MSc Main Dissertation: Surfaces of Constant Width

Andrew David Irving

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Contents

1 Introduction

In this project, we explore surfaces of constant width, building on previous work on constant width in 2 dimensions, although little knowledge is assumed. An obvious example of a surface of constant width is a sphere, but there are many others as we shall see.

Section 2 outlines the fundamental definitions and concepts, introducing the idea of defining a surface in terms of a support function, which is at the very core of this project. The condition for constant width is presented and we make some early conjectures as to the most general form our support functions could take.

In section 3, we look at some simple examples of smooth surfaces of constant width. With graphical displays, it is hoped that this will make the concepts outlined in section 2 seem less abstract, improving the understanding of the reader.

In section 4, we look in depth at curvature in 3 dimensions. This is a vitally important topic, introducing principal curvatures, Gauss cuvature and mean curvature, all of which have great significance in the proofs of many theorems both in this, and subsequent sections.

In section 5, perhaps the most important part of this work, we take a first look at the shape operator and its many applications. We try to use the shape operator in order to modify the support function, that is, we wish to ensure that the surface corresponding to our suggested support function is smooth everywhere.

Section 6 pursues further the work on smoothness in section 5, by examining more closely the parts played by the constant terms in our support function, which also have constraints if we are to find smooth surfaces.

Section 7 asks whether it would be possible for a surface, produced by our chosen support function, to have cuspidal edges or swallowtails on the $x-$axis.

Finally, section 8 has the conclusions and possibilities for further work, whilst section 9 features all the Maple programmes I have used in the making of graphics and lengthly calculations for this project (with annotations).

2 Basic Ideas

My main reference for this section was [K].

2.1 Support Plane & Support Function

Consider a surface T, parametrised by longitude θ and colatitude ϕ (spherical coordinates) where we take an arbitrary tangent plane to our surface at a point $\mathbf{x}$, which we shall call the support plane $l = l(\theta, \phi)$. If we then drop a perpendicular line from $l(\theta, \phi)$ such that it passes through the origin, the length of this normal is called the support function $h = h(\theta, \phi)$.

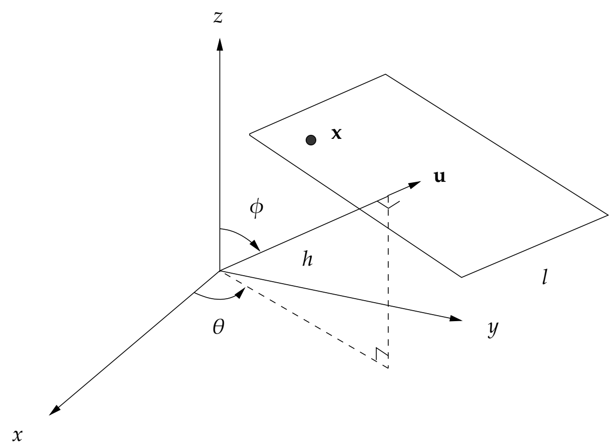

Figure 1: Surface with support function $h(\theta, \phi)$ and support plane $l(\theta, \phi)$.

Here θ is the angle between the x, z-plane (where $x, z > 0$ and $y = 0$) and the normal to $l(\theta, \phi)$ and ϕ is the angle that this makes with the north pole. Support function h and the spherical coordinates θ, ϕ are related to cartesian coordinates x, y, z by,

$$
\begin{aligned}
x &= h \cos\theta \sin\phi && (1) \\
y &= h \sin\theta \sin\phi && (2) \\
z &= h \cos\phi && (3)
\end{aligned}
$$

where $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$. So our position vector in the direction of our normal is clearly (x, y, z) and therfore the unit vector $\mathbf{u}$, as indicated on our diagram, is given by dividing this by its magnitude h throughout. We think of (θ, ϕ) as defining a point $(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$ on the unit sphere.

It is important to note that when $\sin\phi = 0$ (at the poles of our 2-sphere) our surface is not well-defined, so we think of our surface T minus the poles at all times, unless otherwise stated.

The key here is that we are defining a surface by its normals, so if we are given the support function, we can find the equation of the surface using the fact that the surface is the envelope of its tangent planes. The normal to $l\left(\theta,\phi\right)$ here is $h\mathbf{u}$ and so we can say that any tangent vector, that is, a vector in the support plane, will satisfy the equation,

$$(\mathbf{x} - h\mathbf{u}).\mathbf{u} = 0$$

but we know that $\mathbf{u}.\mathbf{u} = |\mathbf{u}|^2 = 1$ so we have that,

$$h = \mathbf{x}.\mathbf{u} \tag{4}$$

is satisfied for all tangent vectors. Therefore the equation of our tangent plane is given by,

$$x\cos\theta\sin\phi + y\sin\theta\sin\phi + z\cos\phi = h\left(\theta,\phi\right)$$

and so, for our family of tangent planes, we want $F\left(x,y,z,\theta,\phi\right) = 0$ where,

$$\begin{aligned} F &= x\cos\theta\sin\phi + y\sin\theta\sin\phi + z\cos\phi - h && (5)\\ \frac{\partial F}{\partial\theta} &= -x\sin\theta\sin\phi + y\cos\theta\sin\phi - h_\theta && (6)\\ \frac{\partial F}{\partial\phi} &= x\cos\theta\cos\phi + y\sin\theta\cos\phi - z\sin\phi - h_\phi && (7)\end{aligned}$$

such that subscripts denote partial derivatives, as they shall throughout unless otherwise stated. The envelope of these tangents is defined as,

$$\mathcal{D}_F = \left\{\mathbf{x} :\ \exists\theta,\phi \text{ with } F = \frac{\partial F}{\partial\theta} = \frac{\partial F}{\partial\phi} = 0\right\}$$

where $\mathbf{x} = (x,y,z)$. Well, if we set equations (5), (6) and (7) equal to 0, then we find that (5) $\times\cos\theta$ − (6) $\times\sin\theta$ gives us x, (5) $\times\sin\theta$ + (6) $\times\cos\theta$ gives y and (5) $\times\cos\phi$ − (7) $\times\sin\phi$ gives z.

Proposition 2.1 *Our surface, defined in terms of our support function $h\left(\theta,\phi\right)$, can be parametrised by,*

$$\begin{aligned} x &= -\frac{h_\theta\sin\theta}{\sin\phi} + h\cos\theta\sin\phi + h_\phi\cos\theta\cos\phi && (8)\\ y &= \frac{h_\theta\cos\theta}{\sin\phi} + h\sin\theta\sin\phi + h_\phi\sin\theta\cos\phi && (9)\\ z &= h\cos\phi - h_\phi\sin\phi. && (10)\end{aligned}$$

So now we would like to know when our surface is an immersion (not singular) and we can use a method known from MATH 443. First consider the following definition of singular points.

Definition
A point **p** on a surface is called regular if the Jacobian of **x** has maximal rank at **p**. Otherwise **p** is called a singular point and the surface is an immersion if it has no singular points.

The parametrisation $\left(\theta, \phi\right)$ of the 2-sphere S^2 is not regular at the poles $\phi = 0, \pi$ so we cannot deduce results about the regularity of T at the poles at this stage. Our map here is $S^2 \to \mathbb{R}^3$ so we consider a 3×2 jacobian of the form,

$$J = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}$$

and our surface is then an immersion if the rank of $J = 2$, note that the rank of $J < 2$ if, and only if, all 2×2 minors are 0. The rank being less than 2 implies that our surface is not regular because it would mean that the cross product of the columns equals 0, i.e. the columns would not be linearly independent.

So in our case, we find that the Jacobian matrix entries are, letting $U = \sin\theta, V = \cos\theta, Y = \sin\phi$ and $Z = \cos\phi$ in all that follows.

$$\frac{\partial x}{\partial \theta} = \frac{-\left(h_{\theta\theta}U + h_\theta V\right)}{Y} + \left(h_\theta V - hU\right)Y + \left(h_{\theta\phi}V - h_\phi U\right)Z \tag{11}$$

$$\frac{\partial x}{\partial \phi} = \frac{-\left(h_{\theta\phi}Y - h_\theta Z\right)U}{Y^2} + \left(h + h_{\phi\phi}\right)VZ \tag{12}$$

$$\frac{\partial y}{\partial \theta} = \frac{\left(h_{\theta\theta}V - h_\theta U\right)}{Y} + \left(h_\theta U + hV\right)Y + \left(h_{\theta\phi}U + h_\phi V\right)Z \tag{13}$$

$$\frac{\partial y}{\partial \phi} = \frac{\left(h_{\theta\phi}Y - h_\theta Z\right)V}{Y^2} + \left(h + h_{\phi\phi}\right)UZ \tag{14}$$

$$\frac{\partial z}{\partial \theta} = h_\theta Z - h_{\theta\phi}Y \tag{15}$$

$$\frac{\partial z}{\partial \phi} = -\left(h + h_{\phi\phi}\right)Y. \tag{16}$$

Note that $Y = \sin\phi$ appears in the denominator of all fraction terms and this goes back to what we said about not including the poles for now. Then our three 2×2 minors,

$$\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix}, \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}, \begin{vmatrix} \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

are equal to (17), (18) and (19) respectively.

$$\frac{\left(h_{\theta\phi}Y - h_\theta Z\right)}{Y}\left(h_{\theta\phi}\frac{Z}{Y} - h_\theta\frac{1}{Y^2} + h_\theta\right) - Z\left(h + h_{\phi\phi}\right)\left(hY + h_\phi Z + h_{\theta\theta}\frac{1}{Y}\right) \tag{17}$$

$$\left(h + h_{\phi\phi}\right)\left(h_{\theta\theta}U + hUY^2 + h_\phi UYZ\right) + \frac{\left(h_{\theta\phi}Y - h_\theta Z\right)U}{Y}\left(h_\theta\frac{Z}{Y} - h_{\theta\phi}\right) \tag{18}$$

$$-V\left(h + h_{\phi\phi}\right)\left(h_\phi YZ + hY^2 + h_{\theta\theta}\right) + \frac{\left(h_{\theta\phi}Y - h_\theta Z\right)V}{Y}\left(h_{\theta\phi} - h_\theta\frac{Z}{Y}\right) \tag{19}$$

Well, it is not obvious when our 3 minors will simultaneously equal 0 (if indeed that is possible) so let us consider some values for θ and ϕ. If we let $\theta_0 = 0$ and $\phi_0 = \frac{\pi}{2}$ then we will have that $U = 0, V = 1, Y = 1, Z = 0$ and by substituting these into (17), (18) and (19) we find that both (17) and (18) equal 0 but that (19) equals,

$$-\left(h + h_{\phi\phi}\right)(h + h_{\theta\theta}) + h_{\theta\phi}^2$$

and we can conclude from this that our surface is not an immersion at $\left(0, \frac{\pi}{2}\right)$ if,

$$\begin{vmatrix} \left(h + h_{\phi\phi}\right) & h_{\theta\phi} \\ h_{\theta\phi} & (h + h_{\theta\theta}) \end{vmatrix} = 0$$

noting that we have multiplied through by -1 in order to rewrite this condition in the following form,

$$det\,(H + hI) = 0$$

where H is the Hessian matrix and I is the identity matrix here.

Proposition 2.2 *Our surface is singular at* $\left(0, \frac{\pi}{2}\right)$ *if and only if* $-h\left(\theta, \phi\right)$ *is an eigenvalue of the Hessian matrix,*

$$H(h) = \begin{pmatrix} h_{\theta\theta} & h_{\theta\phi} \\ h_{\theta\phi} & h_{\phi\phi} \end{pmatrix}$$

which is symmetric and so its eigenvalues are real.

It is interesting to note that the Hessian H depends only on second derivatives of h, so the eigenvalues λ_1 and λ_2 are not affected by the addition of a constant to our support function. Let us see what effect, if any, the adding of a constant to our support function has. For example, if we now let our support function h take the form,

$$h = h_0 + k$$

where k is some constant and h_0 is another support function, then we can use propostion 2.2 to derive a condition on k which will preserve the smoothness of T at $\left(\theta, \phi\right) = \left(0, \frac{\pi}{2}\right)$. Proposition 2.2 says that T is smooth at $\left(0, \frac{\pi}{2}\right)$ if,

$$h\left(0, \frac{\pi}{2}\right) = h_0\left(0, \frac{\pi}{2}\right) + k \neq \lambda_1 \text{ or } \lambda_2$$

where λ_1, λ_2 are the eigenvalues of H and thus, the following proposition is intuitive.

Proposition 2.3 *Our surface is smooth at $\left(0, \frac{\pi}{2}\right)$ if the constant term in our support function h satisfies the condition that k does not equal k_1 or k_2 where,*

$$k_1 = -\lambda_1 - h_0\left(0, \frac{\pi}{2}\right) \tag{20}$$

$$k_2 = -\lambda_2 - h_0\left(0, \frac{\pi}{2}\right) \tag{21}$$

i.e. if we make the magnitude of our constant k sufficiently large, our surface will be smooth at $\left(0, \frac{\pi}{2}\right)$.

2.2 Surfaces of Constant Width

The width of a closed, convex surface in a specified direction is determined by the distance between 2 parallel tangent planes and if the distance between all parallel tangent planes is equal, then we have a surface of constant width (SCW). Our condition then for a SCW is,

$$h\left(\theta, \phi\right) + h\left(\theta + \pi, \pi - \phi\right) = k \tag{22}$$

since the points with parameters $\left(\theta, \phi\right)$ and $\left(\theta + \pi, \pi - \phi\right)$ on the 2-sphere S^2 are diametrically opposite, so the support planes given by these parameter values in $\mathbb{R}^3$ are parallel. Note here that k is a constant equal the width w of our SCW.

Proposition 2.4 *Chords joining surface points on $l\left(\theta, \phi\right)$ and $l\left(\theta + \pi, \pi - \phi\right)$ on a SCW will be common normals to both $l\left(\theta, \phi\right)$ and $l\left(\theta + \pi, \pi - \phi\right)$.*

Proof. From previously, we have that our surface can be parameterised by (8), (9) and (10), let these represent the point of contact between the surface and $l\left(\theta,\phi\right)$. From these we can derive the surface points on $l\left(\theta+\pi,\pi-\phi\right)$ as;

$$\begin{aligned} x' &= \frac{h'_\theta \sin\theta}{\sin\phi} - h'\cos\theta\sin\phi + h'_\phi \cos\theta\cos\phi && (23)\\ y' &= -\frac{h'_\theta \cos\theta}{\sin\phi} - h'\sin\theta\sin\phi + h'_\phi \sin\theta\cos\phi && (24)\\ z' &= -h'\cos\phi - h'_\phi \sin\phi && (25)\end{aligned}$$

where $'$ here denotes that the variable is now being measured at $\left(\theta,\phi\right)=\left(\theta+\pi,\pi-\phi\right)$, e.g. $x'=x\left(\theta+\pi,\pi-\phi\right)$, etc. Note also that we have replaced $\sin(\theta+\pi)$ by $-\sin\theta$, $\cos(\theta+\pi)$ by $-\cos\theta$, $\sin\left(\pi-\phi\right)$ by $\sin\phi$ and $\cos\left(\pi-\phi\right)$ by $-\cos\phi$.

So the direction of the chord joining our parallel tangent planes $l\left(\theta,\phi\right)$ and $l\left(\theta+\pi,\pi-\phi\right)$ is given by $(x'-x, y'-y, z'-z)$ where,

$$\begin{aligned} x'-x &= \frac{\sin\theta}{\sin\phi}\left(h_\theta + h'_\theta\right) - \cos\theta\sin\phi\,(h+h') + \cos\theta\cos\phi\left(-h_\phi + h'_\phi\right) && (26)\\ y'-y &= -\frac{\cos\theta}{\sin\phi}\left(h_\theta + h'_\theta\right) - \sin\theta\sin\phi\,(h+h') + \sin\theta\cos\phi\left(-h_\phi + h'_\phi\right) && (27)\\ z'-z &= -\cos\phi\,(h+h') - \sin\phi\left(-h_\phi + h'_\phi\right) && (28)\end{aligned}$$

but we can use our condition for a SCW, together with its partial derivatives, to simplify equations (26), (27) and (28), i.e. a SCW will satisfy the following conditions.

$$\begin{aligned} h\left(\theta,\phi\right) + h\left(\theta+\pi,\pi-\phi\right) &= k && (29)\\ h_\theta\left(\theta,\phi\right) + h_\theta\left(\theta+\pi,\pi-\phi\right) &= 0 && (30)\\ h_\phi\left(\theta,\phi\right) - h_\phi\left(\theta+\pi,\pi-\phi\right) &= 0 && (31)\end{aligned}$$

By substituting these into (26), (27) and (28), it is clear to see that,

$$(x'-x, y'-y, z'-z) = -k\left(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\right)$$

which is parallel to our unit normal $\mathbf{u}$ (see Figure 1). Therefore, the chords joining $l\left(\theta,\phi\right)$ to $l\left(\theta+\pi,\pi-\phi\right)$ must be orthogonal to both support planes as well, hence they are binormals. ∎

This is of particular significance when we consider definitions of the centre symmetry set (CSS) and focal surface of our SCW.

Definition
The envelope of chords joining points of contact on parallel tangent planes is called the centre symmetry set (CSS) of a surface.

Definition
The envelope of normal lines to our surface is called the focal surface and is analogous to the evolute in $\mathbb{R}^2$.

From Proposition 2.4, we can conclude that, for a SCW, the envelope of chords joining parallel tangent points is equal to the envelope of binormals, that is, the CSS is equal to a double cover of the focal surface.

2.3 Choosing a support function

When looking for a support function which will produce a smooth SCW, there are certain rules by which we must abide. The first thing to say is that we are parametrising by longitude and latitude which means that our parametrisation will be singular at the poles, i.e. referring to equations (1), (2) and (3), we see that for $\phi = 0$ we have that,

$$(x, y, z) = (0, 0, h(\theta, 0))$$

and for $\phi = \pi$, in a similar way (since $\sin(0) = \sin(\pi) = 0$) we find,

$$(x, y, z) = (0, 0, -h(\theta, \pi)) .$$

The problem with which we are presented here is that θ is arbitrary at the poles, i.e. we can see that,

$$(\theta, 0) \quad \rightarrow \quad (0, 0, h(\theta, 0)) \tag{32}$$
$$(\theta, \pi) \quad \rightarrow \quad (0, 0, -h(\theta, \pi)) \tag{33}$$

for any θ. Our model will only work on the basis that, for a given direction, there exists a unique normal. As a result, we want all θ to map to the same points at the poles, where $\phi = 0$ or π. We need to ensure that, neither $h(\theta, 0)$, nor $h(\theta, \pi)$ depend on θ, which gives us our first condition on our support function;

$$h(\theta, 0) = c_1 \ \& \ h(\theta, \pi) = c_2$$

where c_1 and c_2 are constants. A self-evident consequence of this is our second condition, which is as follows,

$$\frac{\partial h}{\partial \theta}(\theta, 0) = \frac{\partial h}{\partial \theta}(\theta, \pi) = 0.$$

Our parametrisation also provides another clear problem, since as we can see from our parametrisation in equations (8), (9) and (10) we need to impose a third condition, that is,

$$\frac{h_\theta}{\sin\phi}$$

must be finite, but this is a problem at the poles since $\sin\phi = 0$ here. One of the most obious ways to solve this problem is to ensure that h_θ is a multiple of $\sin\phi$ so that the numerator and denominator will cancel each other out in this term. So we want something of the form,

$$h\left(\theta,\phi\right) = f\left(\theta,\phi\right)\sin\phi + g\left(\phi\right)$$

where f is a function of θ and ϕ and g is a function of ϕ only. Differentiating with respect to θ gives us h_θ which satisfies our second and third conditions, given f_θ is finite.

$$h_\theta = f_\theta\left(\theta,\phi\right)\sin\phi \Longrightarrow \frac{h_\theta}{\sin\phi} = f_\theta\left(\theta,\phi\right)$$

Developing this through various trials, we conjecture that a support function of the form,

$$h\left(\theta,\phi\right) = \left(p\left(\theta,\phi\right) + r\left(\phi\right)\right)\sin^2\phi + k$$

is the most general, where $p\left(\theta,\phi\right) = a\cos\theta, r\left(\phi\right) = b\cos\phi$ and k is a constant. Our SCW condition (see (22)) is satisfied here, where the width equals $2k$ however, how can we tell whether or not this SCW will be smooth? Perhaps if we look at some examples (see figure 2). These would certainly appear to be smooth, but it is difficult to tell, particularly at the poles. Our next section looks at a simpler example, which we know gives a smooth SCW.

