# Embroidery From Chords of One or Two Circles

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There is a well-known 'embroidery' envelope of lines, a typical example of which is shown in Figure 1, left. Each point $P$ of a circle is joined to the point $Q$ whose angle is a fixed integer multiple $m$ of the angle of $P$. In this note we explore two generalizations of this construction. In §1 we allow $m$ to be a rational number; this produces envelopes such as those illustrated in Figure 1, centre and right. We show how to count the cusps and, a harder problem, to count the self-crossings of these envelopes. In §2 we make some general remarks about envelopes of lines which make it clear that in fact the one-circle envelope is very special—we might almost say 'degenerate'—and in §3 we generalize the construction in a way that removes this degeneracy. In fact we consider envelopes obtained by replacing the circle with two concentric circles: $P$ lies on the inner, say unit circle and $Q$ on the outer circle, radius $r > 1$, again allowing $m$ to be rational. We count the cusps of these envelopes and examine what happens as $r$ changes and as $r \to \infty$. The two-circle envelopes admit different singularities from the one-circle envelopes, specifically the 'butterfly' singularity, and we discuss the 'unfolding' of this butterfly which requires not just $r$ to vary but also the centre of the second circle. Such an 'unfolding' is illustrated in Figure 7.

## 1   An envelope from one circle

A well-known way of forming an envelope from chords of a single circle is as follows (see for example [1, §5.7(2), §7.14(6)]). For each point $(\cos t, \sin t)$, $0 \le t \le 2\pi$, on the unit circle construct the chord from this point to $(\cos(mt), \sin(mt))$ where $m$ is a positive integer $> 1$. (For $t = 0$ we use the tangent at the origin instead of a chord.) These lines have an envelope which has $m - 1$ cusps and no self-crossings. See Figure 1, left, for the case $m = 3$. Now we let $m = \frac{a}{b}$ be any rational number, not equal to 0 or $\pm 1$ and in its lowest terms. We consider three problems, in increasing order of difficulty: (i) find the range of values of $t$ which allows the envelope to close; (ii) find the number of cusps on the envelope; and (iii) find the number of self-crossings on the envelope. Two examples are given also in Figure 1.

We shall prove the following.

**Theorem 1.1** *For the envelope constructed as above, with $m \ne -1, 0, 1$ a rational number $\frac{a}{b}$ in its lowest terms, with $b > 0$,*

*(i) the envelope, starting at $t = 0$, closes for $t = 2b\pi$,*

*(ii) the number of cusps is $|a - b|$, occurring when $(m - 1)t$ is an odd multiple of $\pi$,*

*(iii) the number of self-crossings is*

$$(|a| - 1)|a - b| \quad if \quad |m| < 1$$
$$(b - 1)|a - b| \quad if \quad |m| > 1.$$

First, we shall need a formula for the envelope itself. The line joining $(\cos t, \sin t)$ to $\cos mt, \sin mt$ ($m \ne 1$) has equation

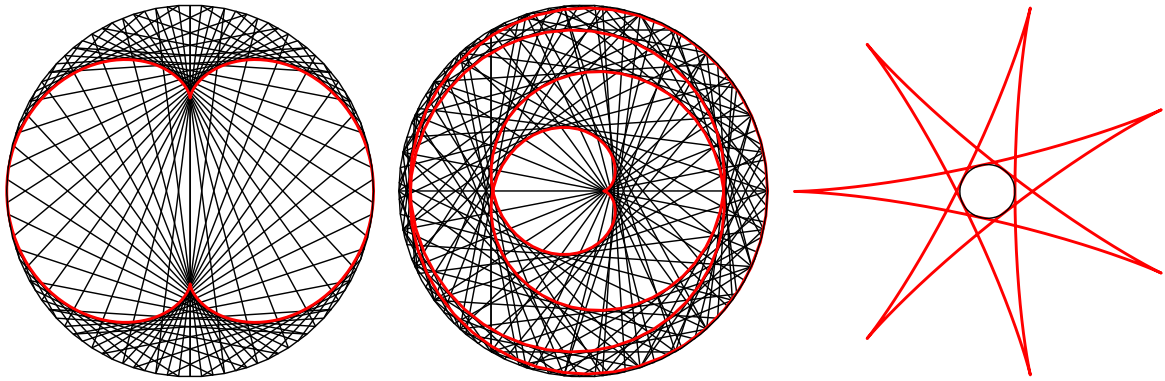$$x(\sin mt - \sin t) - y(\cos mt - \cos t) = \sin(m - 1)t. \tag{1}$$

Figure 1: Left: the embroidery envelope for $m = 3$, with 2 cusps and no self-crossings. Centre: $m = \frac{5}{4}$, which has 1 cusp and 3 self-crossings. Right: $m = -\frac{3}{4}$, with 7 cusps and 14 self-crossings (the lines are not shown here).

The standard way to obtain the envelope of such a family of lines—that is, a curve which is tangent to all the lines—is to differentiate with respect to $t$ and then to solve for $x$ and $y$ as functions of $t$ from the two equations. After a significant amount of trigonometric work this yields the result

$$x(t) = \tfrac{1}{m+1}(m \cos t + \cos mt), \quad y(t) = \tfrac{1}{m+1}(m \sin t + \sin mt). \tag{2}$$

**Remark 1.2** It is important to note that in the process of deriving this parametrization there is a cancellation from numerator and denominator of $1 - \cos(m - 1)t$ which vanishes when $(m - 1)t$ is an even multiple of $\pi$. This corresponds to those values of $t$ for which the two ends of the chord joining $(\cos t, \sin t)$ and $(\cos mt, \sin mt)$ actually *coincide*, but because cancellation takes place in both numerator and denominator it is a 'removable singularity' and the parametrization (2) remains valid at these points. We expand on this in §2 below.

