

At this stage, step one has been completed. In Figure 2, ρ_1, ρ_2 are counterclockwise rotations through θ_1, θ_2 respectively.

The lengths of the added lines are always alternately 100 and l. However, the *angles* between the added lines and the horizontal go up by θ_1 and θ_2 respectively at every step. Thus step two consists of drawing two lines at angles of $2\theta_1, 2\theta_2$ to the horizontal, step three of drawing two lines at angles of $3\theta_1, 3\theta_2$ to the horizontal, etc.

There is some resemblance between the above construction and that in Maurer in [4], but we use a pair of angles and he uses one angle.

The figures in this article are examples of sets of *zags only*, which are extended across the page to give the envelopes a chance to form. Of course one could also consider the zigs alone and obtain analogous pictures and results. Note that in all the figures, the origin has been translated to the 'center' of the zigzag; see §1.1 for details.

We shall need the equation of the j^{th} zag where $j = 0$ means the original horizontal zag of length l. Thus the oriented zag under consideration makes an angle of $j\theta_2$ with the horizontal drawn to the right. The equation is as follows, where the axes have been translated parallel to themselves to pass through the 'center' $\mathbf{c}$ of the zigzag; see §1.1 below for a derivation. For a fuller account, see [2, Ch.11].

$$x \sin j\theta_2 - y \cos j\theta_2 = \frac{50}{\sin \frac{1}{2}\theta_1} \cos(j(\theta_1 - \theta_2) + \tfrac{1}{2}\theta_1) + \tfrac{1}{2} l \cot \tfrac{1}{2}\theta_2. \tag{1}$$

As well as considering all the zags for given θ_1, θ_2 and l it is interesting to select just a subset by starting from $j = k_0$ and increasing j by $\delta > 0$ at each step, that is to consider only the j^{th} zags for $j = k_0 + n\delta,\ n = 0, 1, 2, \ldots$ in (1). In practice we shall usually take $k_0 = 0$, taking 'every δ^{th} zag'. See for example Figure 3 where picking out every eighth zag (b) or every fifth zag (c) give very different results. Of course, if there are for example 100 zags, then drawing *every* zag ($\delta = 1$) is going to produce the same effect as drawing every *third* zag ($\delta = 3$), though the way in which these zags 'step round' the envelope curve may well be different. We shall expand on the latter idea in §3. For the present here is a formula ([2, p.132]) for the total number $s(\delta)$ of zags which occur before the figure closes and repeats. It is assumed that θ_1, θ_2 are fixed in advance so we do not include them in the notation for s.

$$s(\delta) = \frac{360}{(360, \delta\theta_1, \delta\theta_2)}, \tag{2}$$

the round brackets denoting greatest common divisor. Two values of δ (both with $k_0 = 0$) will give the same set of zags precisely when they give the same number of zags (we are dealing here essentially with an additive cyclic group of order $s(1)$, generated by 1, and the subgroup generated by δ).

1.1 Derivation of the equation of a zag

For the time being the origin remains at the beginning of the zigzag. Then define $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}$ by the equations

$$\mathbf{v}_1 = \mathbf{c}_1 - \rho_1 \mathbf{c}_1, \quad \mathbf{v}_2 = \mathbf{c}_2 - \rho_2 \mathbf{c}_2, \quad \mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2.$$

Operating on the first of these equations by $\rho, \rho^2, \ldots$ and adding, we get

$$\mathbf{v}_1 + \rho_1 \mathbf{v}_1 + \rho_1^2 \mathbf{v}_1 + \ldots \rho_1^j \mathbf{v}_1 = \mathbf{c}_1 - \rho_1^{j+1} \mathbf{c}_1,$$