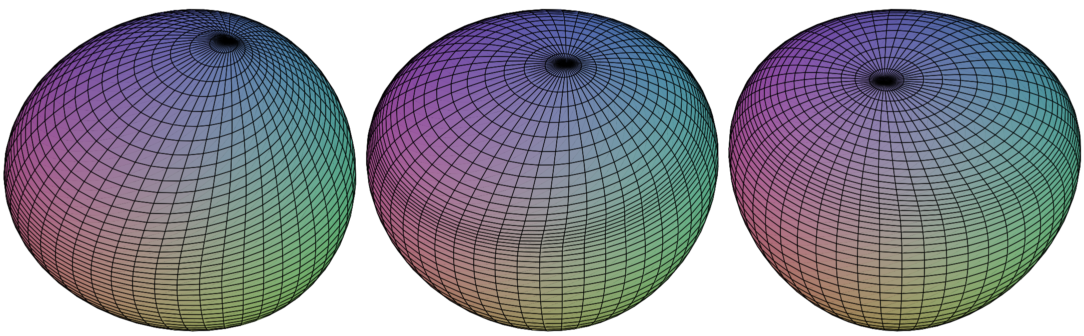

Figure 2: Surfaces with support functions $h = \sin^2\phi\left(12\cos\theta + 3\cos\phi\right) + 30, h = \sin^2\phi\left(5\cos\theta + 8\cos\phi\right) + 26$ and $h = \sin^2\phi\left(2\cos\theta + 9\cos\phi\right) + 28$ respectively.

3 Examples of smooth surfaces of constant width

For this section, my main reference was [Fis].

3.1 Example 1

One can think of this as almost a trivial example in that our support function will be independent of θ, thus removing our most problematic condition, that is that the

$$\frac{h_\theta}{\sin\phi}$$

term, which must be finite. If we consider a support fuction h, which does not depend on θ, then $h_\theta = 0$ and we no longer have to concern ourselves with this term. Choosing h such that it depends only on ϕ effectively allows us to find a curve of constant width (CCW) in the $x, y-$plane and then, as ϕ varies, this CCW will revolve through 180 degrees, thus creating a SCW.

From previous work, we found that an example of a CCW in $\mathbb{R}^2$ could take a support function of the form,

$$h(t) = P\cos Qt + R$$

where $P > 0$ and R were constants and Q was some odd integer. So an example of a SCW in $\mathbb{R}^3$ may take the form,

$$h(\phi) = P\cos Q\phi + R$$

where P, Q and R are as before. For a SCW without singularities however, there are some additional constraints.

Proposition 3.1 *A SCW with a support function of the form $h(\phi) = P\cos Q\phi + R$ where $R > P > 0$ are constants and Q is some odd integer, has no singularities if $|P| < \frac{R}{Q^2-1}$.*

Proof. From Proposition 2.2 we know that $h(\theta, \phi)$ will produce a singular surface if $-h$ is equal to one of the eigenvalues of the Hessian matrix. This condition however becomes greatly simplified for $h = h(\phi)$ since $h_{\theta\theta} = h_{\theta\phi} = 0$ so for our support function h,

$$H(h) = \begin{pmatrix} 0 & 0 \\ 0 & -PQ^2\cos Q\phi \end{pmatrix}$$

i.e. our surface is only singular if $h = 0$ or $h = -h_{\phi\phi}$. Well, if $h = 0$ then this implies that,

$$\cos Q\phi = -\frac{R}{P}$$

but this does not apply to our case since $R > P > 0$ and we know that $-1 \leq \cos x \leq 1$. So let us look at the situation whereby $h = -h_{\phi\phi}$, i.e. in our case,

$$P \cos Q\phi + R = PQ^2 \cos Q\phi$$

which implies that our surface is singular if, and only if,

$$\cos Q\phi = \frac{R}{P\,(Q^2 - 1)}.$$

However, again we know that $-1 \leq \cos x \leq 1$, so our surface will not be singular if,

$$-R < P(Q^2 - 1) < R$$

which simplifies to give us our condition on P such that our SCW is smooth;

$$|P| < \frac{R}{(Q^2 - 1)}.$$

∎

So here are some examples of smooth surfaces of constant width using support functions taking the afore-mentioned form and obeying the condition in Proposition 3.1.

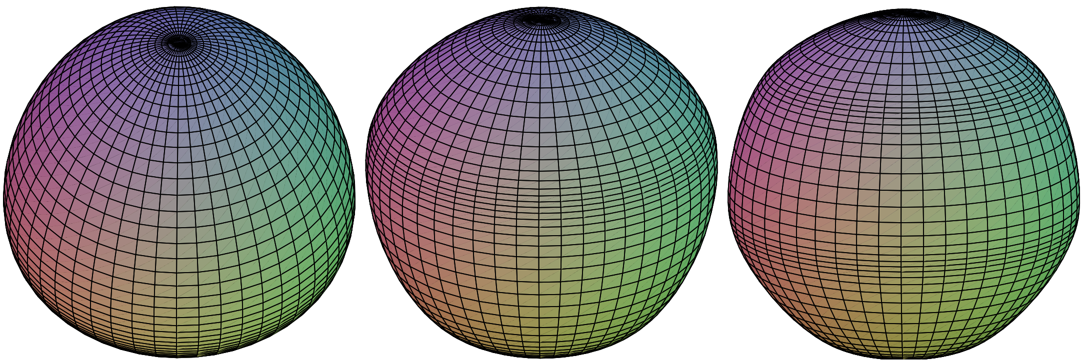

Figure 3: Smooth surfaces of constant width given by $h\left(\phi\right) = 1 + \frac{1}{16}\cos 3\phi$, $h(\phi) = 1 + \frac{1}{48}\cos 5\phi$ and $h(\phi) = 1 + \frac{1}{96}\cos 7\phi$ respectively.

These, and those in figure 2, were constructed using Maple (see Appendix 1). It is note worthy that the condition for smoothness is greatly simplified here, as shown.

Proposition 3.2 *A SCW given by a non-zero θ-independent support function $h = h(\phi)$, has singularities* $\iff$ $h + h_{\phi\phi} = 0$.

4 Curvature

My main reference for this section was [K].

4.1 Principal Curvatures

Curvature at points on a surface is measured by what we call principal curvatures. These are the maximum and minimum values of (normal) curvature of the plane cuves found by taking normal sections to our surface (as explained in the following).

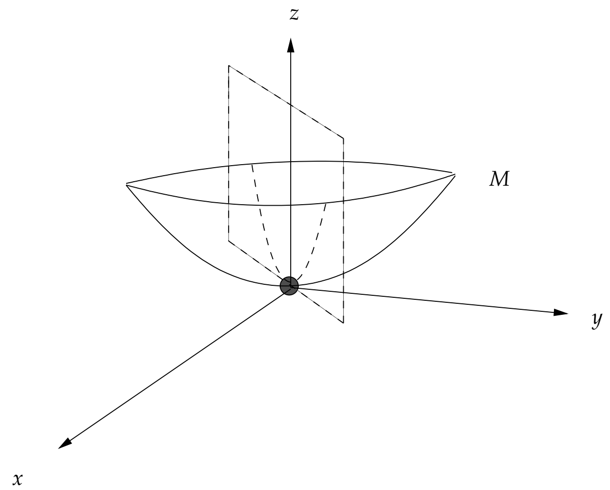

Figure 4: Normal section and plane curve of M represented by dashed lines.

Let us consider a surface in Monge form, i.e. $M = (x, y, z)$ where,

$$z = f(x, y) = \frac{1}{2}\left(ax^2 + 2bxy + cy^2\right) + H.O.T.$$

then we can translate our surface such that the $x, y-$plane ($z = 0$) is a tangent plane to M (since tangency is invariant under translation in euclidean space) and the $z-$axis now represents a normal at the origin. We can express z in the following form,

$$z = \frac{1}{2}\left[\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right] + \ldots$$

where the matrix $II = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is of great significance as we shall see. Let us take normal sections of M at $\mathbf{0}$, that is sections by planes containing the normal at $\mathbf{0}$ (the $z-$axis). We then refer to the intersections of these plane sections with M as plane curves.

Theorem 4.1 *The largest and smallest curvatures of these plane curves at* $\mathbf{0}$ *are exactly the eigenvalues of the matrix II and these eigenvalues are equal to the principal curvatures.*

Proof. First of all, let us make this easier by showing that we can let $b = 0$ in our definition of $z = f(x, y)$. We can do this by rotating the $x, y-$axes anticlockwise through an angle of θ about the origin using the linear transformation $\mathbf{x}' = A\mathbf{x}$. So here we change our basis using,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and we want to rearrange for $\mathbf{x}$ by left multiplying on each side by the inverse of A where,

$$A^{-1} = \frac{1}{detA} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

but the determinant of A equals 1 here, so we have that,

$$x = x'\cos\theta + y'\sin\theta \tag{34}$$

$$y = -x'\sin\theta + y'\cos\theta. \tag{35}$$

By substiuting these into z we can find the coefficient of $x'y'$ (remember we want to show that b can equal 0). The coefficient of $x'y'$ in z is,

$$\frac{1}{2}\left(2a\cos\theta\sin\theta + 2b\left(\cos^2\theta - \sin^2\theta\right) - 2c\left(\sin\theta\cos\theta\right)\right)$$

but this can be simplified, using trig identities, to

$$(a-c)\frac{1}{2}\sin 2\theta + b\cos 2\theta$$

and our question is: can this equal 0 and still give us a solution? Setting this coefficient of $x'y'$ equal to 0 simplifies to,

$$\tan 2\theta = \frac{2b}{(c-a)}$$

and in our coordinates, we have taken $0 \leq \theta < 2\pi$ so then for $0 \leq 2\theta < 4\pi$ we have solutions given by $2\theta, 2\theta + \pi, 2\theta + 2\pi$ and $2\theta + 3\pi$. Hence, when $0 \leq \theta < 2\pi$ we have 4 solutions (just dividing the previous by 2);

$$\text{(i) } \theta \text{ (ii) } \theta + \frac{\pi}{2} \text{ (iii) } \theta + \pi \text{ (iv) } \theta + \frac{3\pi}{2}.$$

From this we can conclude that z in M may take the form,

$$f(x, y) = \frac{1}{2}\left(ax^2 + cy^2\right) + \ldots$$

and note that, with b equal to 0, it is now clear that the eigenvalues of II are

$$\lambda_1 = a \ \& \ \lambda_2 = c$$

since II is now diagonal (and therefore the leading diagonal entries are the eigenvalues). Remember that we claimed λ_1 and λ_2 would be equal to the principal curvatures κ_1 and κ_2 (at the origin), so we must first find the equation of our normal sections.

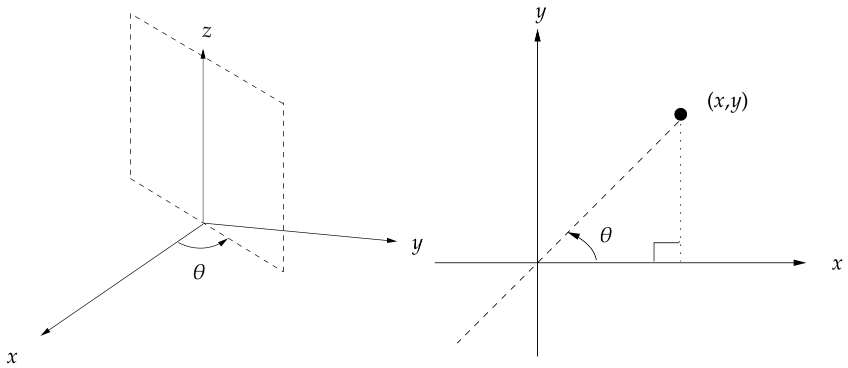

Figure 5: Normal sections with birds-eye view of it on the right.

For the equation of our plane, we consider the right hand side of Figure 5 and we have,

$$\frac{y}{x} = \tan\theta$$

but we know that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ so our nomal sections can be described by the equation,

$$y\cos\theta - x\cos\theta = 0.$$

Let us imagine one of our plane sections (see Figure 6), this is a 2-dimensional object for which the $z-$axis is its vertical axis, so let its horizontal axis be called the $u-$axis and let the plane curve (intersection of plane section and M) be called γ. We need to define unit normals for these axes and, for the $z-$axis, since $x = y = 0$, this is obviously $(0, 0, 1)$.

It is perhaps not so obvious in the case of the $u-$axis but we know that $z = 0$ here, so our axis lies entirely in the $x, y-$plane and therefore it is just like finding the unit vector for a standard planar curve, i.e. our unit vector in the u direction is

$$(\cos\theta, \sin\theta, 0).$$

Therefore, where we have a vertical axis $z = z(u)$ and a horizontal axis $u = u(\theta)$, our plane curve can be parametrised by,

$$\gamma(u) = u(\cos\theta, \sin\theta, 0) + z(0, 0, 1) = (u\cos\theta, u\sin\theta, z).$$

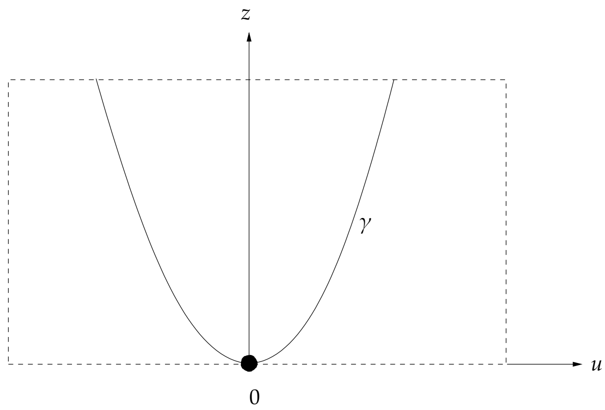

Figure 6: Normal (plane) section together with plane curve γ.

For the (normal) curvature of this plane curve at $\mathbf{0}$, where $\gamma(u) = (u, z(u))$, we can use our standard equation for curvature, that is, when we have a plane curve α parametrised by $\alpha(t) = (X(t), Y(t))$, curvature κ is given by,

$$\kappa = \frac{X'Y'' - X''Y'}{(X'^2 + Y'^2)^{\frac{3}{2}}}$$

where $' = \frac{d}{dt}$. So in our case, the curvature at $\mathbf{0}$ is,

$$\kappa(0) = \frac{z''(0)}{(1 + z'(0))^{\frac{3}{2}}}$$

where $' = \frac{d}{du}$ but the $(z = 0)-$plane is a tangent plane to M at $\mathbf{0}$ which means that, by definition $z'(0) = 0$, hence $\kappa(0) = z''(0)$. So we need to know what $z''(0)$ equals, let us remind ourselves that,

$$z = \frac{1}{2}\left(ax^2 + cy^2\right) + \dots$$

but for our plane section we have that our plane curve γ took $x = u\cos\theta$ and $y = u\sin\theta$ so let us substitute these into z as follows.

$$z = \frac{1}{2}\left(au^2\cos^2\theta + cu^2\sin^2\theta\right) + \dots$$

Well, we want $z''(0)$ which means that any higher order terms would equal 0, so we need not worry about these. We find that

$$z''(0) = \kappa(0) = a\cos^2\theta + c\sin^2\theta \tag{36}$$

and we want to find the maximum and minimum of this. We start by replacing $\sin^2\theta$ by $1-\cos^2\theta$ since then,

$$\kappa(0) = (a-c)\cos^2\theta + c$$

and we know that cosine has a maximum value of 1 and a minimum of -1 (but since here we consider only $\cos^2\theta$, the minimum will occur at 0). Well, setting θ equal to 0 (so $\cos^2\theta = 1$) and θ equal to $\frac{\pi}{2}$ (so $\cos^2\theta = 0$) we find that,

$$\kappa_1(0) = a \;\&\; \kappa_2(0) = c$$

respectively. We don't actually know which one of these is the maximum and which is the minimum, nevertheless the eigenvalues yielded by the II matrix are equal to the principal curvatures of M at the origin. ∎

In addition, we observe that equation (4.1) leads to the well known fact about normal curvatures (taken from MATH349 notes).

Proposition 4.1 *The normal curvature κ_n in the direction of a line on our surface, which makes an angle θ with the (principal) direction of κ_1, is given by*

$$\kappa_n = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta.$$

See Figure 5 to better understand our construction here (the x and y axes would represent the (principal) directions of κ_1 and κ_2 respectively here). Finally, let us prove the following theorem, taken from [CG], before which we need to define Gaussian curvature.

Definition
The Gaussian curvature at a point $\mathbf{p}$ is equal to the product of the principal curvatures κ_1, κ_2 at that point.

Theorem 4.2 *Let κ and g be the normal and Gaussian curvature respectively of a point $\mathbf{p}$ which corresponds to (θ, ϕ) on the 2-sphere S^2. Let $\mathbf{p}$ be a point on a surface with constant width w then,*

$$\frac{\kappa}{g} + \frac{\kappa'}{g'} = w \tag{37}$$

where $'$ denotes measurements taken at a point $\mathbf{p}'$ on the SCW, which corresponds to $(\theta+\pi, \pi-\phi)$ on S^2.

Proof. First of all, let us replace the numerators in the left hand side of (37) using proposition 4.1 and the denominators using definition 4.1. We would like to show that E is equal to the width of our SCW,

$$E = \frac{\kappa_1\cos^2\theta + \kappa_2\sin^2\theta}{\kappa_1\kappa_2} + \frac{\kappa_1'\cos^2\theta + \kappa_2'\sin^2\theta}{\kappa_1'\kappa_2'}$$

where θ is the angle described in proposition 4.1. Now let us separate our fractions,

$$E = \frac{\cos^2\theta}{\kappa_2 + \kappa_2'} + \frac{\sin^2\theta}{\kappa_1 + \kappa_1'} = \cos^2\theta\left(\rho_2 + \rho_2'\right) + \sin^2\theta\left(\rho_1 + \rho_1'\right)$$

where ρ_i, ρ_i' for $i = 1, 2$ are the principal radii of curvature at $\mathbf{p}$ and $\mathbf{p}'$ respectively. According to theorem 5.2 this then becomes,

$$E = \cos^2\theta\,(w) + \sin^2\theta\,(w) = w$$

as required.

5 Shape Operator

For this section, my main reference was [O'N].

The shape operator is a linear map from the tangent plane of a surface to itself measuring the shape of a curve and it is equal to the negative of the covariant derivative.

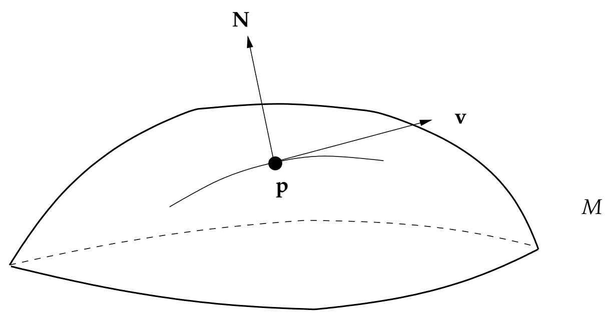

Figure 7: Surface M to which $\mathbf{N}$ is normal and curve to which $\mathbf{v}$ is tangent.