(i) It is clear that with $m = \frac{a}{b}$ in its lowest terms, and $b > 0$, the functions giving $x$ and $y$ will begin to repeat when $t = 2b\pi$. (This is not the same as saying that the smallest $t$ with $x(t) = x(0)$ and $y(t) = y(0)$ is $t = 2b\pi$: the envelope may, as in (iii), have self-crossings for smaller values of $t$.) For $m$ an integer this is of course just a period of $2\pi$. Thus we need to consider in general the interval $0 \le t < 2b\pi$ when discussing cusps or self-crossings of the envelope. In practice we will regard parameter values $t$ as defined modulo $2b\pi$.

(ii) A cusp forms on the curve $(x(t), y(t))$ when its velocity vector $(x'(t), y'(t))$ is zero, the prime $'$ here representing differentiation with respect to $t$. These require both $\sin t + \sin mt = 0$ and $\cos t + \cos m = 0$, which can be rewritten as

$$2 \sin \left(\tfrac{1}{2}(t + mt)\right) \cos \left(\tfrac{1}{2}(t - mt)\right) = 0, \quad 2 \cos \left(\tfrac{1}{2}(t + mt)\right) \cos \left(\tfrac{1}{2}(t - mt)\right) = 0, \tag{3}$$

using the well-known trigonometric formulas for turning sums of sines or cosines into products:

$$\begin{aligned} \cos \theta - \cos \phi &= -2 \sin \left(\tfrac{1}{2}(\theta + \phi)\right) \sin \left(\tfrac{1}{2}(\theta - \phi)\right), \\ \sin \theta - \sin \phi &= 2 \cos \left(\tfrac{1}{2}(\theta + \phi)\right) \sin \left(\tfrac{1}{2}(\theta - \phi)\right). \end{aligned}$$

The equations (3) can only be true simultaneously if $\cos(\tfrac{1}{2}(t - mt)) = 0$, that is $(m - 1)t$ is an odd integral multiple of $\pi$. Thus $|m - 1|t$ takes values $(2k - 1)\pi$ for $k = 1, 2, \ldots$. When $m$ is an integer, we are looking for values of $0 \le t < 2\pi$ and the largest value of $k$ is then $|m - 1|$. In general when $m = \frac{a}{b}$ in lowest terms we have $0 \le t < 2b\pi$ and the largest value of $k$ is $|a - b|$. Hence there are $|a - b|$ cusps in general. (It can be checked that in this situation all cusps are 'ordinary' in the sense that a suitable smooth local

transformation of the plane will take each one to the normal form $(t^2, t^3)$. One way to check this is to show that the vectors $(X'', Y'')$ and $(X''', Y''')$ are independent at each point where $X' = Y' = 0$.)

(iii) Counting the number of self-crossings is significantly harder since we have to find different parameter values, say $t$ and $s > t$, which give the same point $(x(t), y(t))$ on the envelope. Writing down the conditions $x(s) = x(t)$ and $y(s) = y(t)$,

$$m(\cos s - \cos t) = \cos mt - \cos ms, \quad m(\sin s - \sin t) = \sin mt - \sin ms, \tag{4}$$

using again the trigonometric formulas which convert sums into products and dividing the resulting equations, we find the following condition on the *midpoint* of the parameter values $t$ and $s$:

$$\tan\left(\tfrac{1}{2}(t+s)\right) = \tan\left(\tfrac{1}{2}m(t+s)\right), \text{ that is } \tfrac{1}{2}(t+s) = \frac{k\pi}{m-1} \text{ for an integer } k, \ 0 < k \le 2|a-b|. \tag{5}$$

In the division we have cancelled $\sin\left(\tfrac{1}{2}(s-t)\right)$ and $\sin\left(\tfrac{1}{2}m(s-t)\right)$, but given the equations (4) each of the conditions $\sin\left(\tfrac{1}{2}(s-t)\right)$ and $\sin\left(\tfrac{1}{2}m(s-t)\right)$ implies the other, that is, $\tfrac{1}{2}(s-t) = n_1\pi$ and $\frac{a}{2b}(s-t) = n_2\pi$ for integers $n_1, n_2$. This implies $an_1/b$ is an integer and since $a, b$ are coprime that $n_1$ is a multiple of $b$. But $0 \le \tfrac{1}{2}(s-t) < b\pi$ so $n_1 = 0, s = t$ which is a contradiction. Thus the cancellation is justified.

The restriction on the values of $k$ comes from $0 \le \tfrac{1}{2}(s+t) < 2b\pi$ as in (i).

Equation (5) is our first equation, and it connects $t$ and $s$. Writing $u = \tfrac{1}{2}(s-t)$ it shows that

$$\text{The values } s \text{ and } t \text{ giving a crossing are } (\text{mod } 2b\pi)\frac{kb\pi}{|a-b|} \pm u, \ \ 0 < k \le 2|a-b|. \tag{6}$$

In practice we regard the integer $k$ as being defined modulo $2|a-b|$ and $u$ as being defined modulo $b\pi$. It remains to find an equation restricting the values of $u$.

To obtain this second equation, we substitute (5) into one of the original conditions, say into $y(t) = y(s)$ where the sums have been converted to products (and writing $u = \tfrac{1}{2}(s-t)$ as before):

$$m\left(2\sin(-u)\cos\left(\frac{k\pi}{m-1}\right)\right) = 2\sin u \cos\left(\frac{mk\pi}{m-1}\right).$$

Now

$$\cos\left(\frac{k\pi}{m-1}\right) = (-1)^k \cos\left(\frac{k\pi}{m-1} + k\pi\right) = (-1)^k \cos\left(\frac{mk\pi}{m-1}\right),$$

and cancelling $\cos\frac{k\pi}{m-1}$ gives, for the same $k$ as in (5),

$$(-1)^{k+1} m \sin u = \sin(mu), \ \ 0 < u \le b\pi. \tag{7}$$

(If $\cos\frac{k\pi}{m-1} = 0$ then we can use $x(t) = x(s)$ to deduce the same result, assuming $\sin\frac{k\pi}{m-1} \ne 0$; since sine and cosine cannot both be zero we deduce the result without additional conditions.)