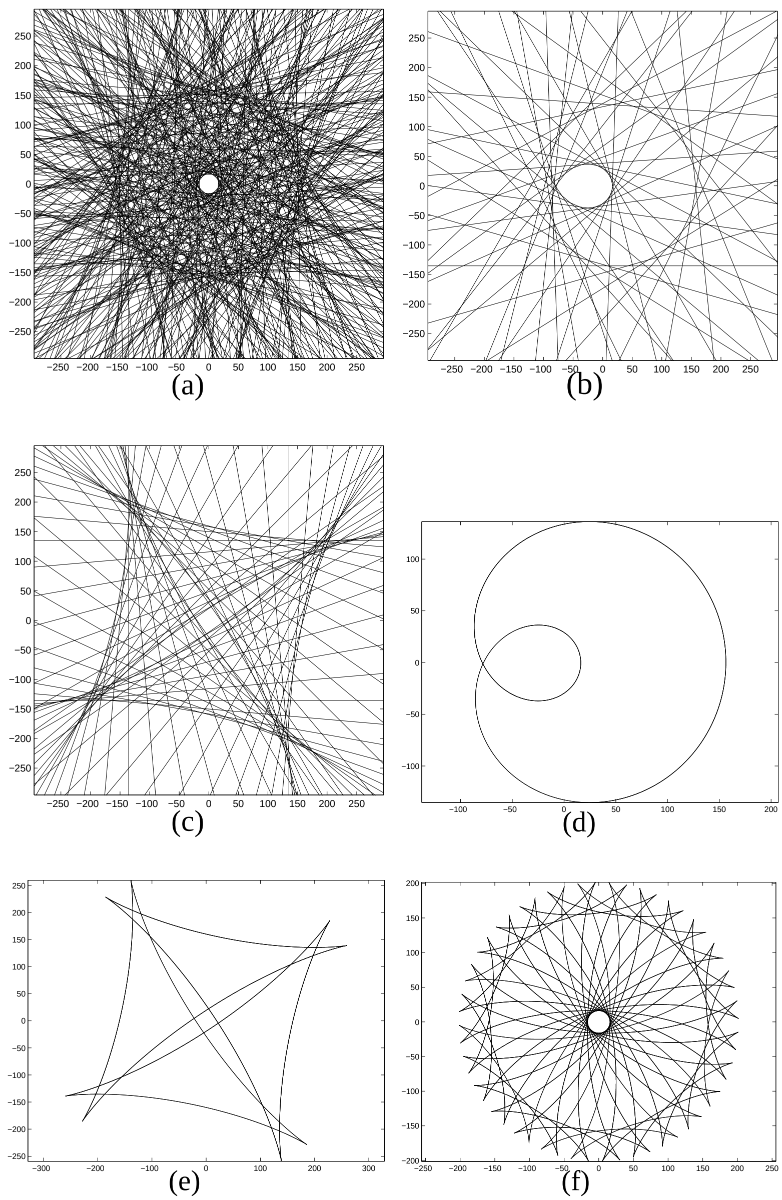

Figure 3: $l = 75$, $\theta_1 = 91°$, $\theta_2 = 47°$. (a) The whole set of 360 zags, making a mess. (b) With $\delta = 8, k_0 = 0$ we pick out one of eight envelope components, with $360/8 = 45$ zags tangent to it. The other seven components are given by $k_0 = 1, \ldots, 7$. (c) With $\delta = 5, k_0 = 0$ we pick out one of five components, with $360/5 = 72$ zags tangent to it. (d) The envelope of (b), generated by a continuous family of lines. (e) The envelope of (c), generated by a continuous family of lines. The whirligig curves in these two examples are determined by the method of §3. (f) A whirligig which is tangent to *all* the zags, that is to all the lines in (a), but which is hardly a visually striking envelope!

with a similar equation having suffix 2 throughout.

Now the point which the zigzag reaches after j steps—the end of the j^{th} zag—is clearly (see Figure 2)

$$\mathbf{v}_1 + \mathbf{v}_2 + \rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2 + \ldots \rho_1^j\mathbf{v}_1 + \rho_2^j\mathbf{v}_2,$$

where as before $j = 0$ means the starting zig and zag both of which are horizontal. Using the above formulae this equals

$$\mathbf{c}_1 + \mathbf{c}_2 - \rho_1^{j+1}\mathbf{c}_1 - \rho_2^{j+1}\mathbf{c}_2.$$

Translating the origin to the point $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$, called the *center* of the zigzag, we can drop the first two terms. Now evaluating $\mathbf{c}_1$ as $(I - \rho_1)^{-1}\mathbf{v}_1$ (where I is the identity), and similarly for ρ_2 we find quite quickly that the two ends of the j^{th} zag are

$$\left(\frac{100}{2\sin\frac{1}{2}\theta_1}\sin\left(j+\tfrac{1}{2}\right)\theta_1 \quad + \quad \frac{l}{2\sin\frac{1}{2}\theta_2}\sin\left(j\pm\tfrac{1}{2}\right)\theta_2,\right.$$
$$\left.-\frac{100}{2\sin\frac{1}{2}\theta_1}\cos\left(j+\tfrac{1}{2}\right)\theta_1 \quad - \quad \frac{l}{2\sin\frac{1}{2}\theta_2}\cos\left(j\pm\tfrac{1}{2}\right)\theta_2\right),$$

where the lower sign is the beginning of the zag and the upper sign is the end.

We can now check that the line (1) has the correct slope $j\theta_2$ and passes through one of the above points (or alternatively passes through both points), and is hence the line along the j^{th} zag. This completes the proof that (1) gives the equation of this zag relative to axes parallel to the original axes but translated to the center $\mathbf{c}$ of the zigzag.