Definition
Let $\mathbf{p}$ be a point on a surface M, then for each tangent vector $\mathbf{v}$ to M at $\mathbf{p}$, we can find the covariant derivative

$$\nabla_{\mathbf{v}}\mathbf{N}.$$

This measures the rate of change of M in the direction of $\mathbf{v}$ and, since the tangent plane to M at $\mathbf{p}$ contains all tangent vectors to M at $\mathbf{p}$, we can that $\nabla_{\mathbf{v}}\mathbf{N}$ tells us how the tangent planes change in the direction of $\mathbf{v}$. It is in this way that $\nabla_{\mathbf{v}}\mathbf{N}$ tells us about the shape of M. Hence the shape operator S of M at $\mathbf{p}$, relative to a suitable basis (same as tangent plane to M) is,

$$S_{\mathbf{p}} = -\nabla_{\mathbf{v}} N.$$

So we would like to find the shape operator of our surface as it has many special properties. Our model is set up such that the surface is defined in terms of its normals and this is a special quality of which we can make great use. We want to find a basis for a curve on our surface, but we already know that our unit normal is,

$$\mathbf{u} = \left(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\right)$$

and so we can find $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$ explicitly,

$$\mathbf{u}_\theta = \left(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0\right) \quad (38)$$

$$\mathbf{u}_\phi = \left(\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\right). \quad (39)$$

Here we are using the fact that if we have a unit normal $\mathbf{N}$ then $\mathbf{N}.\mathbf{N} = 1$ and, when differentiated with respect to its parameter, this gives $\mathbf{N}.\mathbf{N} = 0$ which says that $\mathbf{N}$ and $\mathbf{N}'$ will always be orthogonal. In our case we see that $\mathbf{u}$, $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$ are mutually perpindicular since,

$$\begin{aligned} \mathbf{u}.\mathbf{u}_\theta &= -\sin\theta\cos\theta\sin^2\phi + \sin\theta\cos\theta\sin^2\phi = 0 && (40)\\ \mathbf{u}.\mathbf{u}_\phi &= \cos^2\theta\sin\phi\cos\phi + \sin^2\theta\sin\phi\cos\phi - \sin\phi\cos\phi = 0 && (41)\\ \mathbf{u}_\theta.\mathbf{u}_\phi &= -\sin\theta\cos\theta\sin\phi\cos\phi + \sin\theta\cos\theta\sin\phi\cos\phi = 0 && (42) \end{aligned}$$

therefore $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$ form a basis for our tangent plane.

Now we want to find the shape operator relative to this basis, so we need to find a curve on our surface such that it has $\mathbf{u}_\theta$ or $\mathbf{u}_\phi$ as a tangent. Well, since $\mathbf{u}, \mathbf{u}_\theta$ and $\mathbf{u}_\phi$ are mutually perpindicular, they form a basis for $\mathbb{R}^3$ and we can therefore express a point on our surface in the form,

$$\mathbf{x} = \alpha\mathbf{u} + \beta\mathbf{u}_\theta + \gamma\mathbf{u}_\phi \tag{43}$$

since $\mathbf{x}$ must, by definition, be a linear combination of $\mathbf{u}, \mathbf{u}_\theta$ and $\mathbf{u}_\phi$. From previously, we know that $\mathbf{x} = (x, y, z)$ is expressed in terms of h, h_θ and h_ϕ (see equations (8), (9) and (10)) where $h = \mathbf{x}.\mathbf{u}$ (see equation (4)) which implies that,

$$\begin{aligned} h_\theta &= \mathbf{x}.\mathbf{u}_\theta && (44)\\ h_\phi &= \mathbf{x}.\mathbf{u}_\phi && (45) \end{aligned}$$

so to find α, β and γ we can take the dot product of both sides of equation (43) with $\mathbf{u}$, $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$, thus isolating α, β and γ separately using (40), (41) and (42). We find that,

$$\begin{aligned} \mathbf{x}.\mathbf{u} &= \alpha && (46)\\ \mathbf{x}.\mathbf{u}_\theta &= \beta\sin^2\phi && (47)\\ \mathbf{x}.\mathbf{u}_\phi &= \gamma && (48) \end{aligned}$$

and then we can substitute equations (4), (44) and (45) into the left hand sides of (46), (47) and (48) respectively to give us α, β and γ in terms of h and its partial derivatives, that is;

$$\alpha = h\ ;\ \beta = \frac{h_\theta}{\sin^2\phi}\ ;\ \gamma = h_\phi.$$

By substituting these into equation (43), we conclude that our surface can be parametrised by,

$$\mathbf{x} = h\mathbf{u} + \frac{h_\theta}{\sin^2\phi}\mathbf{u}_\theta + h_\phi\mathbf{u}_\phi$$

i.e. we are expressing our surface in terms of our support function h, our unit normal $\mathbf{u}$ and their derivatives.

Remember, we want a curve in $\mathbb{R}^2$ $\big($in the θ, ϕ – plane$\big)$ which maps, under $\mathbf{x}$, to a curve in $\mathbb{R}^3$ (on our surface) with $\mathbf{u}_\theta$ as a tangent. For simplicity, let us take a straight line in $\mathbb{R}^2$ since this is easy to parametrise.

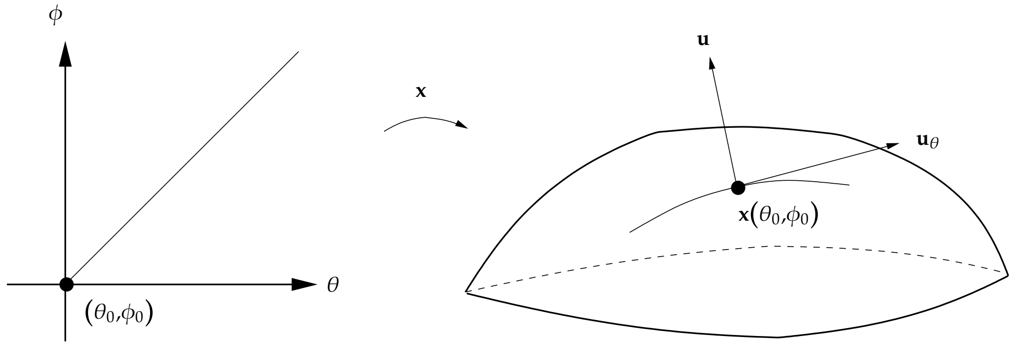

Figure 8: Mapping of straight line in $\mathbb{R}^2$ to a curve in $\mathbb{R}^3$ under $\mathbf{x}$.

Let us parametrise our straight line (see Figure 8) in $\mathbb{R}^2$ by,

$$\big(\theta(t), \phi(t)\big) = \big(a\,(t + \theta_0)\,, b\big(t + \phi_0\big)\big)$$

where a and b are constants. This then maps to a curve on our surface under $\mathbf{x}$,

$$\mathbf{x}\big(\theta(t), \phi(t)\big) = \mathbf{x}\big(a\,(t + \theta_0)\,, b\big(t + \phi_0\big)\big)$$

so that $t = 0$ will give us the point $\mathbf{x}\big(a\theta_0, b\phi_0\big)$. We want the tangent to this to be $\mathbf{u}_\theta$ (and later $\mathbf{u}_\phi$) or some parallel i.e. at $t = 0$ we want,

$$\frac{d}{dt}\mathbf{x}\big(\theta(t), \phi(t)\big) = \lambda \mathbf{u}_\theta$$

and, using the chain rule, this implies,

$$a\mathbf{x}_\theta + b\mathbf{x}_\phi = \lambda \mathbf{u}_\theta \tag{49}$$

where $\mathbf{x}_\theta, \mathbf{x}_\phi$ are being at $\left(a\theta_0, b\phi\right)$ and $a = \frac{d\theta}{dt}, b = \frac{d\phi}{dt}$. So now we need $\mathbf{x}_\theta$ and $\mathbf{x}_\phi$;

$$\begin{aligned}
\mathbf{x}_\theta &= h_\theta \mathbf{u} + h\mathbf{u}_\theta + \frac{h_{\theta\theta}}{\sin^2\phi}\mathbf{u}_\theta + \frac{h_\theta}{\sin^2\phi}\mathbf{u}_{\theta\theta} + h_{\theta\phi}\mathbf{u}_\phi + h_\phi \mathbf{u}_{\theta\phi} &(50)\\
\mathbf{x}_\phi &= h\mathbf{u}_\theta + \frac{h_{\theta\theta}}{\sin^2\phi}\mathbf{u}_\theta - \frac{h_\phi\cos\phi}{\sin\phi}\mathbf{u}_\phi + h_{\theta\phi}\mathbf{u}_\phi + \frac{h_\phi\cos\phi}{\sin\phi}\mathbf{u}_\theta &(51)
\end{aligned}$$

but we want to express these in terms of $\mathbf{u}, \mathbf{u}_\theta$ and $\mathbf{u}_\phi$. We have that,

$$\begin{aligned}
\mathbf{u}_{\theta\theta} &= \left(-\cos\theta\sin\phi, -\sin\theta\sin\phi, 0\right) = -\sin^2\phi\mathbf{u} - \sin\phi\cos\phi\mathbf{u}_\phi &(52)\\
\mathbf{u}_{\theta\phi} &= \left(-\sin\theta\cos\phi, \cos\theta\cos\phi, 0\right) = \frac{\cos\phi}{\sin\phi}\mathbf{u}_\theta &(53)\\
\mathbf{u}_{\phi\phi} &= \left(-\cos\theta\sin\phi, -\sin\theta\sin\phi, -\cos\phi\right) = -\mathbf{u} &(54)
\end{aligned}$$

and by substituting these into equations (50) and (51) as follows,

$$\begin{aligned}
\mathbf{x}_\theta &= \left[h + \frac{h_{\theta\theta}}{\sin^2\phi} + \frac{h_\phi\cos\phi}{\sin\phi}\right]\mathbf{u}_\theta + \left[h_{\theta\phi} - \frac{h_\theta\cos\phi}{\sin\phi}\right]\mathbf{u}_\phi &(55)\\
\mathbf{x}_\phi &= \left[\frac{h_{\theta\phi}}{\sin^2\phi} - \frac{h_\theta\cos\phi}{\sin^3\phi}\right]\mathbf{u}_\theta + \left[h + h_{\phi\phi}\right]\mathbf{u}_\theta &(56)
\end{aligned}$$

we find that we can express $\mathbf{x}_\theta$ and $\mathbf{x}_\phi$ at $\left(\theta_0, \phi_0\right)$ in the forms,

$$\mathbf{x}_\theta = A\mathbf{u}_\theta + B\mathbf{u}_\phi$$

$$\mathbf{x}_\phi = C\mathbf{u}_\theta + D\mathbf{u}_\phi$$

where the values of A, B, C and D are clear by comparison of coefficients, i.e. in the rest of this work we shall use the following notation.

$$A = h + \frac{h_{\theta\theta}}{\sin^2\phi} + \frac{h_\phi\cos\phi}{\sin\phi}, B = h_{\theta\phi} - \frac{h_\theta\cos\phi}{\sin\phi}, C = \frac{h_{\theta\phi}}{\sin^2\phi} - \frac{h_\theta\cos\phi}{\sin^3\phi}, D = h + h_{\phi\phi}$$

Let us substitute these into equation (49),

$$a\left(A\mathbf{u}_\theta + B\mathbf{u}_\phi\right) + b\left(C\mathbf{u}_\theta + D\mathbf{u}_\phi\right) = \lambda\mathbf{u}_\theta$$

and by comparing coefficients, we see that the coefficient of $\mathbf{u}_\phi$ on the left hand side must equal 0 where we have,

$$(aA + bC)\,\mathbf{u}_\theta + (aB + bD)\,\mathbf{u}_\phi = \lambda\mathbf{u}_\theta \qquad (57)$$

so let us choose our constants a and b in such a way that this is the case, i.e. choose $a = D$ and $b = -B$. Then our value for λ is clear and we conclude that our tangent vector to the curve on our surface is,

$$(AD - BC)\,\mathbf{u}_\theta.$$

Remeber that for the covariant derivative, we want the rate of change in our unit normal to the point $\mathbf{x}\left(a\theta_0, b\phi_0\right)$ on the surface in the direction of this tangent vector, i.e. we want,

$$\frac{d}{dt}\left[\mathbf{u}\left(a\,(t+\theta_0)\,, b\left(t+\phi_0\right)\right)\right] = a\mathbf{u}_\theta + b\mathbf{u}_\phi$$

at $t = 0$, but remember, we chose $a = D$ and $b = -B$. Therefore the shape operator associates the vector $B\mathbf{u}_\phi - D\mathbf{u}_\theta$ (the shape operator is the negative of the covariant derivative) to the tangent vector $(AD - BC)\,\mathbf{u}_\theta$ and this can be expressed as follows,

$$S\Big((AD - BC)\,\mathbf{u}_\theta\Big) = -\left(D\mathbf{u}_\theta - B\mathbf{u}_\phi\right). \tag{58}$$

We then wish to look for a curve on our surface to which $\mu\mathbf{u}_\phi$ is a tangent vector and our method for this proceeds in a parallel fashion to the one we have just used. The only difference is that when we get to equation (57), we choose $a = -C$ and $b = A$ so that the coefficient of $\mathbf{u}_\theta$ now equals 0 (and so that $\mu = \lambda$). Working this through provides,

$$S\Big((AD - BC)\,\mathbf{u}_\phi\Big) = -\left(-C\mathbf{u}_\theta + A\mathbf{u}_\phi\right) \tag{59}$$

i.e. the shape operator associates the vector $C\mathbf{u}_\theta - A\mathbf{u}_\phi$ to the tangent vector $(AD - BC)\,\mathbf{u}_\phi$ and since the shape operator is a linear map, we can divide both sides of equations of equations (58) and (59) by $(AD - BC)$ to give us,

$$S\left(\mathbf{u}_\theta\right) = \frac{-1}{(AD-BC)}\left(D\mathbf{u}_\theta - B\mathbf{u}_\phi\right) \tag{60}$$

$$S\left(\mathbf{u}_\phi\right) = \frac{-1}{(AD-BC)}\left(-C\mathbf{u}_\theta + A\mathbf{u}_\phi\right) \tag{61}$$

and we can think of this in terms of a change of basis operation so that our shape operator S in matrix form, relative to basis $\mathbf{u}_\theta, \mathbf{u}_\phi$, will be

$$S = \frac{-1}{(AD-BC)}\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}.$$

Presumably then, we need to ensure that our denominator here is non-zero.

Proposition 5.1 *Our surface is not smooth if $AD - BC = 0$ (proposition 2.2 was a special case of this).*

Proof. If our surface is smooth then the tangents $\mathbf{x}_\theta$ and $\mathbf{x}_\phi$ should never be parallel since, if they were, their cross product would equal 0. In turn, this would mean that the columns of our Jacobian matrix J were not linearly independent and J would not have rank 2 (condition for our surface to be an immersion).

Allow the cross product of $\mathbf{x}_\theta$ and $\mathbf{x}_\phi$ to equal the zero vector $\mathbf{0}$

$$\mathbf{x}_\theta \times \mathbf{x}_\phi = \left(A\mathbf{u}_\theta + B\mathbf{u}_\phi\right) \times \left(C\mathbf{u}_\theta + D\mathbf{u}_\phi\right)$$

and remembering that for 2 vectors $\mathbf{a}$, $\mathbf{b}$ the rules of cross products say that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ this becomes,

$$\mathbf{x}_\theta \times \mathbf{x}_\phi = (AD - BC)\,\mathbf{u}_\theta \times \mathbf{u}_\phi.$$

We would like to know when the right hand side here equals $\mathbf{0}$, i.e. when $\mathbf{x}_\theta$ and $\mathbf{x}_\phi$ are parallel. The right hand side here can only equal $\mathbf{0}$ if $(AD - BC) = 0$ because $\mathbf{u}_\theta \times \mathbf{u}_\phi \neq \mathbf{0}$ since $\mathbf{u}_\theta, \mathbf{u}_\phi$ are perpindicular and therefore never parallel. ∎

Looking at S more closely we see that our factor outside the matrix itself is equal to its determinant, therefore our shape operator can be expressed in a simpler form,

$$S = -\begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1}$$

and the well-known theorem associated with the shape operator S is as follows.

Theorem 5.1 *The eigenvalues of $-S$ are the principal curvatures and the eigenvalues of $-S^{-1}$ are the principal radii of curvature on our surface.*

We can now use this to prove the following theorem.

Theorem 5.2 *The sum of the principal radii of curvature at opposite points on a SCW is equal to the width of that surface.*

Proof. Opposite points on the surface correspond to the points $\left(\theta, \phi\right)$ and $\left(\theta + \pi, \pi - \phi\right)$ on the 2-sphere S^2, so for convenience, let us take a pair of values, say $\left(\theta_0, \phi_0\right) = \left(0, \frac{\pi}{2}\right)$. From therom 5.1, we know that the prinicpal radii of curvature, ρ_1 and ρ_2, are the eigenvalues of $-S^{-1}$ which, for these values, is equal to

$$\begin{pmatrix} h + h_{\theta\theta} & h_{\theta\phi} \\ h_{\theta\phi} & h + h_{\phi\phi} \end{pmatrix} = H + hI$$

where H is the Hessian matrix and our support function h plus its derivatives are measured at $\left(\theta_0, \phi_0\right)$. Let the eigenvalues of $-S^{-1}\left(\theta_0, \phi_0\right)$ be denoted by $\lambda_{1,2}$, that is, they satisfy the equation,

$$|H + hI - \lambda_i I| = 0$$

for $i = 1, 2$ but this is equivalent to saying that,

$$|H - (\lambda_i - h)\, I| = 0$$

i.e. $\mu_i = (\lambda_i - h)$ are eigenvalues of H and therefore satisfy the characteristic equation,

$$\mu_i^2 - \mathrm{Tr}(H)\mu_i + \det(H) = 0. \tag{62}$$

The matrix $-S^{-1}$ for $\left(\theta + \pi, \pi - \phi\right)$ is equal to,

$$\left(\begin{array}{cc} h' + h'_{\theta\theta} & h'_{\theta\phi} \\ h'_{\theta\phi} & h' + h'_{\phi\phi} \end{array} \right) = H' + h'I = H' + (w - h)I$$

where $'$ denotes measurement at $\left(\theta + \pi, \pi - \phi\right)$ and w is the width of our SCW. So now let us consider the Hessian matrix at $\left(\theta + \pi, \pi - \phi\right)$ which we called H' above and, in particular, consider H' at $\left(\theta_0 + \pi, \pi - \phi_0\right) = \left(\pi, \frac{\pi}{2}\right)$,

$$H' = \left(\begin{array}{cc} -h_{\theta\theta} & h_{\theta\phi} \\ h_{\theta\phi} & -h_{\phi\phi} \end{array} \right).$$

Here we have expressed the support function and its derivatives, measured at $\left(\theta_0 + \pi, \pi - \phi_0\right)$ in terms of those measured at $\left(\theta_0, \phi_0\right)$ using our condition for a SCW (see equation (22)) plus its derivatives. We can see that,

$$\mathrm{Tr}(H') = -\mathrm{Tr}(H) \tag{63}$$

$$\det(H') = \det(H) \tag{64}$$

and therefore the eigenvalues of H' must be the roots of the characteristic equation,

$$\mu_i^2 + \mathrm{Tr}(H)\mu_i + \det(H) = 0$$

but this differs to equation (62) by a sign only and when we have 2 quadratic equations of these forms, we know that the roots of one are the negatives of those to the other. We conclude that the eigenvalues of H' are equal to $-\mu_i$, for $i = 1, 2$.