Let us concentrate first on (7), which amounts to counting the number of crossings of two sine curves. We are looking for the number of solutions of the equation

$$\sin(u) = \pm\tfrac{b}{a}\sin\left(\tfrac{a}{b}u\right), \ 0 < u \le b\pi, \tag{8}$$

for a fixed choice of sign $\pm$.

**Lemma 1.3** (i) *The number of solutions of (8) is $|a-1|$ if $|a| < b$ and $b-1$ if $|a| > b$.*
(ii) *Each unordered pair of parameter values $\{t, s\}$ giving a crossing on the envelope arises, first, from an integer $k$ and value $u$ as in (6) and, second, from the integer $k + (a-b)$ and value $b\pi - u$.*
(iii) *The number of values of $k$ giving distinct self-crossings of the envelope is half the total, that is $|a-b|$.*
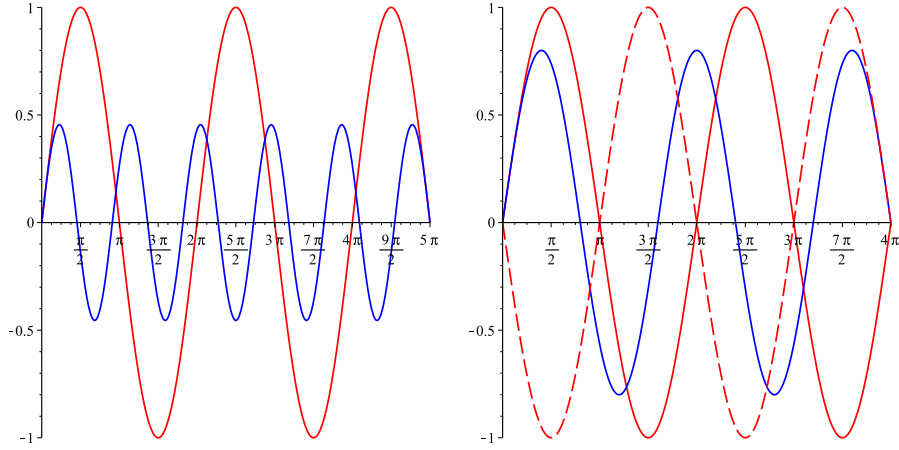
Figure 2: Graphs of $\sin u$ (larger amplitude, solid curve), $-\sin u$ (dashed curve) and $\frac{b}{a}\sin\left(\frac{a}{b}u\right)$ (smaller amplitude curve), over the range $0 \le u \le b\pi$. Left: $a = 11, b = 5$; here the crossings are in pairs, $u$ and $b\pi - u$. Right: $a = 5, b = 4$; here if $u$ is a crossing of $\sin u$ and $\frac{4}{5}\sin\left(\frac{5}{4}u\right)$ then $b\pi - u$ is a crossing of $-\sin u$ and $\frac{4}{5}\sin\left(\frac{5}{4}u\right)$. See Lemma 1.3.

**Proof of lemma** (i) We may assume, for the purpose of counting solutions of (8), the $+$ sign in the equation; also $a > 0$ (else replace $a$ by $-a$); also $a > b$ (if $a < b$ then write $v = \frac{a}{b}u$ and solve for $v$ in the range $0 < v < a\pi$). Figure 2 illustrates a typical case, with $\frac{a}{b} = \frac{11}{5}$. It is clear that the larger amplitude graph, in this case $\sin u$, crosses the smaller amplitude graph $\frac{5}{11}\sin\left(\frac{11}{5}u\right)$ exactly once between consecutive turning points of $\sin u$. But in the range $0 < u \le b\pi$ there are exactly $b$ turning points, which gives the result.

(ii) This is a matter of verifying that

$$\frac{k\pi}{m-1} \pm u = \frac{(k \pm (a-b))\pi}{m-1} \mp (b\pi - u),$$

and noting that $k + (a - b)$ and $k - (a - b)$ are the same modulo $2|a - b|$.

(iii) Note that the parity of $k$ equals that of $k + (a - b)$ if and only if $a - b$ is even. This means that when $u$ is a correct solution of (7) for $k$ then $b\pi - u$ will always be a correct solution of (7) for $k + (a - b)$. Thus exactly half the values of $k$ are needed to produce all unordered pairs of parameter values giving self-crossings on the envelope. $\square$

The situation is illustrated schematically in Figure 3.
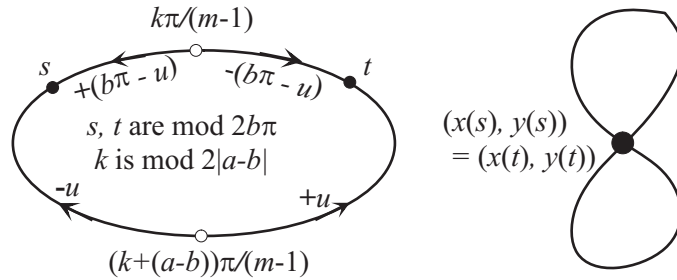


Figure 3: Left: the parameter space of length $2b\pi$ for the envelope, with two parameter values $t, s$ giving a crossing (right) on the envelope. This pair of values is obtained once from values of $k, u$ in (6) and once, in the reverse order, from different values $k + (a - b)$ (or $k - (a - b)$ which is the same modulo $2|a - b|$) and $b\pi - u$.

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**Corollary 1.4** *Combining the results of* (i) *and* (ii) *of the Lemma proves* (iii) *of Theorem 1.1.* □

**Examples 1.5** Here as above $k$ values are mod $2|a - b|$, $u$ values are mod $b\pi$ and $s, t$ values are mod $2b\pi$.