2 Whirligigs

The most general kind of envelope of a continuous family of lines with which we shall compare the zag-envelope is a *whirligig*, defined as follows. Consider a circle, radius R centered at the origin (that is the center $\mathbf{c}$ of the zigzag above). With center at a point on the circumference making an angle $\phi(t)$ with the downward vertical draw another circle, of radius r. Orienting this circle anticlockwise, consider the (oriented) tangent to this circle making an angle $\psi(t)$ with the positive x-axis. See Figure 4. We shall take ϕ, ψ to be *linear* functions of t, so that the speeds of rotation are constant:

$$\phi(t) = at + b, \quad \psi(t) = ct + d, \quad a, b, c, d \text{ constants.} \tag{3}$$

A brief analysis shows that the point of contact of the tangent line with the circle of radius r is

$$(R\sin\phi + r\sin\psi, \quad -R\cos\phi - r\cos\psi),$$

and that the equation of the tangent line is

$$x\sin\psi - y\cos\psi = R\cos(\phi - \psi) + r. \tag{4}$$

The whirligig determined by R, r, a, b, c and d is, then, the envelope of these tangent lines as the spinning circle of radius r moves round the circle of radius R. Whenever we draw a whirligig we shall draw simply the curve itself which is tangent to all these lines. Mathematically, this is obtained by solving the equation (4) and the derivative of (4) with respect to t for the variables x and y.

For the record, here is the resulting parametrization:

$$cx = Rc\sin\phi + rc\sin\psi - Ra\sin(\phi - \psi)\cos\psi, \quad cy = -Rc\cos\phi - rc\cos\psi - Ra\sin(\phi - \psi)\sin\psi.$$

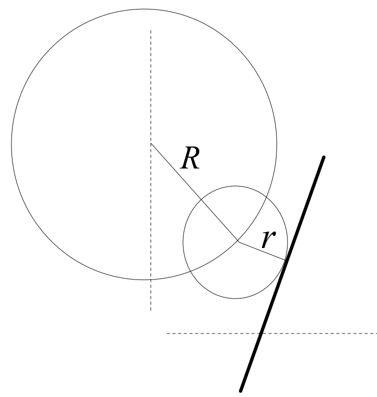

Figure 4: A *whirligig* is the envelope of lines tangent to a rotating circle of radius r whose center moves on a circle of radius R. The angles ϕ, ψ are then functions of 'time' t.

As a simple example, if $a = c \neq 0$ then the whirligig is a circle with center the origin. If $a = 0,\ c \neq 0$, it is a circle with center at some point of the circle radius R.

Remark It is worth noting that what is here called a 'whirligig' appears also in the literature as a 'line-roulette' or more specifically a 'line trochoid'; see for example [3, Ch.17]. The connexion is not immediate since it is usual to require a circle to *roll* on a fixed circle, the rolling circle carrying with it a point, giving a 'point-trochoid', or a line, giving as envelope a 'line-trochoid'. Note that it is not assumed that the moving point (resp. line) is on the circumference of the rolling circle (resp. tangent to the rolling circle).

In fact it is not hard to see that, in Figure 4, we can always find a circle concentric with our fixed circle and a circle concentric with the spinning circle which *do* roll on one another. Taking $a > 0$ there are three cases according as $c < 0,\ 0 < c < a$ or $c > a$ and the reader may enjoy finding the radii of the fixed and rolling circles when the rolling condition is imposed. For example, when $0 < c < a$ the radii are $(a - c)R/c$ and aR/c. Of course r now plays the rôle of telling us the location of the line rigidly attached to the rolling circle whose envelope produces the line-trochoid.

The purpose of introducing whirligigs here is to compare (4) with the equation of the j^{th} zag. If every zag is one of these lines then the envelope of the lines—that is, the whirligig—will be tangent to all the zags and so may serve as an 'envelope of the zags'. On the other hand the whirligig may turn out to be much more complicated than the visually evident 'envelope of the zags'; see Figure 1, lower left, for an example of a whirligig which is, to be sure, tangent to all the zags, but is visually wrong. The correct whirligig is the one shown in Figure 5, left.

Remark It is clear that the whirligigs in the figures often have cusps. Here is a formula for the number of cusps, which is left as a pleasant exercise for the reader:

$$\frac{2|a - c|}{(a, c)}.$$

We often take a and c relatively prime, so the number is then $2|a - c|$.

3 Zigzags and whirligigs

Firstly, a direct comparison between (1) and (4) shows that it makes good sense to take

$$R = \frac{50}{\sin\frac{1}{2}\theta_1}, \quad r = \tfrac{1}{2}l\cot\tfrac{1}{2}\theta_2, \tag{5}$$

and we shall always do this.

Let θ_1, θ_2 and δ be given integers. We consider the zags with $j = 0, \delta, 2\delta, \ldots$. Let

$$\delta\theta_1 \equiv k_1 \quad \text{and} \quad \delta\theta_2 \equiv k_2 \quad \text{mod} \quad 360. \tag{6}$$

We shall usually take the k_i to be the smallest positive residues mod 360, or the residues which are smallest in absolute value.

Proposition *Suppose that a and c are integers and that there exists τ with $a\tau$, $b\tau$ integers and*

$$a\tau \equiv k_1, \quad c\tau \equiv k_2 \quad \textit{mod } 360. \tag{7}$$

Then all the zags ($j = 0, \delta, 2\delta, \ldots$ as above) are lines of the form (4) with the above a, c and $b = \frac{1}{2}\theta_1$, $d = 0$; the zags are therefore tangent to the whirligig given by these values (and R, r as in (5) as usual).