To summarise, we have that μ_i are the eigenvalues of $H = H\left(\theta, \phi\right)$ where $\mu_i = (\lambda_i - h)$, so the principal radii at $\left(\theta, \phi\right)$ are,

$$\rho_1\left(\theta, \phi\right) = \lambda_1 = \mu_1 + h \tag{65}$$

$$\rho_2\left(\theta, \phi\right) = \lambda_2 = \mu_2 + h \tag{66}$$

and that $-\mu_i$ are the eigenvalues of $H' = H\left(\theta + \pi, \pi - \phi\right)$. So the pricipal radii at $\left(\theta + \pi, \pi - \phi\right)$, equal to the eigenvalues of $H' + hI$, are

$$\rho_1\left(\theta + \pi, \pi - \phi\right) = -\mu_1 + h' = -\mu_1 + (w - h) \tag{67}$$

$$\rho_2\left(\theta + \pi, \pi - \phi\right) = -\mu_2 + h' = -\mu_2 + (w - h) \tag{68}$$

and therefore, in conclusion,

$$\rho_1\left(\theta, \phi\right) + \rho_1\left(\theta + \pi, \pi - \phi\right) = \rho_2\left(\theta, \phi\right) + \rho_2\left(\theta + \pi, \pi - \phi\right) = w.$$

∎

5.1 Principal Directions

My main reference for this section was [CG].

The directions in which the principal curvatures, at a point $\mathbf{p}$, occur are called the pricipal directions to that point. They are given by the eigenvectors associated to the eigenvalues of $-S$ or $-S^{-1}$ (the eigenvectors of a matrix G are equal to those of its inverse).

Theorem 5.3 *At points of parallel tangency on our surface, the principal directions are themselves parallel.*

Proof. We would like to find the principal directions without first finding the principal curvatures κ_1, κ_2 (the associated eigenvalues) but how can we do this? By definition, our pricipal directions $\mathbf{e} = (e, f)^T$ must satisfy the equation $-S^{-1}\mathbf{e} = \kappa_i \mathbf{e}$, where $i = 1$ or 2. This equation becomes,

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} \kappa_i e \\ \kappa_i f \end{pmatrix} \tag{69}$$

and multiplying out the matrices on the left hand side,

$$Ae + Cf = \kappa_i e \tag{70}$$

$$Be + Df = \kappa_i f \tag{71}$$

where we want to eliminate our eigenvalue κ_i so that our equations for the eigenvectors don't depend upon the associated eigenvalues (which we don't know explicitly). Making κ_i the subject of (70), (71) and setting them equal to each other gives us an homogeneous equation of degree 2 in e, f only,

$$Be^2 + (D - A)ef - Cf^2 = 0$$

but we can put this into more familiar terms if we divide throughout by f^2 and let $g = \frac{e}{f}$,

$$Bg^2 + (D - A)g - C = 0. \tag{72}$$

Let us say that the roots of this equation are $m_\pm$ and that, since the principal directions at any point $\mathbf{p}$ are orthogonal, if $m_+ = m$ then $m_- = -\frac{1}{m}$ by definition. Now we would like to examin the principal directions corresponding to the "opposite" points in the $\theta, \phi-$plane (points with parallel tangency). Therefore we consider the negative inverse of our shape operator, that is $-S^{-1}$, at $\left(\theta_0, \phi_0\right)$ and $\left(\theta_0 + \pi, \pi - \phi_0\right)$ where $\theta_0 = 0$ and $\phi_0 = \frac{\pi}{2}$. Conveniently, we know these from our proof of theorem 5.2, they are,

$$S^{-1}\left(\theta_0, \phi_0\right) = \begin{pmatrix} h + h_{\theta\theta} & h_{\theta\phi} \\ h_{\theta\phi} & h + h_{\phi\phi} \end{pmatrix} \;\&\; -S^{-1}\left(\theta_0 + \pi, \pi - \phi_0\right) = \begin{pmatrix} w - h - h_{\theta\theta} & h_{\theta\phi} \\ h_{\theta\phi} & w - h - h_{\phi\phi} \end{pmatrix}$$

where h plus derivatives are measured at $\left(\theta, \phi\right) = \left(0, \frac{\pi}{2}\right)$. So if we let $-S^{-1}\left(\theta_0, \phi_0\right)$ take the same form as it did in equation (69) but now with $A = h + h_{\theta\theta}, B = C = h_{\theta\phi}$ and $D = h_{\theta\phi}$ then we have that,

$$-S^{-1}\left(\theta_0 + \pi, \pi - \phi_0\right) = \begin{pmatrix} w - A & C \\ B & w - D \end{pmatrix}.$$

The principal directions at these points can be obtained in the same way as previously, but it is clear that the only difference is that A and D in equation (72) should be replaced by $w - A$ and $w - D$ respectively to obtain,

$$Bg^2 + \left[(w - D) - (w - A)\right] g - C = Bg^2 - (D - A)g - C = 0. \tag{73}$$

Clearly then, equation (73) is equal to (72) but for the sign of its middle term. However, we must remember that our shape operator S is relative to the basis $\mathbf{u}_\theta, \mathbf{u}_\phi$ and thus, so too must our eigenvectors be. Equations (38) and (39) give us $\mathbf{u}_\theta, \mathbf{u}_\phi$ generally and we find that,

$$\mathbf{u}_\theta\left(\theta_0, \phi_0\right) = (0, 1, 0) \tag{74}$$

$$\mathbf{u}_\theta\left(\theta_0 + \pi, \pi - \phi_0\right) = (0, -1, 0) \tag{75}$$

$$\mathbf{u}_\phi\left(\theta_0, \phi_0\right) = (0, 0, -1) \tag{76}$$

$$\mathbf{u}_\phi\left(\theta_0 + \pi, \pi - \phi_0\right) = (0, 0, -1) \tag{77}$$

i.e. we have shown here that $\mathbf{u}_\theta$ points in opposite directions at points of parallel tangency. The significance of this is that if we are to think of our eigenvector corresponding to $-S^{-1}\left(\theta_0, \phi_0\right)$ as $(e, f)^T$ then our eigenvector corresponding to $-S^{-1}\left(\theta_0 + \pi, \pi - \phi_0\right)$ must be thought of as $(-e, f)^T$ or $(e, -f)^T$.

The effect of this is that, whilst equation (72) doesn't change, we replace g by $-g$ in equation (73) since we must substitute e for $-e$ or f for $-f$ and either way g becomes $-g$.

In summary, the principal directions at the points $\mathbf{p}\left(\theta_0, \phi_0\right)$ and $\mathbf{p}\left(\theta_0 + \pi, \pi - \phi_0\right)$ are roots of the same equation (72), therefore the roots of one must be equal to or be a multiple of the other, hence the principal directions are parallel. ∎

5.2 Using a different parametrisation

Recall that our results using $h\left(\theta, \phi\right)$ excluded the poles, so let us try parametrising our surface by σ, τ instead of θ, ϕ where σ is now our longitude and τ is our colatitude. As a consequence, the poles of our surface are now positioned on the positive and negative x−axes (whereas before they were on the positive and negative z−axes).

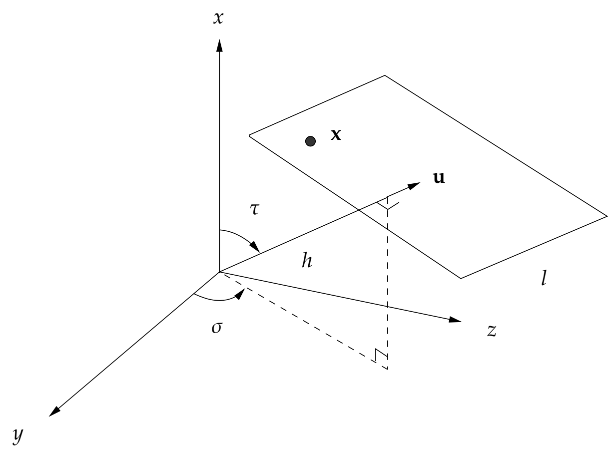

Figure 9: Surface with support function $h\,(\sigma, \tau)$ and support plane $l\,(\sigma, \tau)$.

From this we find, by a simple series of permutations, that a point $\mathbf{x}$ on our surface in terms of σ, τ is given by,

$$\begin{aligned} x &= h \cos \tau && (78) \\ y &= h \cos \sigma \sin \tau && (79) \\ z &= h \sin \sigma \sin \tau && (80) \end{aligned}$$

where $0 \le \sigma < 2\pi$, $0 \le \tau \le \pi$ and $h = h(\sigma, \tau)$ is our support function. By comparing the corresponding components of the unit normals to our surfaces as parametrised by σ, τ and θ, ϕ we equate the following,

$$\begin{aligned} \cos \tau &= \cos \theta \sin \phi && (81) \\ \cos \sigma \sin \tau &= \sin \theta \sin \phi && (82) \\ \sin \sigma \sin \tau &= \cos \phi && (83) \end{aligned}$$

as we would like to find θ, ϕ in terms of σ, τ. The aim of this is that we might determine any additional constraints on our support function $h\left(\theta, \phi\right)$ for a smooth surface. It is easy to see from equation (83) that,

$$\cos \phi = \sin \sigma \sin \tau \tag{84}$$

and we can find $\sin \phi$ by squaring then adding eqautions (81) and (82), the result of which is,

$$\sin^2 \phi = \cos^2 \tau + \cos^2 \sigma \sin^2 \tau.$$

Taking square roots here would usually be a problem as it lacks uniqueness, however $0 \le \phi \le \pi$ and, as a consequence, $\sin \phi \in [0, 1]$ so only the positive square root applies here, hence,

$$\sin \phi = \sqrt{\cos^2 \tau + \cos^2 \sigma \sin^2 \tau}. \tag{85}$$

Finally, from equations (81) and (82) respectively, it is clear that,

$$\begin{aligned} \cos \theta &= \frac{\cos \tau}{\sin \phi} && (86) \\ \sin \theta &= \frac{\cos \sigma \sin \tau}{\sin \phi} && (87) \end{aligned}$$

and what we want to show here is that this σ, τ parametrisation is smooth at the poles $\left(\phi = 0, \pi\right)$. For smoothness, $\cos \phi$ does not appear to be problematic, but $\sin \phi$ is a square root which is not a smooth function so must not use it in odd powers. With regards to $\cos \theta$ and $\sin \theta$, we divide by $\sin \phi$ in both cases and therefore, we have a problem when $\sin \phi = 0$.

Ignoring the square root, $\sin\phi$ is a sum of squares which can only equal 0 if all of these squares equal 0, i.e. we have a problem with our parametrisation if,

$$\cos\tau = \cos\sigma = 0$$

since the sine and cosine of the same angle never equal 0 simultaneously. This will only happen when $(\sigma, \tau) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and these correspond to the north and south poles of our θ, ϕ parametrised surface respectively, where $\phi = 0, \pi$.

We conclude that, for a smooth SCW, $\cos\theta$ and $\sin\theta$ can only be included in our support function $h\left(\theta, \phi\right)$ if great care is taken. However, one obvious way of doing so is by insisting that if a $\sin\theta$ or $\cos\theta$ term appears in $h\left(\theta, \phi\right)$, they should be multiplied by $\sin\phi$ so that the denominators cancel with the numerators (see equations (86) and (87)). Therefore, an adjustment should be made to our proposed support function,

$$h\left(\theta, \phi\right) = \sin^2\phi\left(p\left(\theta, \phi\right) + r\left(\phi\right)\right) + k$$

in so much as $p\left(\theta, \phi\right)$ should be set equal to $a\cos\theta\sin\phi$ (a is constant) rather than $a\cos\theta$ as previously, if our surface is to be smooth. After substiuting for θ, ϕ in h we obtain our new support function,

$$h(\sigma, \tau) = \left(\cos^2\tau + \cos^2\sigma\sin^2\tau\right)(a\cos\tau + b\sin\sigma\sin\tau) + k$$

which is a smooth function of σ, τ. However, we would also like to prove that the surface which corresponds to this is smooth everywhere, that is, including the poles. We can do this by using proposition 5.1, i.e. our surface is not smooth if $\mathbf{x}_\theta \times \mathbf{x}_\phi = AD - BC = 0$ when $\sigma = \tau = \frac{\pi}{2}$ (north pole of the θ, ϕ parametrisation) and $\sigma = \frac{3\pi}{2}, \tau = \frac{\pi}{2}$ (south pole of θ, ϕ parametrisation).

Using Maple (see Appendix 2), we find our shape operator entries A, B, C and D when our support function is, as proposed,

$$h\left(\theta, \phi\right) = \sin^2\phi\left(a\cos\theta\sin\phi + b\cos\phi\right) + k. \tag{88}$$

These are,

$$A\left(\theta, \phi\right) = 2a\cos\theta\sin\phi\cos^2\phi + k + 2b\cos^3\phi \tag{89}$$

$$B\left(\theta, \phi\right) = -2a\sin\theta\sin^2\phi\cos\phi \tag{90}$$

$$C\left(\theta, \phi\right) = -2a\sin\theta\cos\phi \tag{91}$$

$$D\left(\theta, \phi\right) = -2a\cos\theta\sin\phi + 8a\cos\theta\sin\phi\cos^2\phi - 6b\cos\phi + 8b\cos^3\phi + k \tag{92}$$

but we want these in terms of σ, τ (to find $AD-BC$) and for this we use equations (84), (85), (86) and (87). Now we can find $AD - BC$ in σ, τ which is,

$$AD - BC = \left(2a\cos\tau\sin^2\sigma\sin^2\tau + k + 2b\sin^3\sigma\sin^3\tau\right)^2 - 4a^2\cos^2\sigma\sin^4\tau\sin^2\sigma$$

and the question is: will this be non-zero at the poles (of θ, ϕ parametrisation)? To answer, we substitute in the afore-mentioned pole-values, i.e. $\sigma = \frac{3\pi}{2}, \tau = \pm\frac{\pi}{2}$ to reveal that,

$$AD - BC = (k + 2b)^2 \text{ at north pole} \tag{93}$$

$$AD - BC = (k - 2b)^2 \text{ at south pole} \tag{94}$$

and neither of these are equal to 0 unless $k = \pm 2b$.

Proposition 5.2 *Our surface, with support function* $h\left(\theta, \phi\right) = \sin^2\phi\left(a\cos\theta\sin\phi + b\cos\phi\right) + k$ *will be singular at the poles if, and only if,* $k = \pm 2b$.

5.3 Umbilic Points

A point $\mathbf{p}$ on our surface, where the principal curvatures κ_1, κ_2 are equal, is called $\mathbf{p}$ an umbilic point and we would like to find a condition on our surface which will be satisfied by these points.

From previously, we know that the eigenvalues of the negative inverse of our shape operator S are equal to the principal radii of curvature ρ_1, ρ_2 and, since these are the inverses of κ_1, κ_2 respectively, we know that if these are equal, then so too must be κ_1 and κ_2 themselves. Let us consider the characteristic equation of $-S^{-1}$ which has entries A, B, C and D as before,

$$\lambda^2 - (A + D)\lambda + (AD - BC) = 0$$

the roots of which are the principal radii of curvature. This is simply a quadratic equation, which had repeated roots if the discriminant is zero, therefore in our case, repeated roots are found if,

$$(A + D)^2 - 4(AD - BC) = 0$$

and this is our condition for an umbilic point. However, this simplifies significantly if we expand the squared brackets and use the fact that B in our shape operator matrix is equal to $C\sin^2\phi$. Thus our condition becomes,

$$(A - D)^2 + 4C^2\sin^2\phi = 0 \tag{95}$$

but the only way a sum of squares can equal 0 is if both components equal 0 separately, hence we find that umbilic points occur when $A = D$ and $C = 0$, assuming $\sin\phi \neq 0$.

Proposition 5.3 *Away from the poles, we find umbilic points where $A = D$ and $C = 0$.*

Consider the following proposition, noting the form taken by our newly modified support function h (see equation (88)), i.e. we let $q(\theta) = a\cos\theta$ and take out the new factor of $\sin\phi$.

Proposition 5.4 *For our support function $h = h\left(\theta,\phi\right) = q(\theta)\sin^3\phi + r\left(\phi\right)\sin^2\phi + k$, our poles will always be umbilics.*

Proof. At the poles $\sin\phi = 0$ so, by (95), our poles are umbilics if $A = D$, that is, if at $\phi = 0$ and $\phi = \pi$,

$$h + \frac{h_{\theta\theta}}{\sin^2\phi} + \frac{h_\phi\cos\phi}{\sin\phi} = h + h_{\phi\phi}.$$

So we need the values of A and D when $\phi = 0, \pi$ noting that the h terms on either side of the equals sign will cancel. We find that, for our support function h,

$$h_\phi = 3q\sin^2\phi\cos\phi + r_\phi\sin^2\phi + 2r\sin\phi\cos\phi \tag{96}$$

$$h_{\theta\theta} = q_{\theta\theta}\sin^3\phi \tag{97}$$

where $q = q(\theta), r = r\left(\phi\right)$ and when $\phi = 0, \pi$ we can see that, for $h_{\phi\phi}$, all but one term will have a $\sin\phi$ in it so, we may as well take $h_{\phi\phi}$ to equal $2r\cos^2\phi$ since all terms containing $\sin\phi$ equal 0 at $\phi = 0, \pi$ (the poles). Therefore, when $\phi = 0, \pi$ we substitute $\sin\phi = 0$ into A and D to find that,

$$A = D = 2r\cos^2\phi.$$

∎

5.4 Using curvature to confirm smoothness

This is a small subsection in which we demonstrate that our new support function,

$$h\left(\theta,\phi\right) = \sin^2\phi\left(p\left(\theta,\phi\right) + r\left(\theta,\phi\right)\right) + k$$

where $p = p\left(\theta,\phi\right) = a\cos\theta\sin\phi$ and $r = r\left(\phi\right) = b\cos\phi$ is smooth at the poles, whereas our old support function, where p was equal to $a\cos\theta$ but everything else remained the same, was not. We ask the question, what happens when $\phi \to 0$, i.e. what happens when we go closer to the north pole?

To answer this, we approach the north pole along the meridians of our surface. These are curves on our surface joining the north and south pole, where $\phi = 0$ and θ is fixed. We then look at the curvature along these lines which should not depend on θ, i.e. θ should be arbitrary here if our surface is smooth since, as described in our section on choosing a support function, we want all θ to map to the same point at the poles.

From previously, we know that the shape operator matrix is the easiest way of finding the principal curvatures, so let us find it for our old and new support function. For our old support function $h_1 = a\cos\theta\sin^2\phi + b\cos\phi\sin^2\phi + k$, we find the entries for our matrix $-S^{-1}$,

$$-S^{-1} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

using Maple (see Appendix 3),

$$\begin{aligned} A &= \Big(a\cos\theta + b\cos\phi\Big)\Big(1+\cos^2\phi\Big) + k - a\cos\theta - b\cos\phi\sin^2\phi && (98)\\ B &= -a\sin\theta\sin\phi\cos\phi && (99)\\ C &= -\frac{a\sin\theta\cos\phi}{\sin\phi} && (100)\\ D &= \Big(a\cos\theta + b\cos\phi\Big)\Big(3\cos^2\phi - 1\Big) + k - 5b\sin^2\phi\cos\phi && (101) \end{aligned}$$

and we remind ourselves that the eigenvalues of this matrix are equal to the pricipal curvatures κ_1, κ_2. We also know that the product of κ_1 and κ_2 equals the Gaussian curvature g, so it would be easier to check whether or not g is $\theta-$independent than it would be for κ_1, κ_2 since the determinant of a matrix is equal to the product of its eigenvalues, which we then don't need to find explicitly. Putting $\sin\phi = 0$ and $\cos\phi = 1$ (for the north pole) we find that,

$$g = 4bk - a^2 + 3a^2\cos^2\theta + 3ak\cos\theta + 4b^2 + 6ab\cos\theta + k^2$$

noting that $\cos\phi = -1$ (for the south pole) would only change signs and would not affect the dependence on θ in g which shows that surfaces produced by h_1 were not smooth at the poles. On the other hand, for our new support function $h_2 = a\cos\theta\sin^3\phi + b\cos\theta\sin^2\phi + k$, we can, by substiuting $\sin\phi = 0, \cos\phi = 1$ into $\left|S^{-1}\right|$, obtain the Gauss curvature,

$$g = 4bk + 4b^2 + k^2.$$

This is independent of θ, as should be the mean curvature m, which is the mean average of κ_1, κ_2. We use the fact that the trace of a matrix is equal to the sum of its eigenvalues and, using Maple, we find that when $\sin\phi = 0$ and $\cos\phi = 1$,

$$m = 2b + k$$

which is $\theta-$independent and therefore h_2 gives a smooth SCW.