**(1)** $a = 11, b = 5$. Then $a - b = 6$, therefore $k$ and $k + (a - b)$ have the same parity and (7) has the same sign for $k$ and for $k + (a - b)$. Write $u_1^+, u_2^+, u_3^+, u_4^+$ for the $b - 1 = 4$ solutions in increasing order of (8) with the $+$ sign and similarly $u_i^-, i = 1, 2, 3, 4$ with the $-$ sign. Thus $u_1^+ + u_4^+ = b\pi = 5\pi$, $u_2^+ + u_3^+ = 5\pi$ and similarly with $-$.

   The values $k = 1, u_1^+$ and $k = 7, u_4^+$ substituted into (6) give parameter values of the form $t, s$ and $s, t$ respectively, and the envelope points for these two parameter values are the same.

   The values $k = 2, u_1^-$ and $k = 8, u_4^-$ likewise give the same unordered pair of parameter values and the same envelope point; and so on for other values of $k$.

**(2)** $a = 11, b = 4$. Then $a - b = 7$, therefore $k$ and $k + (a - b)$ have opposite parity. Write $u_1^+, u_2^+, u_3^+$ for the $b - 1 = 3$ solutions in increasing order of (8) with the $+$ sign and similarly $u_i^-, i = 1, 2, 3$ with the $-$ sign. Thus $u_1^+ + u_3^- = u_2^+ + u_2^- = u_3^- + u_1^+ = b\pi = 4\pi$.

   The values $k = 1, u_1^+$ and $k = 8, u_3^-$ substituted into (6) give the same unordered pair of parameter values $\{t, s\}$ and the the envelope points for these two values are the same.

   The values $k = 2, u_1^-$ and $k = 9, u_3^+$ substituted into (6) give the same $\{t, s\}$ and the same envelope point; and so on.

## 2   Some general remarks about envelopes of lines

Let
$$F(t, x, y) = A(t)x + B(t)y + C(t), \tag{9}$$
so that $F = 0$ is a family of straight lines, parametrized by $t$, with envelope given by $F = 0$, $F_t = A'x + B'y + C' = 0$, using here and in what follows subscripts to denote partial derivatives. Solving for $x = X(t)$, $y = Y(t)$ gives (omitting the variable $t$)
$$X = \frac{BC' - B'C}{AB' - A'B}, \quad Y = \frac{A'C - AC'}{AB' - A'B},$$
provided of course that $AB' - A'B \neq 0$.

   For the envelope considered above in §1 (see (1)), $A = \sin mt - \sin t, B = -\cos mt + \cos t, C = -\sin(m - 1)t$. Calculation then shows that $AB' - A'B = (m + 1)(1 - \cos(m - 1)t)$ which is zero when $(m - 1)t$ is an even multiple of $\pi$. These are exactly the points for which the two 'ends' of the chord joining $(\cos t, \sin t)$ to $(\cos mt, \sin mt)$ *coincide*. Fortunately this does not invalidate the parametrization (2) and indeed the envelope is actually smooth at these points, the cusps occurring at intermediate values of $t$ where $(m - 1)t$ is an *odd* multiple of $\pi$. It is worth recording the following well-known lemma.

**Lemma 2.1** *In the above notation, suppose that $AB' - A'B \neq 0$ and let $(X(t), Y(t))$ be the resulting parametrization of the envelope, At an envelope point $(x, y)$ with corresponding parameter $t$, given by $F(t, x, y) = F_t(t, x, y) = 0$ the second derivative $F_{tt}(t, x, y)$ vanishes if and only if $X'(t) = Y'(t) = 0$.*

**Proof.** We have $F(t, X(t), Y(t)) = 0$ and $F_t(t, X(t), Y(t)) = 0$ identically as functions of $t$. Differentiating these with respect to $t$ we get $F_t + F_x X' + F_y Y' = 0 = F_{tt} + F_{tx} X' + F_{ty} Y'$. It is clear that if $X' = Y' = 0$ then $F_{tt} = 0$. For the converse, assume $F_{tt} = 0$ at an envelope point, so that $F_t = 0$ too. We have $F_x = A, F_y = B, F_{tx} = A', F_{ty} = B'$ so that the two equations for $X', Y'$ are $AX' + BY' = 0, A'X' + B'Y' = 0$ and these imply $X' = Y' = 0$ since $AB' - A'B \neq 0$. □

Thus away from points where $AB' - A'B = 0$ we can detect singular points of the envelope by the equation $F_{tt} = 0$. When $AB' - A'B = 0$ then in general we might expect the envelope to 'go to infinity'

since the denominator of $X$ and $Y$ vanishes. For the envelope studied in §1 numerator and denominator always vanish together and these 'singularities' are removable. The envelope *never* goes to infinity and its genuine singular points are cusps as determined in Theorem 1.1. In the next section we shall meet an example where the denominator of $X$ or $Y$ can vanish with neither numerator or one numerator vanishing, but never both at once. When one numerator vanishes this means that the envelope 'goes to infinity' in the direction of the $x$- or the $y$-axis, and when neither vanishes it means that the envelope goes to infinity in some other direction.

Naturally singularities of envelopes $(X(t), Y(t))$, detected by the conditions $X' = Y' = 0$, are not always cusps. In the case of the singularities present in one-circle envelopes they are in fact cusps, in the sense that a suitable smooth change of coordinates in the plane will transform them into the standard form $(t^2, t^3)$. But in the next section we shall meet 'higher' singularities, and in those circumstances more derivatives than $F_t$ and $F_{tt}$ will vanish. We expand on this in Remarks 3.5 below.