Remark We can use the zags with $j = k_0, k_0 + \delta, k_0 + 2\delta, \ldots$ by adjusting the values of b and d to $(k_0 + \frac{1}{2})\theta_1$, $k_0\theta_2$ respectively.

Proof Take $\phi = at + \frac{1}{2}\theta_1$, $\psi = ct$, $t = n\tau$ in (4) where a, c, τ satisfy (7). Then the line (4) clearly coincides with the zag

$$x\sin(n\delta\theta_2) - y\cos(n\delta\theta_2) = R\cos(n\delta(\theta_1 - \theta_2) + \tfrac{1}{2}\theta_1) + r$$

for $n = 0, 1, 2, \ldots$.

We shall use this simple proposition to propose whirligigs as possible envelopes of zags. Experiment suggests that a more visually plausible result is obtained if a and c are reasonably small, but the total number of zags drawn (given by $s(\delta)$ as in (2)) is reasonably large. However, 'small' here must be taken with a grain of salt; e.g. Figure 7 shows an example where $a = 17, c = 8$ and the whirligig is clearly right.

An immediate solution to (7) is $a = k_1$, $c = k_2$, $\tau = 1$. Note that we can take out a common factor from a and c in (7) by multiplying τ by the same factor. So we can take out all common factors and assume that a and c are *relatively prime.* Thus

$$a = \frac{k_1}{(k_1, k_2)}, \quad c = \frac{k_2}{(k_1, k_2)}, \quad \tau = (a, c) \tag{8}$$

is a simpler solution. We shall consider other solutions in §5 below.

Before giving a number of examples, it is worth introducing the notion of 'stepping round' the whirligig. Suppose that we have found a and c which are 'correct', as in Figure 3(c) and (e). The family of lines in (c) can be generated by taking $\delta = 5$ or 15 or 30 or 85, or any other δ with $(360, 91\delta, 47\delta) = 5$, which here amounts to just δ being a multiple of 5. (To obtain $a = 5$ and $c = 1$ by the method of this section, we can take $\delta = 85$ or 115; see Example 1 of §4. We can in fact obtain $a = 5, c = 1$ from $\delta = 5$ by the method of §5.) If we take one of these values of δ and draw every δ^{th} zag, starting with the 0^{th} zag, then these will eventually fill up all of Figure 3(c) but will in general dance about over the whirligig curve in (e) rather than stepping along it with the points of contact covering the whirligig just once. When they *do* cover it just once we say that this value of δ makes the zags *step round* the whirligig.

Suppose that a and c, with $(a, c) = 1$, are determined by the method of this section, from a particular value of δ. For the whirligig, the small spinning circle turns $|c|$ times before returning to its starting place. On the other hand consider the zags given by $j = \delta n,\ n = 0, 1, 2, \ldots$. The number of zags before the whole zigzag repeats is given by (2), that is

$$\frac{360}{(360, \delta\theta_1, \delta\theta_2)} = \frac{360}{(360, k_1, k_2)}.$$

We shall take k_1, k_2 to be the least residues of $\delta\theta_1, \delta\theta_2$ mod 360, in the sense of absolute value. A negative value indicates that the selected zags (multiples of δ) turn clockwise instead of anticlockwise. The total number of turns of the zag before returning to the start is therefore the above number times k_2, divided by 360. If drawing every δ^{th} zag is to step round the whirligig defined as in the proposition, we require that $(k_1, k_2) = (360, k_1, k_2)$, which is the same as saying that (k_1, k_2) is a factor of 360:

The stepping round criterion for the method here is that (k_1, k_2) *divides exactly into* 360.

Examples are given in the next Section.

Stepping round is hard to demonstrate with a still picture, but the second Java applet mentioned in the Introduction allows a delay between the drawing of successive zags which makes the idea immediately attractive.

4 Examples

Example 1: $\theta_1 = 91, \theta_2 = 47$. Table 1 shows all the values of $a = k_1/(k_1, k_2), c = k_2/(k_1, k_2)$ which are both ≤ 10, for values of δ from 1 to 180. The number s is the number of steps (here the number of zags) in a complete cycle; as above, this equals $360/(360, k_1, k_2)$. Note that the outlandish $a = 17, c = -11$ of Figure 3(f) is not in the table because of the cutoff value of 10 for a and c.

The first entry in the table, $\delta = 8$, gives Figure 3(b), (d). The second entry, $\delta = 15$, has a and c which are merely the negatives of those in the more interesting entry $\delta = 115$, and the latter has three times as many zags tangent to it. It is shown in Figure 3(c),(e). The third entry, $\delta = 16$, is like $\delta = 8$, but without the feature that the zags step round the whirligig. (This feature is indicated in the table by a '1' in the column headed '?' and its absence by a '0'.) When $\delta = 18$, the number of zags (20) is so small and a and c are relatively large, so this is a complicated whirligig with the zags spaced very far apart (in terms of arclength) along it. From the zags one would never pick out this whirligig as their envelope. For another entry in the table, $\delta = 90$ gives $a = c = -1$, a circle, with just 4 zags tangent to it. Note that there are no values of $\delta \leq 180$ for which the number of steps s is > 72. Thus (at least for this range of δ) it is to be expected that the 72-zag whirligigs are not part of larger ones; in fact that there are $360/72 = 5$ of these which are obtained by choosing different starting points, that is different values of k_0. In the above, k_0 is always chosen to be 0.