6 Smooth surfaces

It would appear that we have shown the surface, produced by $h = a\sin^3\phi\cos\theta + b\sin^2\phi\cos\phi + k$ to be, at all points, smooth. However, when examined more closely, this is clearly not always the case. For instance, it seems that when we choose a relatively small value of k, say where $k < a, b$ and all constants are > 0, our surface develops singularities, but how large does k have to be to maintain the smoothness of our surface?

Proposition 6.1 *When $a > 0$ and $b = 0$ in our support function h, our constant k must be $> 2a$ if our surface is to be smooth.*

Proof. Proposition 5.1 says that our surface is not smooth if $AD - BC = 0$ and we would like to make this a condition on k. Using Maple (see Appendix 4), we can find the value of $AD - BC$ explicitly (in terms of our support function h, where $b = 0$) and this can be written as a quadratic in k, i.e. $k^2 + \Gamma k + \Omega$ where,

$$\Gamma = 10a\cos\theta\sin\phi\cos^2\phi - 2a\cos\theta\sin\phi \tag{102}$$

$$\Omega = 16a^2\cos^2\theta\cos^4\phi - 16a^2\cos^2\theta\cos^6\phi - 4a^2\cos^2\phi + 4a^2\cos^4\phi \tag{103}$$

which can be simplified using trigonometrical identities,

$$k^2 + 2ak\cos\theta\sin\phi\left(5\cos^2\phi - 1\right) + 4a^2\sin^2\phi\cos^2\phi\left(4\cos^2\theta\cos^2\phi - 1\right). \tag{104}$$

Divide this through by a^2 and use a change of variables $x = \frac{k}{a}$ to give us a quadratic equation in x,

$$G = x^2 + 2x\cos\theta\sin\phi\left(5\cos^2\phi - 1\right) + 4\sin^2\phi\cos^2\phi\left(4\cos^2\theta\cos^2\phi - 1\right) \tag{105}$$

where $G = AD - BC$. We claim that G does not equal 0 when $x = \frac{k}{a} > 2$ and since G is a continuous function of x, this would in turn mean that G must either always be > 0 or always < 0 if $k > 2a$. Substituting in some such numerical values for a and k ($k > 2a$) gives $AD - BC > 0$, so we claim that G must be > 0 for all $0 \le \theta < 2\pi$ and $0 \le \phi \le \pi$, where $k > 2a$. Let us assure ourselves that $G = 0$ has real roots by showing that its discriminant is always ≥ 0. We find (using Maple) that the discriminant of (105) is equal to,

$$\mathcal{D} = 4\sin^2\phi\left(\cos^2\theta - 10\cos^2\theta\cos^2\phi + 9\cos^2\theta\cos^4\phi + 4\cos^2\phi\right)$$

and we can ignore the common factor $4\sin^2\phi$ here since it is clearly ≥ 0 as we desire. Replacing $\cos^2\phi$ by $1 - \sin^2\phi$ throughout, this becomes,

$$\mathcal{D} = 9\cos^2\theta\sin^4\phi - 8\cos^2\theta\sin^2\phi - 4\sin^2\phi + 4$$

which can be expressed as a sum of squares in the following manner,

$$\mathcal{D} = \cos^2\theta\left(3\sin^2\phi - 2\right)^2 + 4\sin^2\theta\cos^2\phi$$

which means that the roots of $G = 0$, say k_1 and k_2, are real. Note that this means that for a line through our surface, where θ, ϕ are constant, our surface cannot be without singularities in that direction, for all k.

Our objective here is to find the minimum value of x for which $G \geq 0$ for all θ, ϕ (and we claim this is 2). Therefore, we would like to show that the larger root of $G = 0$ is ≤ 2 for all θ, ϕ. Equation (105) is of the form $G = x^2 + 2\epsilon x + \omega$, so the roots of $G = 0$ can be found using the quadratic equation,

$$x = -\epsilon \pm \sqrt{\epsilon^2 - \omega}$$

and since we've shown that the discriminant ≥ 0, we know that x_+ is the larger root here. Hence we would like to show that,

$$-\epsilon + \sqrt{\epsilon^2 - \omega} \leq 2 \tag{106}$$

but we can take the $-\epsilon$ term across here,

$$\sqrt{\epsilon^2 - \omega} \leq 2 + \epsilon \tag{107}$$

and then we would like to square both sides, but will this create false solutions? In the case of the left hand side, equal to the square root of $\mathcal{D} \geq 0$, the answer is certainly no. However, we need to show that the right hand side, $2 + \epsilon$, is ≥ 0 too, before we can square throughtout. Consider $2 + \epsilon$ in its explicit form,

$$2 + \epsilon = 2 + \cos\theta\sin\phi\left(5\cos^2\phi - 1\right)$$

where for $\sin\phi = 0$, it is clear that $2 + \epsilon$ is positive. Using a similar analysis to previously, if we can show that $2 + \epsilon$ never equals 0, then since it is a continuous function, it must always be > 0 for all $\theta \in [0, 2\pi), \phi \in [0, \pi]$. If $2 + \epsilon$ could equal 0, we have that,

$$2 = \left(1 - 5\cos^2\phi\right)\cos\theta\sin\phi$$

then, making $\cos\theta$ the subject and using $\cos^2\phi = 1 - \sin^2\phi$, this can be expressed in the following way,

$$\cos\theta = \frac{2}{5\sin^3\phi - 4\sin\phi}$$

but we know that $\cos\theta \leq 1$ so we can set an inequality,

$$\frac{2}{5s^3 - 4s} \leq 1$$

where $s = \sin\phi$. Multiplying up by our denominator here, we know that $\sin\phi \in [0, 1]$ and therefore we can square both sides since both must be non-negative, i.e.

$$2^2 = 4 \leq \left(5s^3 - 4s\right)^2 = 25s^6 - 40s^4 + 16s^2. \qquad (108)$$

To recap, this equation should only be satisfied if $2 + \epsilon$ could equal 0. Let us consider the graph of the function on the right hand side of our inequality (108) in figure 10.

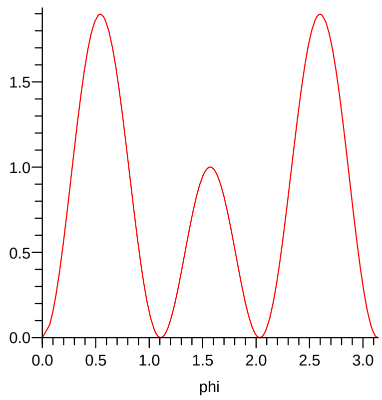

Figure 10: Graph of function $L = 25\sin^6\phi - 40\sin^4\phi + 16\sin^2\phi$.

By the vertical axis, it is clear that our fucntion is never ≥ 4, therefore (108) is not satisfied, thus $2 + \epsilon$ is never 0 and we can conclude that $2 + \epsilon$ is always positive. So now we can return to equation (107) and square both sides to give,

$$\epsilon^2 - \omega \leq (2 + \epsilon)^2 = 4 + 4\epsilon + \epsilon^2$$

but the ϵ^2 terms cancel and so, taking the $-\omega$ term across the inequality, this says,

$$0 \leq 4 + 4\epsilon + \omega$$

where the right hand side turns out to be equal to $G(x = 2)$. To summarise here, in order to show that the larger root of $G = 0$ is ≤ 2, we had to prove that (106) would hold, but we have now shown that this is equivalent to showing that $G(x = 2) \geq 0$ for all θ, ϕ.

When we substitute $x = 2$ into $G(x)$ we find that there is a common factor of 4 so we need to prove that,

$$\frac{1}{4}G(2) = 1 + \cos\theta\sin\phi\left(5\cos^2\phi - 1\right) + \sin^2\phi\cos^2\phi\left(4\cos^2\theta\cos^2\phi - 1\right) \geq 0$$

where we can use the changes of variables,

$$y = \cos\theta\sin\phi \tag{109}$$

$$z = \cos^2\phi \tag{110}$$

which combine to give us,

$$\cos\theta = \frac{y}{\sin\phi} = \frac{y}{\sqrt{1-z}}.$$

Now let $\frac{1}{4}G(2) = H(y,z)$ and, using our changes of variables, we obtain the following

$$H(y,z) = 1 + y\,(5z - 1) + z\,(1 - z)\left(\frac{4y^2z}{1-z} - 1\right)$$

which cancels down to,

$$H(y,z) = 1 - y + z^2\left(1 + 4y^2\right) + z\,(5y - 1)\,.$$

Here, our function $H = H(y,z)$ is a polynomial and we need to find the turning points of H and verify that $H \geq 0$ for all of them. We do this by plotting the boundaries of our variables y, z and then looking for the corresponding values under the mapping of H (see figure 11).

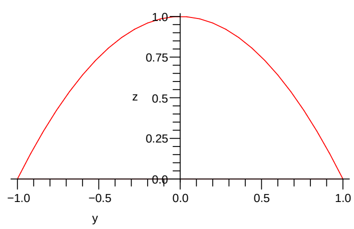

Figure 11: Graph of y, z region.

From their definitions, we know that $z \in [0,1]$ and $y \in [-1,1]$ and so our function will be bounded in this region, but we also have a relationship between y and z, i.e.

$$\frac{y}{\sqrt{1-z}} = \cos\theta \leq 1$$

which means that our curved boundary is given by $\cos\theta = 1$. Therefore, the y,z region will have $z = 0$ and $y = \sqrt{1-z}$ or, if you like, $y^2 + z = 1$ as its boundaries. On the first boundary $z = 0$, we have $H(y, z = 0) = 1 - y$ which is ≥ 0 since $-1 \leq y \leq 1$.

On the other boundary $z = 1 - y^2$ so, substituting this into H, we find (using Maple) that,

$$H\left(y, z = 1 - y^2\right) = (y-1)\left(y^3 - y - 1\right)(1+2y)^2$$

but we can't be sure that this is always ≥ 0. However, by taking a factor of -1 out of both of the first 2 brackets, this becomes,

$$H\left(y, z = 1 - y^2\right) = (1-y)\left(1 + y - y^3\right)(1+2y)^2$$

where the first and third brackets are clearly ≥ 0. So it only remains to show that the second bracket is ≥ 0, let us graph it as a function (see figure 12).

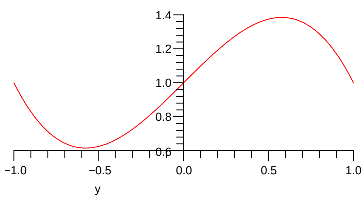

Figure 12: Graph of $f(y) = 1 + y - y^3$.

It is clear that, where the vertical axis here is $f(y) = 1 + y - y^3$, our second bracket is always ≥ 0 (note that the vertical axis starts from $f(y) = 0.6$). So we have shown that, on the boundary $H \geq 0$, so now we must look inside the bounded region for turning points. Turning points can be found by taking partial deivatives of our function with respect to its variables and setting them equal to 0, i.e. we want to solve,

$$H_y = 8yz^2 + 5z - 1 \tag{111}$$

$$H_z = 2z\left(4y^2 + 1\right) + 5y - 1 \tag{112}$$

for y, z when both equal 0. but this gives us $8\left(yz^2 - y^2z\right) + 3z = 0$ from which we cannot isolate one vaiable or find one in terms of the other. However, if we instead consider zH_z and yH_y, which still equal 0, we have,

$$yH_y = 8y^2z^2 + 5yz - y \quad (113)$$

$$zH_z = 2z^2\left(4y^2 + 1\right) + 5yz - z \quad (114)$$

and setting these to be equal gives us that $y = z - 2z^2$. Substitute this into $H_y = 0$ to find an equation in z only, i.e.

$$16z^4 - 8z^3 - 5z + 1 = 0$$

from which we find the roots (using Maple), namely $z_1 = 0.193$ and $z_2 = 0.838$ (both given to 3 s.f. as will all figures that follow). We then substitute these into $y = z - 2z^2$, finding the corresponding values of y, these are $y_1 = 0.118$ and $y_2 = -0.568$ for $z = z_1, z_2$ respectively. These are our turning points in H. Finally we substitute these values into H with the desire that $H \geq 0$ for both ,

$$H(y_1, z_1) = 0.842 \ \& \ H(y_2, z_2) = -0.0414$$

but $H(y_2, z_2)$ is less than 0. However, we notice that y_2, z_2 do not satisfy the condition $y^2 + z \leq 1 \left(y_2^2 + z_2 = 1.16\right)$ which means that this turning point of H lies outside our y, z region (see Figure 11) and therefore $H(y_2, z_2)$ is not relevant here.

To conclude, we have shown that $H \geq 0$ for all $y \in [-1, 1], z = [0, 1]$ which means that $G \geq 0$ when $x = 2$, where $x = \frac{k}{a}$. Therefore the larger point at which our smoothness condition $\mathbf{x}_\theta \times \mathbf{x}_\phi = AD - BC \neq 0$ fails is some value of $k \leq 2a$. ∎

6.1 What about when b ≠ 0?

This is a most interesting case as we have an insight into constraints on the constant k in our support function h. In this section, we have found that when $b = 0$, our surface is only smooth if $k > 2a$ $(a > 0)$ whilst Proposition 5.2 told us that our surface would not be smooth at the poles if $k = \pm 2b$ (which could easily be interpreted as $k > 2b$ for smooth poles).

It should be made clear that, in the later case, we made no assumptions on a, b in our support function (except that they were both non-zero) yet it transpired that our smoothness condition (see Proposition 5.1) did not depend on a at the poles. With this in mind, perhaps it would not be unreasonable to conjecture that, for our surface T to be smooth everywhere, our constant k should respect the following condition.

Proposition 6.2 *When $a, b > 0$ in our support function h, our constant k must be $> 2a + 2b$ if our surface is to be smooth.*

Proof. We begin in the same way we did previously, that is, we use Maple (see Appendix 4) to express $AD - BC$ as a quadratic in k, which takes the form $k^2 + \Lambda k + \Psi$ where,

$\Lambda = \Gamma - 6b\cos\phi + 10b\cos^3\phi$

$\Psi = \Omega - 16ab\cos\theta\sin\phi\cos^3\phi + 32ab\cos\theta\sin\phi\cos^5\phi - 12b^2\cos^4\phi + 16b^2\cos^6\phi$

and Γ, Ω are as they were defined in (102) and (103) respectively. We know that $k^2+\Lambda k+\Psi = 0$ has real roots since, through various trials, we find that the discriminant can be expressed as a sum of squares,

$$\mathcal{D} = 4\sin^2\phi\left[a\cos\theta\left(3\sin^2\phi - 2\right) + 3b\sin\phi\cos\phi\right]^2 + 16a^2 sin^2\theta\sin^2\phi\cos^2\phi$$

so $\mathcal{D} \geq 0$. We want to find the minimum value of k for which $AD - BC \geq 0$ and we claim that this is $k = 2a + 2b$. If we substitute $k = 2a + 2b$ into $k^2 + \Lambda k + \Psi$ and show that for $a, b > 0$ this will always be ≥ 0, we can prove this. Having made this substitution, we use Maple to express the result as a Taylor series in a, b which takes the form $T(a, b) = \Phi_1 a^2 + \Phi_2 ab + \Phi_3 b^2$, where Φ_i $(1 \leq i \leq 3)$ are coefficients in θ, ϕ as below.

$\Phi_1 = 16\cos^2\theta\cos^4\phi + 20\cos\theta\sin\phi\cos^2\phi - 16\cos^2\theta\cos^6\phi - 4\cos\theta\sin\phi - 4\cos^2\phi + 4\cos^4\phi + 4$

$\Phi_2 = 20\cos\theta\sin\phi\cos^2\phi - 16\cos\theta\sin\phi\cos^3\phi - 4\cos\theta\sin\phi - 12\cos\phi + 20\cos^3\phi +$
$32\cos\theta\sin\phi\cos^5\phi + 8$

$\Phi_3 = 16\cos^6\phi - 12\cos^4\phi - 12\cos\phi + 20\cos^3\phi + 4$

Divide $T(a, b)$ through by b^2 and make a change of variable $\lambda = \frac{a}{b}$ so that we have a quadratic in λ, say $T(\lambda) = \Phi_1\lambda^2 + \Phi_2\lambda + \Phi_3$ and we wish to show that $T(\lambda) \geq 0$ for all $\lambda > 0$, where $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$. The most obvious way to do this is to look for the minimum value of $T(\lambda)$ in hope that it is ≥ 0. We do know, at least, that $\Phi_1 \geq 0$ since Φ_1 is equal to $T(a, 0)$, i.e. Φ_1 equals $AD-BC$ when $b = 0$, which we have shown is ≥ 0 for all θ, ϕ when $k \geq 2a$.

It is clear then, if we can also show that Φ_2 and Φ_3 are ≥ 0, it is given that $T(\lambda) \geq 0$ under our previously stated conditions. If we let $c = \cos\phi$ and substitute this into Φ_3 then we can factorise using Maple, i.e.

$$\Phi_3 = 4\left(c^2 - c + 1\right)(c + 1)^2(2c - 1)^2$$

and this has roots $c = \frac{1}{2}$ (twice) and $c = -1$ (twice) noting that the first bracket in Φ_3 has no real roots. These roots correspond to $\phi = \frac{\pi}{3}$ and $\phi = \pi$ respectively and these are the only points for $0 \le \phi \le \pi$ where $\Phi_3 = 0$. Ignoring the first bracket (as it doesn't have real roots) we have that Φ_3 is a product of squares and therefore ≥ 0 everywhere.

It is left only to prove that $\Phi_2 \ge 0$ in order to prove that $T(\lambda) \ge 0$, so again we use Maple to factorise Φ_2, the terms of which have a common factor of $4\left(\cos\phi + 1\right)$ which is always ≥ 0 so we have removed it to give P.

$$P = \cos\theta \sin\phi \left(8\cos^4\phi - 8\cos^3\phi + 4\cos^2\phi + \cos\phi - 1\right) + 5\cos^2\phi - 5\cos\phi + 2$$

Let us find the turning points of P, that is, all maxima and minima in terms of $0 \le \theta < 2\pi, 0 \le \phi \le \pi$ and show that $P > 0$ for all of them. We have that P is a function of 2 variables, θ and ϕ, therefore it has turning points when $P_\theta = P_\phi = 0$. Finding these points is made easier if we factorise our partial derivatives using Maple (see Appendix 5).

$$P_\theta = -\sin\theta \sin\phi \left(2\cos\phi - 1\right)\left(4\cos^3\phi - 2\cos^2\phi + \cos\phi + 1\right)$$

For potential turning points, we look for places where $P_\theta = 0$ and we can see from the $\sin\theta\sin\phi$ term that $\theta = 0, \pi$ or $\phi = 0, \pi$ will produce such a result. From the first bracket, $\cos\phi = \frac{1}{2}$ is seen to be a root, therefore $\phi = \frac{\pi}{3}$ is a solution of $P_\theta = 0$. The second bracket is more complicated, but we can plot this as a separate function, say P_3 using Maple.