# 3   An envelope from two circles

There is an interesting extension of the above construction to produce a *family of envelopes* in which transitions occur increasing or decreasing the number of cusps and self-intersections. We take two circles centred at the origin, of radii 1 and $r > 0$, and join the point $(\cos t, \sin t)$ of the first circle to the point $(r \cos mt, r \sin mt)$ of the second. The line joining these two points has equation $F(x, y) = 0$, for a given $t$ and $r$, where

$$F(x, y) = x(r \sin mt - \sin t) - y(r \cos mt - \cos t) - r \sin(m - 1)t, \tag{10}$$

and the envelope of these lines, obtained from $F = \partial F / \partial t = 0$ is, for fixed $r$,

$$
\begin{aligned}
x_r(t) &= \frac{r \left[ \cos mt (r \cos(m - 1)t - 1) + m \cos t (\cos(m - 1)t - r) \right]}{r(m + 1) \cos(m - 1)t - (mr^2 + 1)}, \\
&\\
y_r(t) &= \frac{r \left[ \sin mt (r \cos(m - 1)t - 1) + m \sin t (\cos(m - 1)t - r) \right]}{r(m + 1) \cos(m - 1)t - (mr^2 + 1)}.
\end{aligned}
\tag{11}
$$

Note that in this case the factor $\cos((m - 1)t) - 1$, which was cancelled from numerator and denominator when deducing the parametrization (2) in the one-circle case, appears here explicitly: when $r = 1$ the denominator in (11) becomes exactly $(m + 1)(\cos(m - 1)t - 1)$.

We are interested, as before, in the case where $m = \frac{a}{b}$, a rational number in its lowest terms, with $b > 0$.

**Remark 3.1** The two-circle envelope above can also be constructed from the family of straight lines joining the point $(\cos bT, \ \sin bT)$ on the unit circle to the point $(r \cos aT, \ r \sin aT)$ on the concentric circle of radius $r$: just write $t = bT$ so that $mt = (a/b)bT = aT$. The range of values of $T$ is $0 \le T < 2\pi$. The formula corresponding to (10) is

$$\widetilde{F}(x, y) = x(r \sin aT - \sin bT) - y(r \cos aT - \cos bT) - r \sin(a - b)T.$$

When studying the two-circle envelope we may assume $r > 1$. To see this note that replacing $r$ by $1/r$, $x$ by $x/r$, $y$ by $y/r$ and interchanging $a$ and $b$ the formula for $\widetilde{F}$ is merely multiplied by $1/r^2$. Thus the family of lines for given values of $r, a, b$ is obtained from that for $1/r, b, a$ by scaling with centre at the origin. The envelopes are therefore the same up to a magnification or contraction and in particular all geometrical features such as cusps and crossings will be the same.  □

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When discussing the general two-circle envelope we shall therefore assume $r > 1$ since the case $r = 1$ in this context is special and is already covered in §1.

Note that the denominator in (11) can be zero, indicating that the envelope has "gone to infinity". This occurs when

$$\cos(m-1)t = \frac{r^2 m + 1}{r(m+1)}. \tag{12}$$

The denominator in *this* expression is not zero! Values of $t$ exist satisfying (12) if and only if the right-hand side lies in the closed interval $[-1, 1]$. (As above we exclude the special case $r = 1$: the $r = 1$ envelope has a simpler parametrization (2) and never goes to infinity.) Some work with inequalities, using $r > 1$ and $m \neq -1, 0, 1$, shows that this is equivalent to $1 < r \leq \frac{1}{|m|}$ and in particular is only possible if $|m| < 1$. Summing this up:

**Proposition 3.2** *The 2-circle envelope parametrized by (11), where $r > 1$, $m = \frac{a}{b}, b > 0$ is in lowest terms, goes to infinity if and only if $|m| < 1$ and $1 < r \leq \frac{1}{|m|}$. The values of $t$ for which this happens are given by (12) which has $2|b - a|$ solutions for $0 \leq t < 2b\pi$, except for $r = \frac{1}{|m|}$ when the number is $|b - a|$.* □

There is an illustration of this in Figure 4.



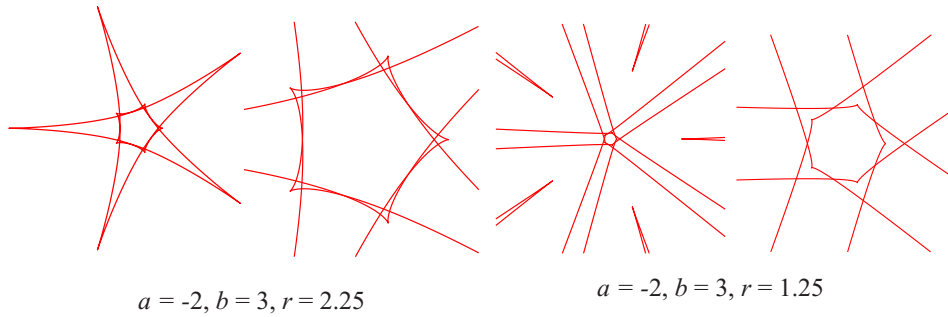$a = -2, b = 3, r = 2.25$          $a = -2, b = 3, r = 1.25$

Figure 4: Left: $r > \frac{1}{|m|} = 1.5$ and the envelope is finite. The central part is enlarged. Right: the envelope goes to infinity 10 $(= 2(b - a))$ times, because $r < \frac{1}{|m|}$ (see Proposition 3.2). The central part of the envelope is again enlarged for clarity.