For the remaining examples, we shall not give so much detail.

Example 2: $\theta_1 = 45°,\ \theta_2 = 15°$. (Take l=50.)

δ	k_1	k_2	a	c	?	s	Comment
1	45	15	3	1	1	24	correct visually: Figure 5, left
7	-45	105	-3	7	1	24	wrong visually: Figure 1, lower left

δ	k_1	k_2	a	c	?	s	δ	k_1	k_2	a	c	?	s
8	8	16	1	2	1	45	88	88	176	1	2	0	45
15	-75	-15	-5	-1	1	24	90	-90	-90	-1	-1	1	4
16	16	32	1	2	0	45	96	96	-168	4	-7	1	15
18	-162	126	-9	7	1	20	99	9	-27	1	-3	1	40
20	20	-140	1	-7	1	18	100	100	20	5	1	1	18
24	24	48	1	2	1	15	108	108	36	3	1	1	10
30	-150	-30	-5	-1	1	12	115	25	5	5	1	1	72
32	32	64	1	2	0	45	120	120	-120	1	-1	1	3
36	36	-108	1	-3	1	10	126	-54	162	-1	3	0	20
40	40	80	1	2	1	9	130	-50	-10	-5	-1	1	36
45	135	-45	3	-1	1	8	135	45	-135	1	-3	1	8
48	48	96	1	2	0	15	138	-42	6	-7	1	1	60
54	-126	18	-7	1	1	20	140	140	100	7	5	1	18
56	56	112	1	2	0	45	144	144	-72	2	-1	1	5
60	60	-60	1	-1	1	6	145	-125	-25	-5	-1	0	72
63	-27	81	-1	3	0	40	150	-30	-150	-1	-5	1	12
64	64	128	1	2	0	45	160	160	-40	4	-1	1	9
72	72	144	1	2	1	5	162	-18	54	-1	3	1	20
75	-15	-75	-1	-5	1	24	168	168	-24	7	-1	1	15
80	80	160	1	2	0	9	170	-10	70	-1	7	1	36
84	84	-12	7	-1	1	30	180	180	180	1	1	1	2
85	175	35	5	1	0	72							

Table 1: $\theta_1 = 91$, $\theta_2 = 47$. This table shows, for δ up to 180, values of a and c, obtained by the method of §3 and both ≤ 10 in modulus, giving whirligigs which are tangent to all the s resulting zags. Low values of a, c and high values of s tend to give recognizable envelopes. The column headed '?' contains a 1 if the zags *step round* the whirligig and a 0 otherwise.

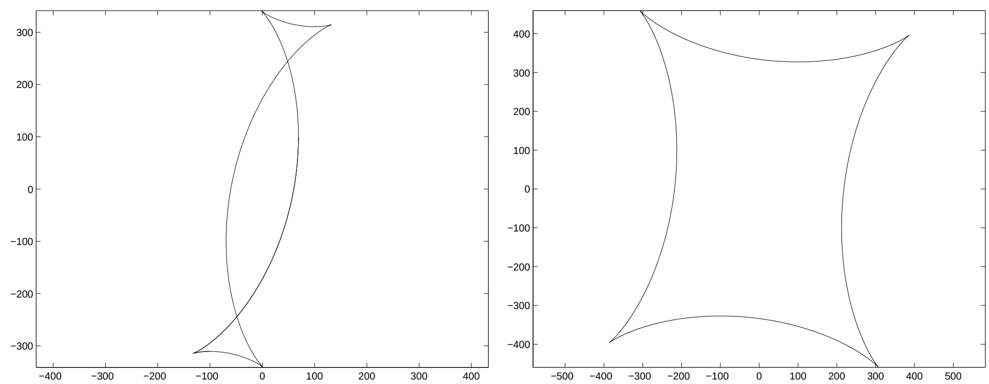

Figure 5: Left: the 'correct' envelope for the family of lines in Figure 1, lower left (see Example 2). Here $\theta_1 = 45°, \theta_2 = 15°, l = 50$. Right: one of the three 'correct' envelope components for the family in Figure 1, upper right (see Example 4). Here $\theta_1 = 21°, \theta_2 = 47°, l = 50$.