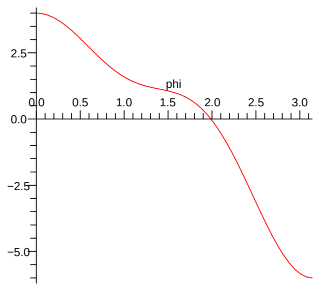

Figure 13: The graph of $P_3 = 4\cos^3\phi - 2\cos^2\phi + \cos\phi + 1$.

Maple finds that $\phi = 1.988068135$ is the exact root, so now let us look at P_ϕ.

$$P_\phi = 8\cos\theta\cos^5\phi - 32\cos\theta\sin^2\phi\cos^3\phi - 8\cos\theta\cos^4\phi + 24\cos\theta\sin^2\phi\cos^2\phi + 4\cos\theta\cos^3\phi$$

$$-8\cos\theta\sin^2\phi\cos\phi - 10\sin\phi\cos\phi - \cos\theta\sin^2\phi + \cos\theta\cos^2\phi + 5\sin\phi - \cos\theta\cos\phi$$

Let us ask, when do P_θ and P_ϕ simultaneously equal 0? We already know the values for which $P_\theta = 0$ so we just need to substitute these into P_ϕ and find the values at which it also equals 0. Let us plot $P_\phi\left(0, \phi\right)$ and find its roots, of which there are 3.

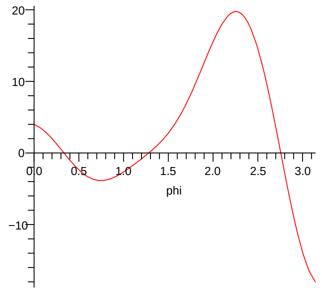

Figure 14: The graph of $P_\phi\left(0, \phi\right)$.

These are found to be $\phi_1 = 0.3335534097, \phi_2 = 1.280949579$ & $\phi_3 = 2.756476602$, so we need to show that for $\theta = 0$ and these 3 values of ϕ, our function $P > 0$ and, by direct substitution, we obtain,

$$P\left(0, \phi_1\right) = 2.769017328,\ P\left(0, \phi_2\right) = 0.480278222 \ \&\ P\left(0, \phi_2\right) = 16.10400599$$

all of which are > 0. Now do the same for our second value, $\theta = \pi$, plotting $P_\phi\left(\pi, \phi\right)$ which also has 3 roots.

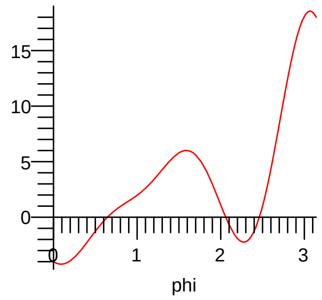

Figure 15: The graph of $P_\phi\left(\pi, \phi\right)$.

These are found to be $\phi_4 = 0.6435011088,\ \phi_5 = 2.059570464$ & $\phi_6 = 2.462405301$ and, by direct substitution, we obtain,

$$P\left(\pi, \phi_4\right) = 0.275520000,\ P\left(\pi, \phi_5\right) = 4.894398896 \ \&\ P\left(\pi, \phi_6\right) = 4.304097159$$

all of which are > 0. Putting $\phi = 0$ into P_ϕ gives $P_\phi(\theta, 0) = 4\cos\theta$ and clearly the roots of this are $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{3\pi}{2}$, so our corresponding values of P are,

$$P(\theta_1, 0) = P(\theta_2, 0) = 2.$$

Next we try $\phi = \pi$ in P_ϕ, which gives $P_\phi(\theta, \pi) = -18\cos\theta$ and clearly the roots again here are θ_1, θ_2. Corresponding values of P are,

$$P(\theta_1, \pi) = P(\theta_2, \pi) = 12$$

and so now we can try $\phi = \frac{\pi}{3}$ which, when substituted into P_ϕ, gives $P_\phi\left(\theta, \frac{\pi}{3}\right) = -\frac{9}{4}\cos\theta$, the only roots of which are θ_1 and θ_2. Substituting these into P gives,

$$P\left(\theta_1, \frac{\pi}{3}\right) = P\left(\theta_2, \frac{\pi}{3}\right) = \frac{3}{4}$$

which is greater than 0 and our final potential turning point value $\phi = 1.988068135$, when substitued into P_ϕ has no real roots and is therefore not a turning point of P (see figure 16).

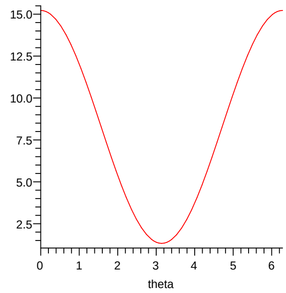

Figure 16: The graph of $P_\phi(\theta, 1.988068135)$.

In conclusion, all maxima and minima of $P > 0$, therefore $\Phi_2 \geq 0$ and $T(\lambda) \geq 0$. Recall that $T(\lambda)$ was equal to $\mathbf{x}_\theta \times \mathbf{x}_\phi = AD - BC$ when k was equal to $2a + 2b$ and so we have shown that $AD - BC$ is always ≥ 0 when $k = 2a + 2b$ which leaves us the following possibilities.

Since $AD - BC = k^2 + \Lambda k + \Psi = 0$ is a quadratic with either one or two real roots, it is a parabola, thus if $AD - BC$ is ≥ 0 for $k = 2a + 2b$, then $k = 2a + 2b$ must either be greater than or equal to the larger root of $AD - BC = 0$ or, less than or equal to the smaller root. Consider also the case where $AD - BC = 0$ has a single repeated root (where $\mathcal{D} = 0$) where this root equals $k = 2a + 2b$. In this case, $k > 2a + 2b$ or $k < 2a + 2b$ would give a smooth surface, as $AD - BC$ would be non-zero for any k not equal to $2a + 2b$ (note $a, b > 0$ in all arguments here).

If $AD - BC = 0$ has 2 unequal roots, then to decide whether $k = 2a + 2b$ is potentially the larger or smaller of those roots is straight foward. If, as we claim, it is potentially the larger, then for some value of $k < 2a + 2b$ we could find that $AD - BC < 0$ and this would not be the case if it were potentially the smaller root. So let us plot $AD - BC$ for some explicit values, say we let a and b both equal 2 ($a, b > 0$) and let $k = 2a + 2b = 8$, together with a smaller value, say $k = a + b = 4$.

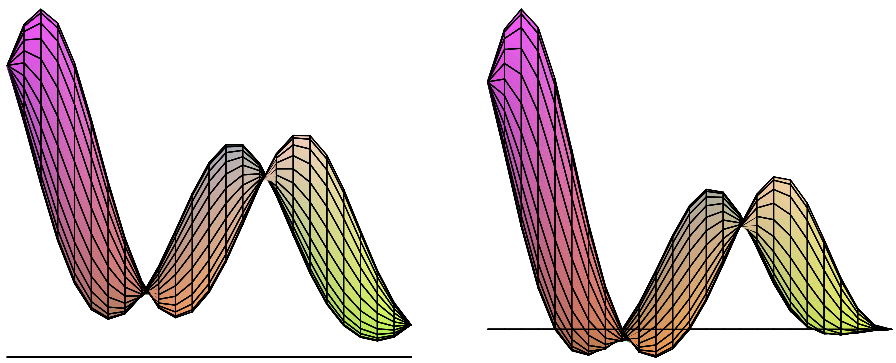

Figure 17: The graphs of $AD - BC$ where $k = 2a + 2b$ (left) and $k = a + b$ (right).

In figure 17, the grey horizontal lines represent $AD - BC = 0$ so $AD - BC$ is negative when the graph falls below this line and clearly in the case where $k = a + b$, we have $AD - BC < 0$ for some θ, ϕ. Therefore $k = 2a + 2b$ could only possibly be the larger of 2 roots (although in this example it would appear to actually be greater than the larger root since $AD - BC$ doesn't seem to be 0 anywhere).

If it is only poosible for $k = 2a + 2b$ to be the larger root of $AD - BC = 0$ and not the smaller, we conclude that for $k > 2a + 2b$ we have $AD - BC > 0$ for all $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$, thus our surface is always smooth when k, in our chosen support function h, is greater than $2a + 2b$, where a and b are both positive.

7 Cuspidal Edges & Swallowtails

We would like to find conditions for which our surface exhibits cuspidal edges or swallowtails repectively. In order to do this, we shall explore the possibility of A_2 and A_3 singularities, as a versally unfolded A_2 (A_3) singularity has discriminant which is locally diffeomorphic to a cuspidal edge (swallowtail). Let us begin by briefly reminding ourselves of our surface's construction (see figure 1).

The condition for a point on our surface, say $\mathbf{x}$, to lie on our support plane (tangent plane in $\mathbb{R}^3$ perpindicular to our unit normal $\mathbf{u}$) gives us the equation of the tangent plane itself. This condition says that the vector joining $\mathbf{x}$ to $h\mathbf{u}$ is perpindicular to $\mathbf{u}$, i.e.

$$(\mathbf{x} - h\mathbf{u}) . \mathbf{u} = 0$$

and we wish to draw form this our family of planes F. Simplifying we have,

$$F\left(\theta, \phi, x, y, z\right) = \mathbf{x}.\mathbf{u} - h$$

and, as we have done throughout, we shall look specifically at the point given by $\theta = 0, \phi = \frac{\pi}{2}$. However, during the course of our calculations in this section, we shall use Taylor series and so, it would be more convenient to use $(0, 0)$ as our base point.

Parametrising by longitude θ and latitude ψ allows us to do this, i.e. $\left(\theta, \phi\right) = \left(0, \frac{\pi}{2}\right)$ is equaivalent to $(\theta, \psi) = (0, 0)$ since $\psi = \frac{\pi}{2} - \phi$ (relationship between latitude and colatitude). Therefore, our unit normal $\mathbf{u}$ needs to be expressed in terms of θ, ψ,

$$\mathbf{u} = (\cos\theta\cos\psi, \sin\theta\cos\psi, \sin\psi)$$

where our family $F = F(\theta, \psi, x, y, z)$ and our resulting surface is,

$$\mathcal{D}_F = \left\{\mathbf{x} : F = \frac{\partial F}{\partial\theta} = \frac{\partial F}{\partial\psi} = 0\right\}.$$

We hope that this will be a 2-manifold such that the only singularities of $\mathcal{D}_F$ result from the projection taking $\mathbb{R}^5 \to \mathbb{R}^3$. For our envelope we need F_θ and F_ψ, so using $F = \mathbf{x}.\mathbf{u} - h$ we find,

$$F_\theta = \mathbf{x}.\mathbf{u}_\theta - h_\theta \quad (115)$$

$$F_\psi = \mathbf{x}.\mathbf{u}_\psi - h_\psi \quad (116)$$

where,

$$\mathbf{u}_\theta = (-\sin\theta\cos\psi, \cos\theta\cos\psi, 0) \quad (117)$$

$$\mathbf{u}_\psi = (-\cos\theta\sin\psi, -\sin\theta\sin\psi, \cos\psi) \quad (118)$$

and noting that our support function is now $h = h(\theta, \psi)$ which we need explicitly for our example later. Using our previous support function, that is,

$$h\left(\theta, \phi\right) = \sin^2\phi\left(a\cos\theta\sin\phi + b\cos\phi\right) + k$$

we find h as a function of θ, ψ by making the following substitutions,

$$\sin\psi = \sin\left(\frac{\pi}{2} - \phi\right) = \cos\phi \qquad (119)$$

$$\cos\psi = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi \qquad (120)$$

therefore our support function, which we shall now refer to as h, in our new coordinates is,

$$h(\theta, \psi) = \cos^2\psi\,(a\cos\theta\cos\psi + b\sin\psi) + k. \qquad (121)$$

We can then find $\mathbf{x}$ in terms of this new h (and its derivatives) in the same way we found it previously, the calculations being more or less identical so we shall not repeat them, but simply state that,

$$\mathbf{x} = h\mathbf{u} + \frac{h_\theta}{\cos^2\psi}\mathbf{u}_\theta + h_\psi\mathbf{u}_\psi$$

where, for our h,

$$h_\theta = -a\sin\theta\cos^3\psi \qquad (122)$$

$$h_\psi = -3a\cos\theta\sin\psi\cos^2\psi + b\cos^3\psi - 2b\sin^2\psi\cos\psi. \qquad (123)$$

Now that we have $\mathbf{x}, \mathbf{u}, h$ and their derivatives explicitly we can find our Jacobian, i.e. $\mathcal{D}_F$ will map to the discriminant in $\mathbb{R}^3$ and we are trying to find out when this dicriminant, and therefore our surface, will exhibit singularities. The Jacobian of F, F_θ and F_ψ will be a 3×5 matrix, since our map is $\mathbb{R}^5 \to \mathbb{R}^3$, taking the form,

$$J = \begin{pmatrix} F_\theta & F_\psi & F_x & F_y & F_z \\ F_{\theta\theta} & F_{\theta\psi} & F_{\theta x} & F_{\theta y} & F_{\theta z} \\ F_{\psi\theta} & F_{\psi\psi} & F_{\psi x} & F_{\psi y} & F_{\psi z} \end{pmatrix}$$

and these entries can be found explicitly as follows.

$$\begin{aligned} F_x &= \cos\theta\cos\psi, F_y = \sin\theta\cos\psi, F_z = \sin\psi & (124) \\ F_\theta &= -x\sin\theta\cos\psi + y\cos\theta\cos\psi - h_\theta & (125) \\ F_\psi &= -x\cos\theta\sin\psi - y\sin\theta\sin\psi + z\cos\psi - h_\psi & (126) \\ F_{\theta\theta} &= -x\cos\theta\cos\psi - y\sin\theta\cos\psi - h_{\theta\theta} & (127) \\ F_{\theta\psi} &= x\sin\theta\sin\psi - y\cos\theta\sin\psi - h_{\theta\psi} & (128) \\ F_{\psi\psi} &= -x\cos\theta\cos\psi - y\sin\theta\cos\psi - z\sin\psi - h_{\psi\psi} & (129) \\ F_{\theta x} &= -\sin\theta\cos\psi, F_{\theta y} = \cos\theta\cos\psi, F_{\theta z} = 0 & (130) \\ F_{\psi x} &= -\cos\theta\sin\psi, F_{\psi y} = -\sin\theta\sin\psi, F_{\psi z} = \cos\psi & (131) \end{aligned}$$

More specifically, we would like to examine $J\left(F, F_\theta, F_\psi\right)$ at the point $(\theta, \psi) = (0, 0)$ on our surface. Bare in mind that a point on our surface at $\mathbf{0}$ is given by,

$$\mathbf{x} = h(\mathbf{0})\,\mathbf{u}(\mathbf{0}) + h_\theta(\mathbf{0})\,\mathbf{u}_\theta(\mathbf{0}) + h_\psi(\mathbf{0})\,\mathbf{u}_\psi(\mathbf{0})$$

where our unit normal $\mathbf{u}$ and its derivatives are orthonormal at $\mathbf{0}$, that is, they are mutually perpinicular with unit length. In fact we find that $\mathbf{u}(\mathbf{0}) = (1, 0, 0)$, $\mathbf{u}_\theta(\mathbf{0}) = (0, 1, 0)$, $\mathbf{u}_\psi(\mathbf{0}) = (0, 0, 1)$ and thus,

$$\mathbf{x}(\mathbf{0}) = (x(\mathbf{0}), y(\mathbf{0}), z(\mathbf{0})) = \left(h(\mathbf{0}), h_\theta(\mathbf{0}), h_\psi(\mathbf{0})\right)$$

and so our Jacobian at $(\theta, \psi) = (0, 0)$ is given by,

$$J(\mathbf{0}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -h(\mathbf{0}) - h_{\theta\theta}(\mathbf{0}) & -h_{\theta\psi}(\mathbf{0}) & 0 & 1 & 0 \\ -h_{\theta\psi}(\mathbf{0}) & -h(\mathbf{0}) - h_{\psi\psi}(\mathbf{0}) & 0 & 0 & 1 \end{pmatrix}$$

and this is an immersion if the rank is maximal, which would be 3 here. On the other hand, this is not an immersion if all 3×3 minors are equal to 0, however self-evidently this is not the case here as one of our minors is the identity matrix I_3, i.e.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

and so we conclude that $\mathcal{D}_F$ is always an immersion at the point $\mathbf{0}$ on our surface T. Thus we can regard F as an unfolding where $F(\theta, \psi, x(\mathbf{0}), y(\mathbf{0}), z(\mathbf{0})) = F\left(\theta, \psi, h(\mathbf{0}), h_\theta(\mathbf{0}), h_\psi(\mathbf{0})\right) = f(\theta, \psi)$ and ask when this has an A_2 or A_3 singularity.

If indeed it is we should then ask whether $f = f(\theta, \psi)$ is versally unfolded by x, y, z as, if it is, we can then talk about cuspidal edges and swallowtails. Earlier we found that $F = \mathbf{x}.\mathbf{u} - h$ so we use this to obtain,

$$f = h(\mathbf{0}) \cos\theta \cos\psi + h_\theta(\mathbf{0}) \sin\theta \cos\psi + h_\psi(\mathbf{0}) \sin\psi - h \tag{132}$$

and then we state (without proof) the well-known theorem that all A_k singularities, where $1 \leq k \leq 3$, of a function, say $g = g(x, y)$ can be reduced to one of the following forms.

$$A_1 \;:\; g = x^2 \pm y^2 \tag{133}$$

$$A_2 \;:\; g = x^2 + y^3 \tag{134}$$

$$A_3 \;:\; g = x^2 \pm y^4 \tag{135}$$

So we wish to reduce our function to these forms (if possible) but before doing so, we must express our functions of θ, ψ as Taylor series around our base point $(\theta, \psi) = (0, 0)$, baring in mind that we are not interested in terms beyond order 4. We know that,

$$\sin x = x - \frac{x^3}{3!} + \ldots \tag{136}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \tag{137}$$

and the Taylor series expansion of h around $(\theta, \psi) = (0, 0)$ is,

$$T(h) = h(\mathbf{0}) + h_\theta(\mathbf{0})\,\theta + h_\psi(\mathbf{0})\,\psi + \tfrac{1}{2}h_{\theta\theta}(\mathbf{0})\,\theta^2 + h_{\theta\psi}(\mathbf{0})\,\theta\psi + \tfrac{1}{2}h_{\psi\psi}(\mathbf{0})\,\psi^2 +$$

$$\tfrac{1}{6}h_{\theta\theta\theta}(\mathbf{0})\,\theta^3 + \tfrac{1}{2}h_{\theta\theta\psi}(\mathbf{0})\,\theta^2\psi + \tfrac{1}{2}h_{\theta\psi\psi}(\mathbf{0})\,\theta\psi^2 + \tfrac{1}{6}h_{\psi\psi\psi}(\mathbf{0})\,\psi^3 + \tfrac{1}{24}h_{\theta\theta\theta\theta}(\mathbf{0})\,\theta^4 +$$

$$\tfrac{1}{6}h_{\theta\theta\theta\psi}(\mathbf{0})\,\theta^3\psi + \tfrac{1}{4}h_{\theta\theta\psi\psi}(\mathbf{0})\,\theta^2\psi^2 + \tfrac{1}{6}h_{\theta\psi\psi\psi}(\mathbf{0})\,\theta\psi^3 + \tfrac{1}{24}h_{\psi\psi\psi\psi}(\mathbf{0})\,\psi^4 + \ldots$$

and so for an A_1 we need only concern ourselves with the quadratic terms and we try to make a perfect square, as in (133). Substitution of the quadratic terms of these series into f gives, after some simplification,

$$f = -\frac{1}{2}\left(h(\mathbf{0}) + h_{\theta\theta}(\mathbf{0})\right)\theta^2 - h_{\theta\psi}(\mathbf{0})\,\theta\psi - \frac{1}{2}\left(h(\mathbf{0}) + h_{\psi\psi}(\mathbf{0})\right)\psi^2 + \ldots$$

and we can complete the square on the first 2 terms here (we want to remove the $\theta\psi$ term in order to make a perfect square). Let us make some changes of variables for the sake of convenience, i.e. let K, L and M equal the coefficients of $\theta^2, \theta\psi$ and ψ^2 in f respectively here so that $f = K\theta^2 + L\theta\psi + M\psi^2 + \ldots$ here.