$a = -2, b = 3, r = 1.036$          $a = -2, b = 3, r = 1$          $a = 7, b = 4, r = 1.09$          $a = 7, b = 4, r = 1$
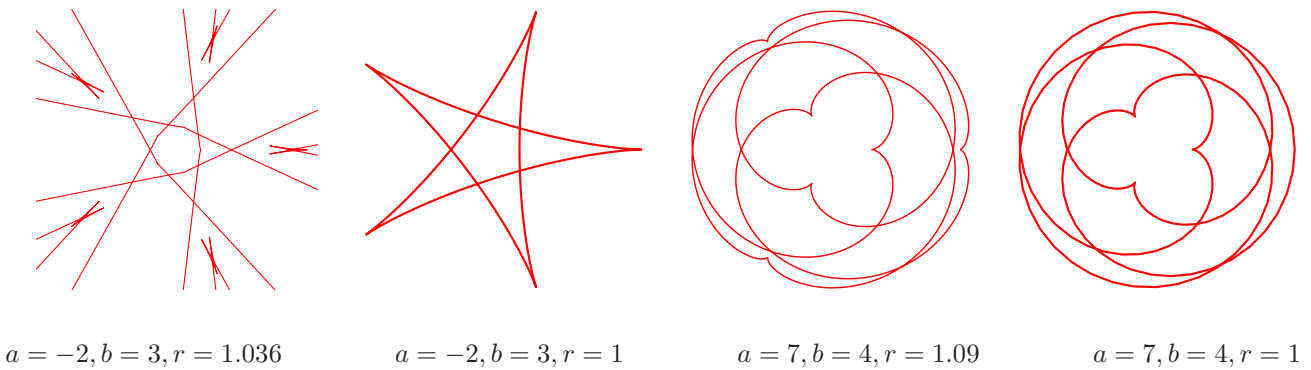
Figure 5: Left pair, $a = -2, b = 3, |m| = \frac{2}{3} < 1$: continuing from Figure 4. Left: here $r < \left|\frac{m-2}{2m-1}\right| = \frac{8}{7}$ which results in 10 additional cusps, as in Proposition 3.4. Right: the limiting envelope when $r = 1$, a highly discontinuous change. Right pair, $a = 7, b = 4, m = \frac{7}{4} > 1$: the less drastic change as $r \to 1$, when half the cusps mutate into smooth points of the envelope.

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## 3.1 Cusps on the 2-circles envelope

When counting the cusps on the 2-circles envelope it is helpful to use Lemma 2.1. At points given by (11), we have

$$F_{tt} = rm \sin(m-1)t \left( m - 2 + r(m+1)\cos(m-1)t + r^2(1-2m) \right) \tag{13}$$

and cusps (or possibly 'higher singularities': see Remarks 3.5) occur when this is zero.

The factor $\sin(m-1)t$ is equal to zero if $|m-1|t$ takes values $k\pi$ for $k = 0, 1, \ldots$. Thus, taking $m = \frac{a}{b}, b > 0$ in lowest terms and looking at the range $0 \le t < 2b\pi$, there are $2|a-b|$ solutions for this term, corresponding to $2|a-b|$ 'general cusps' for any $r > 1$.

The additional factor of (13) equals zero when

$$\cos(m-1)t = \frac{(2m-1)r^2 + (2-m))}{r(m+1)}. \tag{14}$$

Values of $t$ satisfying this expression exist if and only if the right-hand side lies within the interval $[-1, 1]$. Evaluating the resulting inequalities gives the result that extra cusps occur in the range $1 < r \le |\frac{m-2}{2m-1}|$ if $|m| < 1$. Conversely, for $|m| > 1$, there are no extra cusps besides the $2|a-b|$ general cusps for all $r > 1$. When $r$ takes the extreme value $|\frac{m-2}{2m-1}|$ then it is easy to check that $\cos(m-1)t = \pm 1$, so that $\sin(m-1)t = 0$ and the values of $t$ have coincided with those for the 'general cusps' above.

**Remarks 3.3** (1) The values of $t$ in (14) cannot coincide with those in (12) (given as usual $m \ne 1, r > 1$), but the cusps given by $\sin(m-1)t = 0$ can be at infinity, in fact when $m = |1/r|$. An example occurs at the intermediate value $r = 1.5$, between the illustrations in Figure 4, when all the cusps have 'gone to infinity'.

(2) The situation as $r \to 1$ is rather strange, and highly discontinuous, especially when $|m| < 1$, as illustrated in Figure 5. As $r$ decreases to 1 it necessarily passes through the values $\frac{1}{|m|}$ and $\left|\frac{m-2}{2m-1}\right|$ which cause the envelope to go to infinity and acquire additional cusps. All these disappear at $r = 1$ and the envelope becomes a finite curve. The situation is slightly less drastic for $|m| > 1$ when half of the $2|a-b|$ cusps mutate into smooth points. See Figure 5 again.

We should expect that the case $r = 1$, regarded as a special case of general $r$, is *very* special, as it arises from the coincidence of the two circles generating the envelope.

From the above discussion we have the following.

**Proposition 3.4** *The two-circle envelope parametrized by (11), where as usual $m = \frac{a}{b}$ is in its lowest terms, $r > 1$ and the range of $t$ is $[0, 2b\pi)$, has cusps as follows.*

(i) $2|a-b|$ *cusps always (we call these the 'general cusps';*

(iia) *when $|m| < 1$ and $1 < r < \left|\frac{2-m}{2m-1}\right|$, an additional $2|a-b|$ cusps;*

(iib) *when $|m| < 1$ and $r = \left|\frac{2-m}{2m-1}\right|$, an additional $|a-b|$ cusps (but see Remark 3.5(1) below).* $\qquad\square$

Two further examples are given in Figure 6.

**Remarks 3.5** (1) At the transitional value $r = \left|\frac{2-m}{2m-1}\right|$ in Proposition 3.4(iib) above the singularities of the envelope are strictly not cusps of the kind which 'look like' $(t^2, t^3)$. In fact further partial derivatives of $F$ beyond $F_{tt}$ vanish and the singularities when this happens are called 'swallowtail' when just $F_{ttt} = 0$ and 'butterfly' when also $F_{tttt} = 0$. Some routine calculation shows that, when $F = F_t = F_{tt} = 0$ then

(i) $F_{ttt} = 0$ if and only if $r = \left|\frac{m-2}{2m-1}\right|$.