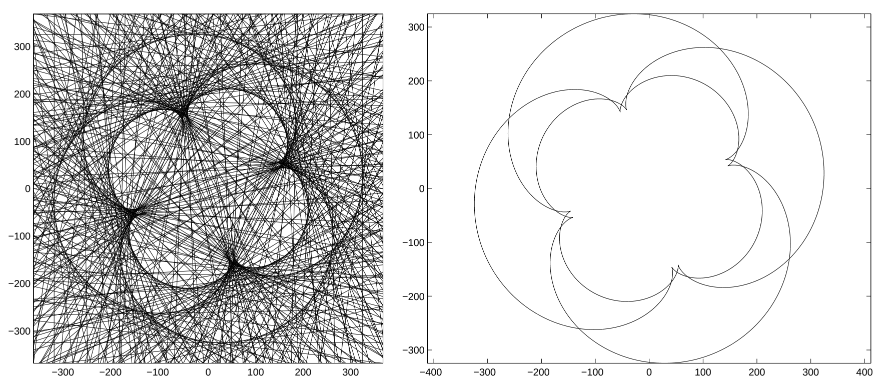

Figure 6: Left: $\theta_1 = 21°, \theta_2 = 49°, l = 50$. The envelope has one piece, though it is hard to be sure what it looks like. Right: the envelope produced as the envelope of a continuous family of lines. See Example 5.

Example 3: $\theta_1 = 77°,\ \theta_2 = 22°$. (Take $l = 50$.)

δ	k_1	k_2	a	c	?	s	Comment
131	7	2	7	2	1	360	See Figure 1, upper left
72	144	144	1	1	0	5	Regular pentagon and circle: $k0$ changes radius

Example 4: $\theta_1 = 21°,\ \theta_2 = 47°$ (take $l = 50$). Here $\delta = 1$ gives all of Figure 1, upper right, and $\delta = 3$ gives just one-third of the zags, which are tangent to one of the three curves visible in this figure. The value $\delta = 39$ gives the same zags as $\delta = 3$ but the method of §3 shows that $a = 3,\ c = 1$ provides an appropriate whirligig as in Figure 5, right. To get the other components with $\delta = 39$ we take $k_0 = 1, 2$ and adjust b and d as in the Remark in §3. The value $\delta = 69$ gives the same zags again as $\delta = 3$ but in addition steps round the figure.

Example 5: $\theta_1 = 21°, \theta_2 = 49°$ (take $l = 50$). Here $\delta = 1$ gives $a = 3, c = 7$ by the method of §3. See Figure 6. In order to step round the envelope we can take $\delta = 103$.

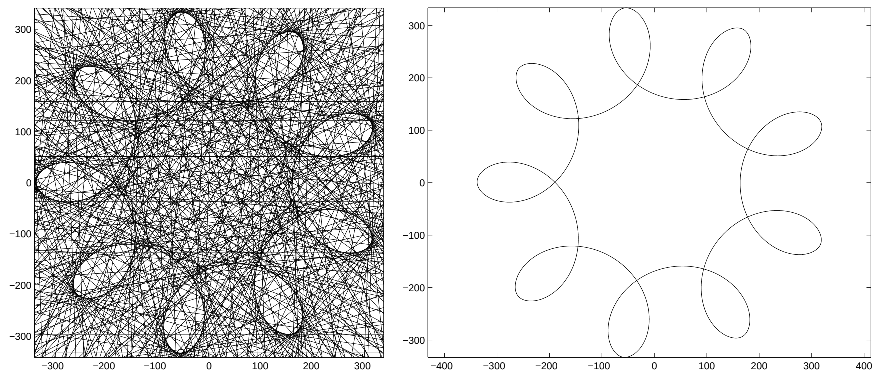

Figure 7: The case $\theta_1 = 23°, \theta_2 = 32°, l = 50$ where we need $a = 17, c = 8$ to generate the envelope (right). See Example 6.

Example 6: $\theta_1 = 23°, \theta_2 = 32°$ (take $l = 50$). Here, as in Figure 7, we need to take the larger values $a = 17,\ c = 8$, which are given by $\delta = 79$ using the method of §3. This value of δ also steps round the resulting whirligig.

5 An alternative method

The formula (8) is not the only solution to the equations (7) for finding a whirligig which is tangent to all the zags. For example, with $\theta_1 = 91°, \theta_2 = 47°, \delta = 5$ it misses the good solution $a = 5,\ c = 1$ which gives the same picture as Figure 3(c),(e).

Here is a sketch of another possible method. To simplify notation we rewrite (8) as

$$a\tau \equiv u, \quad c\tau \equiv v \mod w. \tag{9}$$

We shall assume in what follows that τ is an *integer* and that $(a, c) = 1$. We seek to *minimise the value of* a. Let $(\tau, w) = h$; then (9) implies $h|u,\ h|v$. Write $u = u_1 h, v = v_1 h, w = w_1 h, \tau = \tau_1 h$ so that $(\tau, w_1) = 1$ and (9) can be replaced by

$$a\tau_1 \equiv u_1, \quad c\tau_1 \equiv v_1 \mod w_1.$$

It now follows, using $(a, c) = 1$, that u_1, v_1, w_1 cannot all have a common factor so that in fact $h = (u, v, w)$.