When we complete the square on the first two terms in f, there will always be a correction or error term, this will become part of the new coefficient of ψ^2 as shown,

$$f = K\left(\theta - \frac{L}{2K}\right)^2 + \left(M - \frac{L^2}{4K}\right)\psi^2 + \dots$$

and this leads us to the first of 2 possible cases here.

7.1 Case 1: $K \neq 0$, f has type A_1?

Let us first assume that $K \neq 0$ since we divide by it in order to complete the square, this forms the first of our conditions, i.e.

$$K \neq 0 \Longrightarrow h(\mathbf{0}) + h_{\theta\theta}(\mathbf{0}) \neq 0$$

and we can use the change of variable $\theta' = \theta - \frac{L}{2K}$. It is important to note that this, as must each, change of variable we use be a local diffeomorphism and we can check this with the Jacobian of the map $\mathbb{R}^2 \to \mathbb{R}^2$ taking (θ, ψ) to (θ', ψ') or vice versa, although it seems unnecessary to make the change $\psi' = \psi$. This Jacobian is given by,

$$\begin{pmatrix} \frac{\partial\theta'}{\partial\theta} & \frac{\partial\theta'}{\partial\psi} \\ \frac{\partial\psi'}{\partial\theta} & \frac{\partial\psi'}{\partial\psi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

at $(\theta, \psi) = (0, 0)$ and is therefore non-singular (so our change of variable is indeed a local diffeomorphism). Using this we can express f is the following way,

$$f = K\theta'^2 + \left(M - \frac{L^2}{4K}\right)\psi'^2 + \dots$$

and we then use a fifth change of variable, let $N = M - \frac{L^2}{4K}$ and thus,

$$f = K\theta'^2 + N\psi'^2 + \dots$$

which leads us to our second condition, i.e. we assume that the coefficient of $\psi'^2 \neq 0$, that is,

$$N \neq 0 \Longrightarrow (h(\mathbf{0}) + h_{\theta\theta}(\mathbf{0}))\left(h(\mathbf{0}) + h_{\psi\psi}(\mathbf{0})\right) - h^2_{\theta\psi}(\mathbf{0}) \neq 0.$$

Now it only remains to make two more changes of variables for f to take the form of an A_1 singularity, they are as follows,

$$\text{Let } \theta'' = \begin{cases} \sqrt{K}\theta' & \text{if } K > 0 \\ \sqrt{-K}\theta' & \text{if } K < 0. \end{cases}$$

$$\text{Let } \psi'' = \begin{cases} \sqrt{N}\psi' & \text{if } N > 0 \\ \sqrt{-N}\psi' & \text{if } N < 0. \end{cases}$$

and so we have 4 possibilities for f.

(pf 1) $f = \theta''^2 + \psi''^2 + \ldots$

(pf 2) $f = \theta''^2 - \psi''^2 + \ldots$

(pf 3) $f = -\theta''^2 + \psi''^2 + \ldots$

(pf 4) $f = -\theta''^2 - \psi''^2 + \ldots$

but (pf 3) and (pf 4) are just the negatives of (pf 2) and (pf 1) respectively, so simple changes of variables would give us the same results, i.e. f can be described by (pf 1) and (pf 2) alone. In conclusion we have shown that f can take the form of an A_1, that is,

$$f = \theta''^2 \pm \psi''^2 + \ldots$$

under the 2 afore-mentioned conditions, firstly that,

$$h(\mathbf{0}) + h_{\theta\theta}(\mathbf{0}) \neq 0$$

and also our second condition,

$$(h(\mathbf{0}) + h_{\theta\theta}(\mathbf{0}))\left(h(\mathbf{0}) + h_{\psi\psi}(\mathbf{0})\right) - h^2_{\theta\psi}(\mathbf{0}) \neq 0$$

which corresponds exactly to the work that lead to the smoothness condition, proposition 2.2, in the first section of this work (as we might have expected). This says that f cannot take the form of an A_1 unless the matrix $(H + hI)$ is non-singular at $\mathbf{0}$, where H is the Hessian matrix of our support function h and $I = I_2$ is the identity matrix.

7.2 Case 1: $K \neq 0$, f has type A_2?

Now we would like to uncover the condition(s) on f for an A_2 and, from (134), we can see that this requires the coefficient of ψ'^2 to equal 0, as we only want one squared term and we are exploring the case where the coefficient of θ'^2 does not equal 0 here. Therefore our first condition for an A_2 is that the matrix $(H + hI)$ is singular at $\mathbf{0}$ and so now we want to examine f with only its θ''^2 term plus any terms of order 3,

$$f = \theta''^2 - \frac{1}{6}(h_\theta + h_{\theta\theta\theta})\theta^3 - \frac{1}{6}\left(h_\psi + h_{\psi\psi\psi}\right)\psi^3 - \frac{1}{2}\left(h_\theta + h_{\theta\psi\psi}\right)\theta\psi^2 - \frac{1}{2}h_{\theta\theta\psi}\theta^2\psi + \ldots$$

but we see here that we need θ in terms of θ'' before we complete the square for the θ''^2 and cubic terms (note h and all its partial derivatives are measured at $\mathbf{0}$ here). Referring back to our previous changes of variables, it is not difficult to see that

$$\theta = \frac{\theta''}{\sqrt{\pm K}} + \frac{L\psi}{2K}$$

and, after a lot of simplification, we can find f in terms of θ'' and ψ only,

$$f = \theta''^2 + \theta''^3 \left\{ -\tfrac{1}{6} \left(h_\theta + h_{\theta\theta\theta} \right) \left(\tfrac{1}{\sqrt{\pm K}} \right)^3 \right\} +$$

$$\psi^3 \left\{ -\tfrac{1}{6} \left(h_\theta + h_{\theta\theta\theta} \left(\tfrac{L}{2K} \right)^3 - \tfrac{1}{6} \left(h_\psi + h_{\psi\psi\psi} \right) - \tfrac{1}{2} \left(h_\theta + h_{\theta\psi\psi} \right) \left(\tfrac{L}{2K} \right) - \tfrac{1}{2} h_{\theta\theta\psi} \right) \left(\tfrac{L}{2K} \right)^2 \right\} +$$

$$\theta''^2\psi \left\{ -\tfrac{1}{2} \left(h_\theta + h_{\theta\theta\theta} \right) \left(\tfrac{L}{2K} \right) \left(\tfrac{1}{\sqrt{\pm K}} \right)^2 - \tfrac{1}{2} h_{\theta\theta\psi} \left(\tfrac{1}{\sqrt{\pm K}} \right)^2 \right\} +$$

$$\theta''\psi^2 \left\{ -\tfrac{1}{2} \left(h_\theta + h_{\theta\theta\theta} \right) \left(\tfrac{L}{2K} \right)^2 \left(\tfrac{1}{\sqrt{\pm K}} \right) - \tfrac{1}{2} \left(h_\theta + h_{\theta\psi\psi} \right) \left(\tfrac{1}{\sqrt{\pm K}} \right) - h_{\theta\theta\psi} \left(\tfrac{1}{\sqrt{\pm K}} \right) \left(\tfrac{L}{2K} \right) \right\} + \ldots$$

where h and all its partial derivatives are measured at $\mathbf{0}$ here. Completing the square gives,

$$f = \left(\theta'' + \frac{1}{2} \left[\theta''^2 \{\ldots\} + \theta''\psi \{\ldots\} + \psi^2 \{\ldots\} \right] \right)^2 + \psi^3 \{\ldots\} + \ldots \tag{138}$$

where the coefficients here are the same as those previous under expansion and our correction term here will introduce new terms of degree 4 which we must take into account when we go on to look at A_3. For now we are only interested in the coefficient of ψ^3 as a simple change of variable, i.e. let $\theta''' = (\ldots)$ in (138), will give us our required quadratic term here. Thus, our condition for A_2 will be that the coefficient of ψ^3 in (138) is non-zero, since we would wish to make a change of variable of the form $\psi'' = \psi \{\ldots\}^{\frac{1}{3}}$ where $\{\ldots\}$ is the coefficient of ψ^3 in (138), in order for f to take the following form, as in (134).

$$f = \theta'''^2 + \psi''^3 + \ldots$$

Remember here that cubes have unique roots, so we don't find that $f = \theta'''^2 \pm \psi''^3 + \ldots$ and we conclude that f is an A_2 if $K \neq 0$ and that the coefficient of $\psi^3 \neq 0$, i.e.

$$\left(h_\psi (\mathbf{0}) + h_{\psi\psi\psi} (\mathbf{0}) \right) + 3 \left(h_\theta (\mathbf{0}) + h_{\theta\psi\psi} (\mathbf{0}) \right) \left(\frac{L}{2K} \right) + 3 h_{\theta\theta\psi} (\mathbf{0}) \left(\frac{L}{2K} \right)^2 + \left(h_\theta (\mathbf{0}) + h_{\theta\theta\theta} (\mathbf{0}) \right) \left(\frac{L}{2K} \right)^3 \neq 0$$

and note that we have multiplied up by -6 here.

7.3 Case 2: $K = 0$, f has type A_1?

Let us return to the point at which we first assumed that $K \neq 0$ during the course of our A_1 calculations and instead, let us take K to equal 0, i.e. we had,

$$f = K\theta^2 + L\theta\psi + M\psi^2 + \dots$$

so, if we allow $K = 0$, then we have $f = L\theta\psi + M\psi^2 + \dots$ but f cannot take the form of a perfect square here. We conclude that if $K = 0$, then L must also equal 0 if we are to express f as a perfect square and therefore, our alternative conditions for f to be an A_1 singularity are,

$$K = 0 \Longrightarrow h(\mathbf{0}) + h_{\theta\theta}(\mathbf{0}) = 0 \ \& \ L = 0 \Longrightarrow h_{\theta\psi}(\mathbf{0}) = 0.$$

7.4 Case 2: $K = 0$, f has type A_2?

As a consequence of taking the value of K to be zero, we must know look for f in an unfamiliar form, i.e. not $f = \theta^2 + \psi^3 + \dots$, but rather $f = \psi^2 + \theta^3 + \dots$, where our f, with $K = L = 0$, is of the form,

$$f = M\psi^2 - \frac{1}{6}(h_\theta + h_{\theta\theta\theta})\theta^3 - \frac{1}{6}\left(h_\psi + h_{\psi\psi\psi}\right)\psi^3 - \frac{1}{2}\left(h_\theta + h_{\theta\psi\psi}\right)\theta\psi^2 - \frac{1}{2}h_{\theta\theta\psi}\theta^2\psi + \dots$$

noting that h and all its partial derivatives are measured at $\mathbf{0}$ here and we are not interested in terms of degree > 3 here. Firstly we change vaiable here, let $\psi''' = \sqrt{M}\psi$ and now we can complete the square in the same way that we did for the $K \neq 0$ case, i.e.

$$f = \left(\psi''' - \frac{1}{2}\left[\frac{1}{6}\left(h_\psi + h_{\psi\psi\psi}\right)\psi^2 + \frac{1}{2}\left(h_\theta + h_{\theta\psi\psi}\right)\theta\psi\right]\right)^2 - \frac{1}{6}(h_\theta + h_{\theta\theta\theta})\theta^3 + \dots \tag{139}$$

noting that there will again be new terms of degree 4 as a result of the correction term. In conclusion, as was the case when K did not equal 0, our condition for an A_2 here is that the coefficient of our variable, which has no terms of degree less than 3, is non-zero. Therefore, if we let the coefficient of θ^3 in (139) be U, then our 3 conditions for f to be an A_2 are as follows.

$$K = 0 \Longrightarrow h(\mathbf{0}) + h_{\theta\theta}(\mathbf{0}) = 0 \tag{140}$$

$$L = 0 \Longrightarrow h_{\theta\psi}(\mathbf{0}) = 0 \tag{141}$$

$$U \neq 0 \Longrightarrow h_\theta(\mathbf{0}) + h_{\theta\theta\theta}(\mathbf{0}) \neq 0 \tag{142}$$

7.5 What do these 2 cases mean in terms of chosen support function h?

Since most of the conditions for A_1 here are just the opposite of some or all of the conditions for A_2, let us look at these alone. In case one, we had that $K \neq 0$ and in case 2 we had that

$K = 0$ and in terms of our support function (see equation (121)) we find that $K = a + k - a = k$. We want to check these first conditions for f to be an A_2 against our support function, they are,

$$\text{let } k \begin{cases} \neq 0, & \text{for case 1} \\ = 0, & \text{for case 2} \end{cases}$$

neither of which are unreasonable. Our second condition, imposed on case 1 only, is that the matrix $(H + hI)$ must be singular at $(\theta, \psi) = (0, 0)$ where this matrix is equal to,

$$\begin{pmatrix} k & 0 \\ 0 & k - 2a \end{pmatrix}$$

and so its determinant is equal to $k(k - 2a)$ which means that our matrix is singular if $k = 0$ or $k = 2a$, therefore our second condition on our support function is that we must,

$$\text{let } k \begin{cases} = 2a, & \text{for case 1} \\ = 0, & \text{for case 2} \end{cases}$$

using the fact that we've just shown $k \neq 0$ in case 1 and of course, this means that $a \neq 0$ for case 1. The final condition for case 1 was,

$$\left(h_\psi(\mathbf{0}) + h_{\psi\psi\psi}(\mathbf{0})\right) + 3\left(h_\theta(\mathbf{0}) + h_{\theta\psi\psi}(\mathbf{0})\right)\left(\frac{L}{2K}\right) + 3h_{\theta\theta\psi}(\mathbf{0})\left(\frac{L}{2K}\right)^2 + \left(h_\theta(\mathbf{0}) + h_{\theta\theta\theta}(\mathbf{0})\right)\left(\frac{L}{2K}\right)^3 \neq 0$$

which is very complicated for the general case, but becomes very simple for our specific support function. In fact, it simplifies to the following condition,

$$k^3 b \neq 0$$

which tells us that $k \neq 0$ (we already knew this but, importantly, there is no contradiction) and $b \neq 0$ either. An example of a surface satisfying such conditions is shown in figure 18.

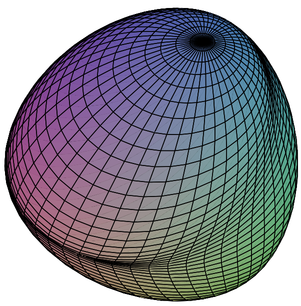

Figure 18: Surface with an A_2 singularity given by $h(\theta, \psi) = \cos^2\psi\,(3\cos\theta\cos\psi + \sin\psi) + 6$.

The next condition for A_2 in case two is that $L = 0$ which implies that $h_{\theta\psi}(\mathbf{0}) = 0$ and this condition is automatically satisfied for our h, i.e. a, b and k are arbitrary in (121) for this condition. Finally, when $K \neq 0$, for f to be an A_2, we require that $U \neq 0$, which implies that $h_\theta(\mathbf{0}) + h_{\theta\theta\theta}(\mathbf{0}) \neq 0$ generally, but for our h, we find that $U = 0$ for arbitrary a, b and k. This is a contradiction, so f cannot be an A_2 when $K = 0$.

Proposition 7.1 *For a support function of the form,* $h(\theta, \psi) = \cos^2\psi\,(a\cos\theta\cos\psi + b\sin\psi) + k$*, f has type A_2 at $(\theta, \psi) = (0, 0) \iff k = 2a \neq 0$ and $b \neq 0$.*

With regards to the conditions for A_1, case one requires a non-zero constant k in our support function (121) and that k does not equal $2a$, which seems consistent with our work in the section on smooth surfaces and, notably, proposition 6.1. Case 2 says that f is an A_1 if $k = 0$ and, whilst this is fine in theory, we must remember that our support function (121) was constructed such that it would produce a surface with width $2k$ and a surface with zero width is not desirable in this piece of work.

Proposition 7.2 *For a support function of the form,* $h(\theta, \psi) = \cos^2\psi\,(a\cos\theta\cos\psi + b\sin\psi) + k$*, f has type A_1 at $(\theta, \psi) = (0, 0)$ if $k \neq 0$ or $2a$.*

7.6 Does f have type A_3?

With propositions 7.1 and 7.2 in mind, we shall now look for a condition under which f has type A_3, where $K \neq 0$ (case 1) only. These calculations can be made far simpler, hence easier to understand, if we use Maple (see Appendix 6) and consider results only for our chosen support function (see equation (121)), rather than for general h.

Firstly, we expand $h(\theta, \psi)$ plus the sine and cosine functions in f where,

$$f = x_0\cos\theta\cos\psi + y_0\sin\theta\cos\psi + z_0\sin\psi - h(\theta, \psi)$$

as Taylor series around the base point $(\theta, \psi) = (0, 0)$ up to and including terms of order 5,

$$f = (x_0 - a - k) + y_0\theta + (z_0 - b)\psi - \tfrac{1}{2}(x_0 - a)\theta^2 - \tfrac{1}{2}(x_0 - 3a)\psi^2 - \tfrac{1}{6}y_0\theta^3 - \tfrac{1}{6}(z_0 - 7b)\psi^3 -$$

$$\tfrac{1}{2}y_0\theta\psi^2 + \tfrac{1}{24}(x_0 - a)\theta^4 + \tfrac{1}{24}(x_0 - 21a)\psi^4 + \tfrac{1}{4}(x_0 - 3a)\theta^2\psi^2 + \tfrac{1}{120}y_0\theta^5 + \tfrac{1}{120}(z_0 - 61b)\psi^5 +$$

$$\tfrac{1}{24}y_0\theta\psi^4 + \tfrac{1}{12}y_0\theta^3\psi^2 + \dots$$

noting that, in equation (132), we wrote f when considering a general support function h, i.e. x_0, y_0 and z_0 here were written as $h(\mathbf{0}), h_\theta(\mathbf{0})$ and $h_\psi(\mathbf{0})$ respectively in (132). For our chosen h, $x_0 = a + k$, $y_0 = 0$ and $z_0 = b$ so, by substituting these into f we obtain,

$$f = -\frac{1}{2}k\theta^2 - \frac{1}{2}(k - 2a)\psi^2 + b\psi^3 + \frac{1}{4}(k - 2a)\theta^2\psi^2 + \frac{1}{24}k\theta^4 + \frac{1}{24}(k - 20a)\psi^4 - \frac{1}{2}b\psi^5 + \dots$$

and we can see from this quite clearly our conditions for A_1, A_2 and A_3, when $K \neq 0$ (which implies $k \neq 0$ in our specific h). To elaborate, we have seen that when f takes this form, that is, when f has no $\theta\psi$ term, it has type A_1 (when $k \neq 0$) if the coefficient of $\psi^2 \neq 0$, i.e. f has type A_1 when $k \neq 0$ and,

$$-\frac{1}{2}(k-2a) \neq 0 \Longrightarrow k \neq 2a$$

as shown for proposition 7.2. Furthermore, when $k \neq 0$ in f, it has type A_2 if the coefficient of $\psi^2 = 0$, which implies that $k = 2a$. Finally, since there are no cubic terms, other than $b\psi^3$, the only other condition for f to have type A_2 is that b, the coefficient of ψ^3, is non-zero (as shown in proposition 7.1).