(ii) $F_{tttt} = 0$ too if and only if in addition $\sin(m-1)t = 0$ so that the values of $t$ coincide with those
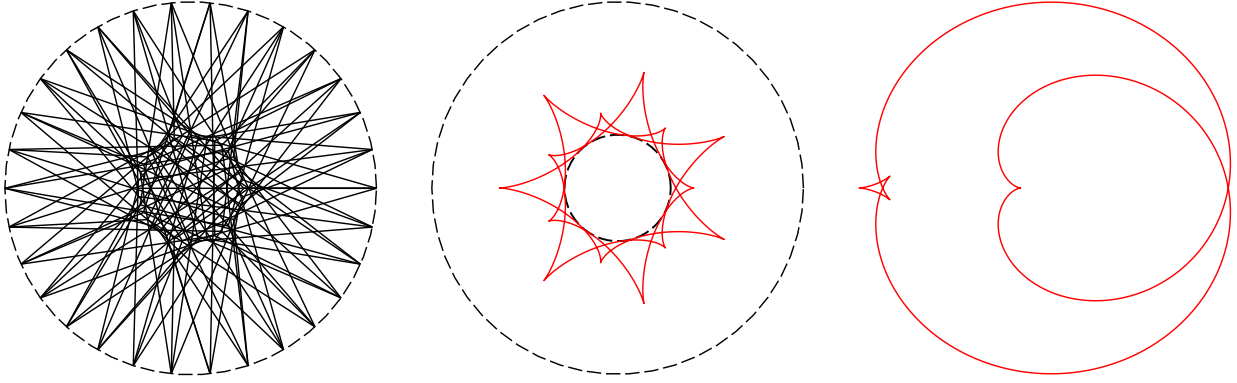
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Figure 6: Left and centre: the case $a = -4, b = 3, r = 3.5$, with $2|a - b| = 14$ cusps. On the left the lines forming the envelope are drawn and also the circle radius $r$ is drawn dashed. In the centre the envelope is the solid curve and the circles of radii 1 and $r$ are drawn dashed. Right: the case $a = 3, b = 4, r = 2$ with 4 cusps, which is more than $2|a-b|$. The collection of three nearby cusps at the left of this envelope is called a 'butterfly' configuration. As $r \to 2.5$ this configuration collapses to a single cusp. In order to realize the full 'unfolding' of a butterfly singularity we need to introduce a second continuous parameter besides $r$. In fact it is enough to move the centre of the second circle away from the origin to the point $(0, d)$. See Remark 3.5 for some notes on this. See also Figure 7.
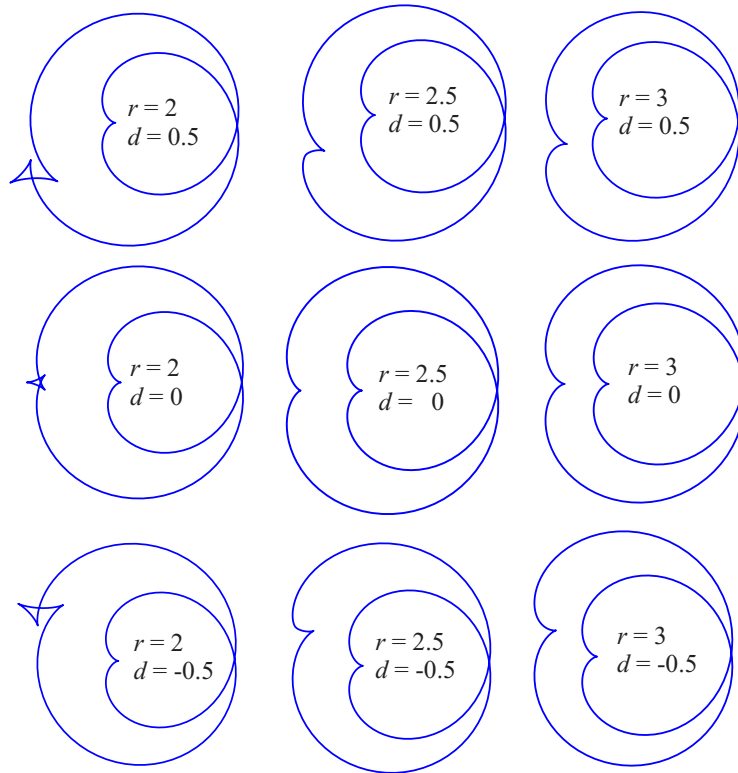


Figure 7: $a = 3, b = 4$ and the pair $(r, d)$ moving around the central value $(2.5, 0)$ where the second circle has radius $r$ and centre $(0, d)$. This shows the "butterfly" singularity for $(r, d) = (2.5, 0)$ being unfolded by the parameters $(r, d)$. The inner cusp on the envelope is unaffected by these changes.

giving the 'general cusps' of Proposition 3.4 and the cusps have merged into a 'butterfly' singularity. (iii) The fifth patial derivative of $F$ with respect to $t$ never vanishes for the values of $m$ considered here

$(m \neq -1, 0, 1)$.

(2)  We can treat the equation (10) as $F(t, x, y, r)$, a 3-parameter family of functions of $t$. For fixed $r$ the *discriminant set* is the envelope in the $(x, y)$ plane as above, and when $r$ varies the discriminant set $\{(x, y, r) : F = \partial F/\partial t = 0\}$ is a surface in 3-space whose slices $r = $ constant are the envelopes discussed above. Moving the slice we observe the transitions on the envelope, and these can be investigated by the methods of [1] as unfoldings. However in order to realize the full 'unfolding' of the 'butterfly' singularities on the envelope it is necessary to introduce a second parameter; we shall move the centre of the second circle to the point $(0, d)$. Naturally we could move it to $(c, d)$ instead but the single additional parameter $d$ turns out to be sufficient.