Since $(\tau_1, w_1) = 1$ we can find the inverse $s_1 \equiv \tau_1^{-1} \bmod w_1$ and then $a \equiv s_1 u_1,\ c \equiv s_1 v_1 \bmod w_1$ is a solution. So we proceed as follows:

Let $g = (u_1, w_1,\ u_1/g = u_2,\ w_1/g = w_2$ and, to make a as small as possible, choose for s_1 the number $u_2^{-1} \bmod w_2$. Then $a \equiv g \bmod w_1$ (so take $a = g$) and $c \equiv s_1 v_1 \bmod w_1$. Note that there is a possibility here that $(s_1, w_1) > 1$, even though $(s_1, w_2) = 1$. This would prevent us from deducing that a and c satisfy (9), since we could not choose $\tau_1 \equiv s_1^{-1} \bmod w_1$. There is also the possibility that a and c are not, in fact, relatively prime. ($\theta_1 = 73°,\ \theta_2 = 26°,\ \delta = 36$ makes $(a, c) = 2$ by this method.) So when using this method we need to check both of these conditions en route.

Of course we can 'minimise c' by the same method. As an example, let $\theta_1 = 91°,\ \theta_2 = 47°$, as in Table 1 where the method of §3 was used to find a and c. Then $\delta = 5, 25, 35, 65, \ldots$ all give $s = 72$ steps, as in Figure 3(c), and the new method correctly predicts $a = 5,\ c = 1$ is a solution here.

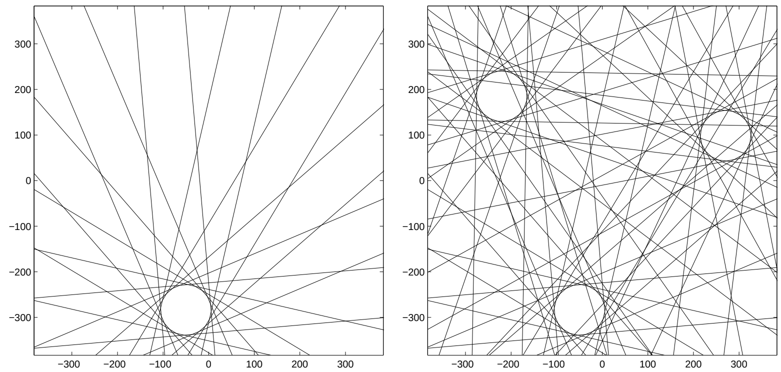

Figure 8: Two examples where the zags are all tangent to one or more circles. Here, θ_1 is a factor of 360, in fact 20°, and $\delta = 18$ on the left, $\delta = 6$ on the right. Here $\theta_2 = 49°, l = 50$.

Note that the 'stepping round' criterion is slightly different now. We need $\tau s(\delta) = w = 360$ and since $s(\delta) = 360/(u, v, w)$ and $\tau = \tau_1(u, v, w)$ we need $\tau_1 = 1$ (or -1).

The stepping round criterion for the method here is that $\tau_1 = \pm 1$.

6 The special case when the zags are tangent to circles

A glance at Figures 8, 9 and 10 shows that there are a number of situations where the zags are all tangent to one or more circles. We consider some of these here.

Case 1. Suppose $\delta\theta_1$ is a multiple of 360. Then clearly we can take $a = 0$ in (7), that is the angle ϕ in the whirligig construction (Figure 4) is *constant*. This means that the zags are all tangent to one circle, as in Figure 8, left.

More generally, the j^{th} zag (1) coincides with the line (4), for the usual values of R and r as in (5), when

$$\phi \equiv (j + \tfrac{1}{2})\theta_1, \quad \psi \equiv j\theta_2 \mod 360.$$

Write $j = k_0 + n\delta,\ n = 0, 1, 2, \ldots$. If $m\delta\theta_1$ is a multiple of 360 for an integer m then $n = 0, 1, 2, \ldots, m-1$ will give distinct ϕ and the zags will be tangent to m circles in turn which will then repeat. See Figure 8, right, where $\theta_1 = 20°,\ \theta_2 = 49°,\ l = 50$.

Case 2. A *different* way of identifying (1) and (4) is to take

$$\phi \equiv j(2\theta_2 - \theta_1) - \tfrac{1}{2}\theta_1, \quad \psi \equiv j\theta_2 \mod 360.$$

This makes use of the evenness of the cosine function. If now $\delta(2\theta_2 - \theta_1)$ is a multiple of 360, then $j = k_0 + n\delta$ will give a constant ϕ (mod 360), namely $\phi \equiv k_0(2\theta_2 - \theta_1) - \frac{1}{2}\theta_1$. Thus all the zags will be tangent to one circle. For example, Figure 9, right, we have $\theta_1 = 34°,\ \theta_2 = 37°$ (and $l = 30$), so that $2\theta_2 - \theta_1 = 40 = 360/9$. Then $\delta = 9$ gives one circle and $\delta = 1$, by an argument similar to Case 1, gives nine circles, as shown in the figure. (If $m\delta(2\theta_2 - \theta_1) = 360N$ where $(m, N) = 1$, then we get m circles.)

The enigmatic Figure 1, lower right, is a strange hybrid: here $\theta_1 = 20°,\ \theta_2 = 46°$ (and $l = 50$) so that $2\theta_2 - \theta_1 = 72 = 360/5$, but also, as in Case 1, $\theta_1 = 360/18$. The case $\delta = 1$ is shown in Figure 9, left, and we see the expected five circles. In Figure 1, lower right, we have $\delta = 3$. Since

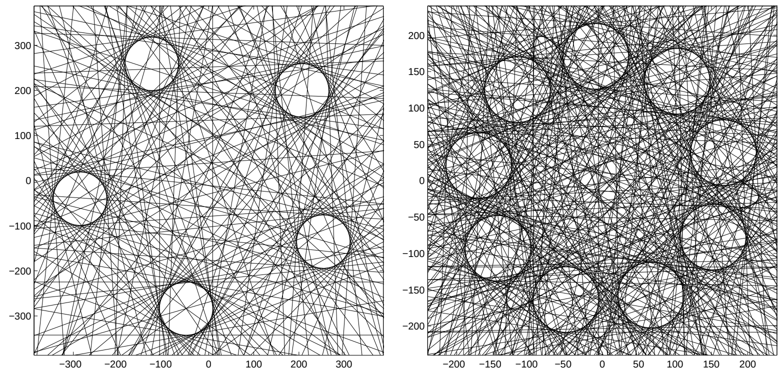

Figure 9: Left: five circles produced by $\theta_1 = 20°, \theta_2 = 46°, \delta = 1$, making $2\theta_2 - \theta_1 = 2\pi/5$. Taking $\delta = 5$ reduces to a single circle. Right: nine circles produced by $\theta_1 = 34°, \theta_2 = 37°, \delta = 1$ making $2\theta_2 - \theta_1 = 2\pi/9$. Taking $\delta = 9$ reduces to a single circle.

$20 \times 3 = 360/6$ we might expect six circles as in Case 1 , but $72 \times 3 = 3 \times 360/5$ so perhaps there are also five Case 2 circles present! What do you think?

Case 3. There is another case where the zags are all tangent to one or more circles. If $\delta(\theta_1 - \theta_2)$ is a multiple of 2π then the cosine term on the right hand side of (1) is *constant*. We can then make (1) match (4) with $R = 0$ and

$$r = \frac{100}{2\sin\frac{1}{2}\theta_1}\cos(k_0(\theta_1 - \theta_2) + \tfrac{1}{2}\theta_1) + \tfrac{1}{2}l\cot\tfrac{1}{2}\theta_2.$$

Thus all the zags are tangent to one circle, centered at $\mathbf{c}$. If $k_0 = 0$ then the radius of the circle is

$$\tfrac{1}{2}100\cot\tfrac{1}{2}\theta_1 + \tfrac{1}{2}l\cot\tfrac{1}{2}\theta_2.$$

If, in fact, $m\delta(\theta_1 - \theta_2)$ is a multiple of 2π with the integer m as small as possible then there will be m *concentric* circles. An example is shown in Figure 10, where $l = 50, \theta_1 = 19°, \theta_2 = 73°$. Here $\theta_1 - \theta_2 = -54°$ and $\delta = 20$ is the smallest number making $\delta(\theta_1 - \theta_2)$ a multiple of 360, so we obtain (left) one circle with all zags tangent to it. The right hand figure shows $\delta = 10$ giving two circles.

7 A concluding problem

What exactly is it about θ_1, θ_2 and δ which allows the existence of a reasonably simple whirligig tangent to the visible envelope? We want, in rough terms, a and c to be small but the number $s(\delta)$ of zags to be large. I do not know the full answer to this.

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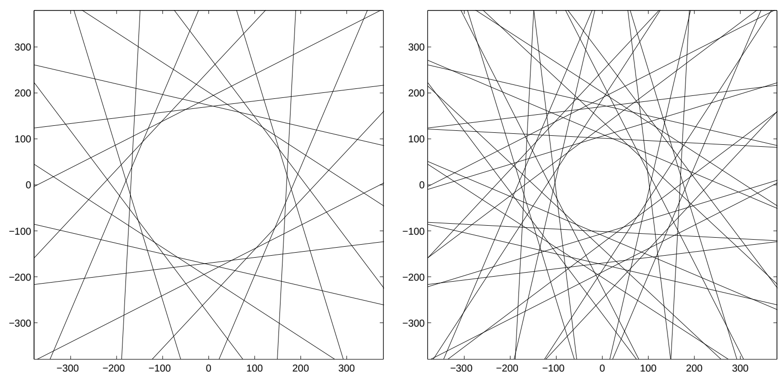

Figure 10: An example where all zags are tangent to (left) one circle and (right) two *concentric* circles.

References

[1] H. Abelson and A. A. diSessa, *Turtle Geometry*, M.I.T.Press 1980.

[2] Ke Chen, Peter Giblin and Alan Irving, *Mathematical Explorations with MATLAB*, Cambridge University Press 1999.

[3] E.H.Lockwood, *A Book of Curves*, Cambridge University Press 1961.

[4] P.M.Maurer, 'A rose is a rose...', *Amer. Math. Monthly* 94 (1987), 631–645.