## 3.2  The 2-circles envelope for $r \to \infty$

As $r \to \infty$ only the $r^2$ terms in the expressions for $x_r(t)$ and $y_r(t)$ above are significant; the limiting envelope is as follows.

$$
\begin{aligned}
x_\infty(t) &= \tfrac{1}{m}\left[m \cos t - \cos mt \cos(m-1)t\right] = \tfrac{1}{2m}\left[(2m-1)\cos t - \cos(2m-1)t\right], \\
y_\infty(t) &= \tfrac{1}{m}\left[m \sin t - \sin mt \cos(m-1)t\right] = \tfrac{1}{2m}\left[(2m-1)\sin t - \sin(2m-1)t\right],
\end{aligned}
\tag{15}
$$

where the second expression is in each case obtained from the well-known formulae for turning products of sines and cosines into sums.

Clearly the expressions (15) are almost the same as (2): replace $m$ in (2) by $2m - 1$. But there is an important change of sign. This sign change can be accomplished by means of a rotation and a possible change of parameter. In fact by direct calculation we have the following.

**Proposition 3.6**  *The "$r = \infty$" envelope (15) for a given $m$ can be rotated by an angle $\alpha$ and reparametrized by replacing $t$ by $t + \delta$, to coincide (as a set of points in the plane) with the one-circle envelope (2) in which $m$ is replaced by $2m - 1$, whenever there are integers $n_1, n_2$ such that*

$$
\alpha = \frac{4mn_1 - 2(n_1 + n_2) - 1}{2(m-1)}\pi = \frac{(4a - 2b)n_1 - 2bn_2 - b}{2(a - b)}\pi, \quad \delta = \frac{2(n_2 - n_1) + 1}{2(m-1)}\pi,
$$

*where $m = \frac{a}{b}$.*  $\square$

**Corollary 3.7**  *Write $m = \frac{a}{b}$ in (15), where as usual $a, b$ are coprime.*
*(i)  We may take $\alpha = 0$ in the proposition if and only if $b$ is a multiple of 4. In that case no rotation is required.*
*(ii)  If $b$ is even but not a multiple of 4 then $\alpha = 0$ is not possible but $\alpha = \pi$ is a possible value.*
*(iii)  The smallest rotation $\alpha_0$ in absolute value satisfying the formula of the proposition is the absolute value of*

$$
\frac{-b \mod 2(2a, b)}{2(a - b)}\pi,
$$

*where the round brackets $(2a, b)$ stand for the greatest common divisor of $2a$ and $b$. Since $(a, b) = 1$ this is 1 ($b$ odd) or 2 ($b$ even).*

**Proof** of the corollary: in (i) with $b$ a multiple of 4, take $n_1 = \frac{1}{4}b$ and since $a - \frac{1}{2}b$ is odd, write it as $2n_2 + 1$, giving $\alpha = 0$. The converse is clear: if $\alpha = 0$ then clearly $b$ is even and writing $b = 2b_1$ we quickly deduce $b_1$ is even.
For (ii) with $b$ even but not a multiple of 4, write $\frac{1}{2}b = 2n_1 - 1$ and since $a - \frac{1}{2}b$ is even write it as $2n_2$, making $\alpha = \pi$. Conversely if $\alpha = \pi$ is a possible value then we find $2(2n_1 - 1)a = b(2n_1 + 2n_2 - 1)$ so

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that $b$ is even, but $b$ cannot be a multiple of 4 by (i).

For (iii) note that the values of $(4a-2b)n_1-2bn_2$ are exactly the multiples of $(4a-2b, 2b) = 2(2a-b, b) = 2(2a, b)$. Thus the numerator of the formula for $\alpha$ can be reduced to the remainder on dividing $-b$ by this GCD and not to any smaller value in modulus. The denominator of the formula for $\alpha$ is constant. $\square$

For example, Figure 8, the left pair, shows $m = \frac{7}{6}$ in (15) and $m = 2 \times \frac{7}{6} - 1 = \frac{2}{3}$ in (2), illustrating (ii) of the Corollary. Both examples in the figure illustrate (iii).
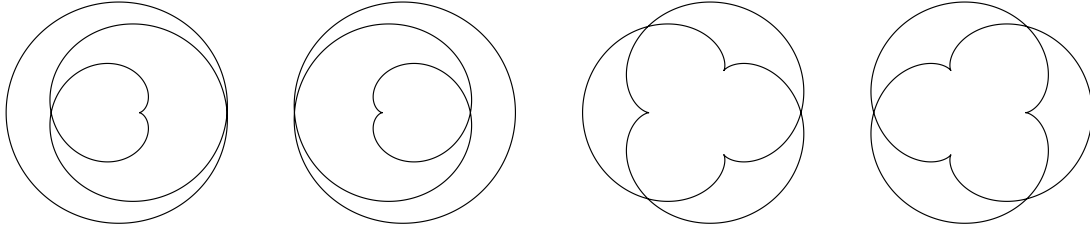


Figure 8: Left pair: An "$r = \infty$" envelope (left) for $m = \frac{7}{6}$ in which the denominator is even but not a multiple of 4, and a 1-circle envelope (right) for $m = \frac{4}{3}$, identical by a half-turn as in Corollary 3.7. Right pair: taking $n_1 = 1, n_2 = 0$ in the formula for $\alpha$ in Proposition 3.6 gives a rotation of $\frac{1}{3}\pi$ for the $r = \infty$ envelope with $m = \frac{7}{10}$ (left) and the 1-circle envelope with $m = \frac{2}{5}$ (right).

# References

[1] J.W.Bruce and P.J.Giblin, *Curves and Singularities*, Second edition, Cambridge University Press 1992.