With reference to (135) and baring in mind that $k \neq 0$ and $k = 2a$ from previous calculations, we need the coefficient of $\psi^3 = 0$, which means that $b = 0$ when f has type A_3. Substituting these values of k and b into f, we have,

$$f = -a\theta^2 + \frac{1}{12}a\theta^4 - \frac{3}{4}a\psi^4 + \dots \tag{143}$$

and in fact, $b = 0$ is the only additional condition needed since $k = 2a \neq 0$ makes the coefficient of ψ^4 non-zero.

Proposition 7.3 *For a support function of the form,* $h(\theta,\psi) = \cos^2\psi\,(a\cos\theta\cos\psi + b\sin\psi) + k$, *$f$ has type A_3 at $(\theta,\psi) = (0,0)$ if $k = 2a \neq 0$ and $b = 0$.*

7.7 Versal Unfoldings

Let us recall the following definition from MATH 443.

Definition
Suppose F' is a versal unfolding (r parameters) of a singularity f' of type A_k. Then $\mathcal{D}_{F'}$ is locally diffeomorphic to $\mathcal{D}_G$ of any other unfolding G with r parameters of a singularity of type A_k.

When a singularity is of type A_k we can make our calculations in the vector space of $(k+1)$–jets. In our case, we have a family F with 3 parametrers x, y, z and 2 variables θ, ψ, i.e. $F(\theta,\psi,x,y,z)$ is a 3-parameter unfolding of our 2 variable function $f = f(\theta,\psi)$.

This unfolding is versal if f_θ, f_ψ plus any multiples by monomials of the form $\theta^i\psi^j$, together with F_x, F_y and F_z, taken at our base point $(\theta,\psi) = (0,0)$, span the vector space of all monomials in θ, ψ of degree $\leq k+1$ (when they have been truncated at degree $k+1$).

For the case where f has type A_2, we need to express F_x, F_y, F_z, f_θ and f_ψ as 3-jets where F_x, F_y, F_z are defined in (124) and,

$$f = -a\theta^2 + b\psi^3 + \frac{1}{12}a\theta^4 - \frac{3}{4}a\psi^4 - \frac{1}{2}b\psi^5 + \dots$$

remembering that $k = 2a$ for type A_2 and that we truncate everything at degree 3. We then set up a table, whereby the columns headings represent potential elements in the 3-jets of the row headings and we insert the appropriate coefficients into the table itself.

	1	θ	ψ	θ^2	$\theta\psi$	ψ^2	θ^3	$\theta^2\psi$	$\theta\psi^2$	ψ^3
F_x	1	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	0	0
F_y	0	1	0	0	0	0	$-\frac{1}{6}$	0	$-\frac{1}{2}$	0
F_z	0	0	1	0	0	0	0	0	0	$-\frac{1}{6}$
f_θ	0	$-2a$	0	0	0	0	$\frac{a}{3}$	0	0	0
f_ψ	0	0	0	0	0	$3b$	0	0	0	$-3a$
θf_θ	0	0	0	$-2a$	0	0	0	0	0	0
ψf_ψ	0	0	0	0	0	0	0	0	0	$3b$
ψf_θ	0	0	0	0	$-2a$	0	0	0	0	0
θf_ψ	0	0	0	0	0	0	0	0	$3b$	0
$\theta\psi f_\theta$	0	0	0	0	0	0	0	$-2a$	0	0
$\theta^2 f_\theta$	0	0	0	0	0	0	$-2a$	0	0	0

This is an 11 by 10 matrix, which we would like to show has maximal rank of 10, since this would mean that F versally unfolds f of type A_2. We do this using a method in which we look for any row or column in which there is only one non-zero entry, we then cross out both the row and column of that entry. For example, we can see that the bottom row of our matrix has only one non-zero entry, namely $-2a$, so we cross out both the bottom row and the θ^3 column.

A natural question to ask at this point would be, why are we doing this? The answer is, because we would like to show that this matrix has non-zero determinant and clearly, this is a very time consuming task, unless we can reduce the matrix, which is what we do when crossing out rows and columns. If, after reducing our matrix, the remaining matrix is non-singular, we can say that F versally unfolds f of type A_2.

We carry out the following operations on our matrix,

(op 1) Cross out $\theta^2 f_\theta$ row, so must cross out θ^3 column (non-zero entry was $-2a$).

(op 2) Cross out $\theta\psi f_\theta$ row, so must cross out $\theta^2\psi$ column (non-zero entry was $-2a$).

(op 3) Cross out θf_ψ row, so must cross out $\theta\psi^2$ column (non-zero entry was $3b$).

(op 4) Cross out ψf_θ row, so must cross out $\theta\psi$ column (non-zero entry was $-2a$).

(op 5) Cross out ψf_ψ row, so must cross out ψ^3 column (non-zero entry was $3b$).

(op 6) Cross out θf_θ row, so must cross out θ^2 column (non-zero entry was $-2a$).

(op 7) Cross out f_ψ row, so must cross out ψ^2 column (non-zero entry was $3b$).

(op 8) Cross out F_z row, so must cross out ψ column (non-zero entry was 1).

(op 9) Cross out F_x row, so must cross out 1 column (non-zero entry was 1).

and these leave us with a 2 by 1 matrix, namely,

	θ
F_y	1
f_θ	$-2a$

which has rank 1, since we don't have 2 non-zero entries. We carried out 9 operations which means that, if this reduced matrix has rank 1, then our 11 by 10 matrix must have rank equal to $1 + 9 = 10$. This is maximal which means that when f has type A_2, it is versally unfolded by x, y and z at $(\theta, \psi) = (0, 0)$. A versally unfolded A_2 has discriminant locally diffeomorphic to a cuspidal edge, therefore, figure 18 shows a surface with a cuspidal edge.

Proposition 7.4 *For a support function of the form, $h(\theta, \psi) = \cos^2\psi\,(a\cos\theta\cos\psi + b\sin\psi) + k$, our surface exhibits a cuspidal edge at $(\theta, \psi) = (0, 0) \iff k = 2a \neq 0$ and $b \neq 0$.*

For the case of A_3, that is, where $k = 2a \neq 0$ and $b = 0$ in f, we need to express F_x, F_y, F_z, f_θ and f_ψ as 4-jets. We can do this in a parallel fashion to the A_2 method, setting up a table, whereby the columns headings represent potential elements in the 4-jets of the row headings and we insert the appropriate coefficients in the table itself.

	1	θ	ψ	θ^2	$\theta\psi$	ψ^2	θ^3	$\theta^2\psi$	$\theta\psi^2$	ψ^3	θ^4	$\theta^3\psi$	$\theta^2\psi^2$	$\theta\psi^3$	ψ^4
F_x	1	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	0	0	$\frac{1}{24}$	0	$\frac{1}{4}$	0	$\frac{1}{24}$
F_y	0	1	0	0	0	0	$-\frac{1}{6}$	0	$-\frac{1}{2}$	0	0	0	0	0	0
F_z	0	0	1	0	0	0	0	0	0	$-\frac{1}{6}$	0	0	0	0	0
f_θ	0	$-2a$	0	0	0	0	$\frac{a}{3}$	0	0	0	0	0	0	0	0
f_ψ	0	0	0	0	0	0	0	0	0	$-3a$	0	0	0	0	0
θf_θ	0	0	0	$-2a$	0	0	0	0	0	0	$\frac{a}{3}$	0	0	0	0
ψf_ψ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-3a$
ψf_θ	0	0	0	0	$-2a$	0	0	0	0	0	0	$\frac{a}{3}$	0	0	0
$\theta^2 f_\theta$	0	0	0	0	0	0	$-2a$	0	0	0	0	0	0	0	0
$\psi^2 f_\theta$	0	0	0	0	0	0	0	0	$-2a$	0	0	0	0	0	0
$\psi^3 f_\theta$	0	0	0	0	0	0	0	0	0	0	0	0	0	$-2a$	0
$\theta\psi^2 f_\theta$	0	0	0	0	0	0	0	0	0	0	0	0	$-2a$	0	0
$\theta\psi f_\theta$	0	0	0	0	0	0	0	$-2a$	0	0	0	0	0	0	0
$\theta^3 f_\theta$	0	0	0	0	0	0	0	0	0	0	$-2a$	0	0	0	0
$\theta^2\psi f_\theta$	0	0	0	0	0	0	0	0	0	0	0	$-2a$	0	0	0

This is a 15 by 15 matrix which we would like to show has maximal rank 15 as this would show that f of type A_3 is versally unfolded by F. In our table we see that the bottom 7 rows have single non-zero entries so let us cross out those entries, including their columns which leaves us with the following 8 by 8 matrix.

	1	θ	ψ	θ^2	$\theta\psi$	ψ^2	θ^3	ψ^3
F_x	1	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0
F_y	0	1	0	0	0	0	$-\frac{1}{6}$	0
F_z	0	0	1	0	0	0	0	$-\frac{1}{6}$
f_θ	0	$-2a$	0	0	0	0	$\frac{a}{3}$	0
f_ψ	0	0	0	0	0	0	0	$-3a$
θf_θ	0	0	0	$-2a$	0	0	0	0
ψf_θ	0	0	0	0	$-2a$	0	0	0
$\theta^2 f_\theta$	0	0	0	0	0	0	$-2a$	0

Here, the bottom 4 rows have single non-zero entries so let us cross out those entries, including their columns which leaves us with the following 4 by 4 matrix.

	1	θ	ψ	ψ^2
F_x	1	0	0	$-\frac{1}{2}$
F_y	0	1	0	0
F_z	0	0	1	0
f_θ	0	$-2a$	0	0

This 4 by 4 matrix clearly has zero determinant, so it does not have maximal rank, hence nor does the original 15 by 15 matrix. Therefore when f has type A_3, it is not versally unfolded by f and, in conclusion, our surface, defined by our h (see (121)), never exhibits a swallowtail at the point $(\theta, \psi) = (0, 0)$.

8 Conclusions, further work and acknowledgements

In this project, we have learnt much about surfaces of constant width and their properties. We began by outlining the fundamental definitions and concepts, introducing the idea of defining a surface in terms of a support function, which was key to this work. The condition for constant width was presented and we made some early conjectures as to the most general form our support functions could take.

In section 3, we looked at some simple examples of smooth surfaces of constant width. The aim here was to make the concepts outlined in section 2 seem less abstract and to improve the understanding of the reader. In section 4, we looked in depth at curvature in 3 dimensions, which was a vitally important topic. We introduced principal curvatures, Gauss cuvature and mean curvature, all of which had great significance in the proofs of many theorems and were utilised in this and subsequent sections.

Section 5 introduced the shape operator and its many uses. For exmaple, we went on to look at parallel directions, proving that they were always parallel at points of parallel tangency on the surface. The remainder of the section tried to use the shape operator to modify the support function as we wanted to ensure that the surface corresponding to our suggested support function was smooth everywhere. Section 6 then pursued this work further, by examining more closely the parts played by the constant terms in our support function, which also had constraints for smooth surfaces. Section 7 found that it would be possible for a surface, produced by our chosen support function, to have cuspidal edges on the $x-$axis, but not swallowtails.

Certainly there are many possibilities for further research into this area, in particular, we did not look closely at centre symmetry sets in 3 dimensions, nor focal surfaces. Perhaps we could look for a parametrisation of the CSS and the focal surface, in terms of the support function h. Also, it would be wrong to assume that our $h = a\cos\theta\sin^3\phi + b\cos\phi\sin^2\phi + k$ is the most general form h could take for a SCW as it would seem, for example, that $h = a\cos P\theta\sin^3\phi + b\cos\phi\sin^2\phi + k$ (P odd, ≥ 1) would give a SCW, though we haven't checked for smoothness (also, see Appendix 7). This is a very interesting area with much literature available, although a lot of it is written in german and not everyone would be fortunate enough to have a supervisor fluent in german!

At this point, I would particularly like to thank Prof. Peter Giblin, who has supervised all of my projects this year and given up huge amounts of his time to help me. I know that he has a lot of other students who rely upon him as much as I did and I shall always be truely grateful for his kindness and generosity. Thank you also to my office mates Jack and Matt

who, along with many others, put up with my shyness and made me laugh a lot. Finally, I must mention Prof. Peter Newstead who obtained a bursary for me this year and made sure I had a place on this course. He leaves Liverpool this year to retire and I know he will be sorely missed by evreryone, I will never be able to thank him enough.

9 Maple Appendices

9.1 Appendix 1

Here we use a support function completely independent of θ, i.e. we have basically revolved a CCW around the $x, y-$plane.

```
restart;
with(plots):with(plottools):
```

Consider our surface, parametrised by x, y, z as follows.

```
x:=h(theta,phi)*cos(theta)*sin(phi)-diff(h(theta,phi),theta)
*sin(theta)/sin(phi)+diff(h(theta,phi),phi)*cos(phi)*cos(theta);
y:=h(theta,phi)*sin(theta)*sin(phi)+diff(h(theta,phi),theta)
*cos(theta)/sin(phi)+diff(h(theta,phi),phi)*cos(phi)*sin(theta);
z:=h(theta,phi)*cos(phi)-diff(h(theta,phi),phi)*sin(phi);
```

Define the support function $h = h\left(\theta, \phi\right)$, modifying slightly for figures 2 and 18.

```
h(theta,phi):=P*cos(Q*phi)+R;
```

Consider different values of P, Q and R.

```
P1:=1/16;Q1:=3;R1:=1;
P2:=1/48;Q2:=5;R2:=1;
P3:=1/96;Q3:=7;R3:=1;
```

Substitute these into h to give 3 different support functions.

```
h1:=subs(P=P1,Q=Q1,R=R1,h(theta,phi));
h2:=subs(P=P2,Q=Q2,R=R2,h(theta,phi));
h3:=subs(P=P3,Q=Q3,R=R3,h(theta,phi));
```

Define x, y, z using these values calling them X, Y, Z to avoid confusion.

```
X1:=subs(h(theta,phi)=h1,P=P1,Q=Q1,R=R1,x);
Y1:=subs(h(theta,phi)=h1,P=P1,Q=Q1,R=R1,y);
Z1:=subs(h(theta,phi)=h1,P=P1,Q=Q1,R=R1,z);
X2:=subs(h(theta,phi)=h2,P=P2,Q=Q2,R=R2,x);
Y2:=subs(h(theta,phi)=h2,P=P2,Q=Q2,R=R2,y);
Z2:=subs(h(theta,phi)=h2,P=P2,Q=Q2,R=R2,z);
X3:=subs(h(theta,phi)=h3,P=P3,Q=Q3,R=R3,x);
Y3:=subs(h(theta,phi)=h3,P=P3,Q=Q3,R=R3,y);
Z3:=subs(h(theta,phi)=h3,P=P3,Q=Q3,R=R3,z);
```

Define and display 3 surfaces.

```
plot1:=plot3d([X1,Y1,Z1],theta=0..2*Pi,phi=0..Pi,grid=[50,50]):
plot2:=plot3d([X2,Y2,Z2],theta=0..2*Pi,phi=0..Pi,grid=[50,50]):
plot3:=plot3d([X3,Y3,Z3],theta=0..2*Pi,phi=0..Pi,grid=[50,50]):
display(plot1);display(plot2);display(plot3);
```

9.2 Appendix 2

This programme relates to checking that our new support function is smooth everywhere (including the poles).

```
restart;
with(plots):
```

Consider our support function in θ, ϕ.

```
h1:=a*cos(theta)*sin(phi)^3+sin(phi)^2*b*cos(phi)+k;
```

Let $s1 = \sin\theta, s2 = \sin\phi, c1 = \cos\theta$ and $c2 = \cos\phi$. This is so that we have expressions for θ, ϕ in terms of σ, τ.

```
s2:=sqrt(cos(tau)^2+cos(sigma)^2*sin(tau)^2);
c2:=sin(sigma)*sin(tau);
s1:=cos(sigma)*sin(tau)/s2;
c1:=cos(tau)/s2;
```

Corresponding support function in σ, τ.

```
h:=(cos(tau)^2+cos(sigma)^2*sin(tau)^2)*(a*cos(tau)
b*sin(sigma)*sin(tau))k;
```

Shape operator entries in θ, ϕ, noting that D is protected in Maple (it means something different), so we use DD instead.

```
A:=simplify(h1+diff(diff(h1,theta),theta)/sin(phi)^2
+diff(h1,phi)*cos(phi)/sin(phi));
B:=simplify(diff(diff(h1,theta),phi)-diff(h1,theta)*cos(phi)/sin(phi));
C:=diff(diff(h1,theta),phi)/sin(phi)^2-diff(h1,theta)*cos(phi)/sin(phi)^3;
DD:=simplify(h1+diff(diff(h1,phi),phi));
```

Smoothness condition for θ, ϕ.

```
smooth1:=simplify(A*DD-B*C);
```

But now we want this in σ, τ.

```
A2:=subs(sin(theta)=s1,cos(theta)=c1,sin(phi)=s2,cos(phi)=c2,A);
B2:=subs(sin(theta)=s1,cos(theta)=c1,sin(phi)=s2,cos(phi)=c2,B);
C2:=subs(sin(theta)=s1,cos(theta)=c1,sin(phi)=s2,cos(phi)=c2,C);
DD2:=subs(sin(theta)=s1,cos(theta)=c1,sin(phi)=s2,cos(phi)=c2,A);
```

Smoothness condition for σ, τ.

```
smooth2:=A2*DD2-B2*C2;
```

Our θ, ϕ parametrisation is certainly smooth away from the poles, but we need to check the poles themselves, where $\sigma = \frac{\pi}{2}, \tau = \frac{\pi}{2}$ and where $\sigma = \frac{3\pi}{2}, \tau = \frac{\pi}{2}$.

```
smooth3a:=evalf(subs(sigma=Pi/2,tau=Pi/2,smooth2));
smooth3b:=evalf(subs(sigma=3*Pi/2,tau=Pi/2,smooth2));
```

Consider values of a, b and k for plotting. Note that k is sufficiently large here.

```
a1:=2;b1:=3;k1:=10;
```

Consider $smooth3a, smooth3b$ and $smooth2$ for these values.

```
smooth3aa:=subs(a=a1,b=b1,k=k1,smooth3a);
smooth3bb:=subs(a=a1,b=b1,k=k1,smooth3b);
```

```
smooth4:=subs(a=a1,b=b1,k=k1,smooth2):
```
Plot *smooth*2 for these values.
```
plot3d(smooth4,sigma=0..2*Pi,tau=0..Pi);
```

9.3 Appendix 3

This programme checks the smoothness of the SCW produced by 2 different support functions using curvature.
```
restart;
```
We need to envoke an extra package when using linear algebra tools in Maple.
```
with(linalg):with(plots):
```
As part of our support function, we have p and r, where $h = (p + r)sin^2\left(\phi\right) + k$. For our new support function (change this accordingly for the old one) we have p as follows.
```
p:=a*cos(theta)*sin(phi);r:=b*cos(phi);
```
Partial derivatives required for Shape Operator matrix S entries.
```
p1:=diff(p,theta);p11:=diff(p1,theta);p2:=diff(p,phi);
p22:=diff(p2,phi);r2:=diff(r,phi);r22:=diff(r2,phi);p12:=diff(p1,phi);
```
Our shape operator had entires $a_{11} = A, a_{12} = C, a_{21} = B$ and $a_{22} = DD$.
```
A:=(p+r)*(1+cos(phi)^2)+k+p11+sin(phi)*cos(phi)*(p2+r2);
B:=sin(phi)*cos(phi)*p1+p12*sin(phi)^2;
C:=p1*cos(phi)/sin(phi)+p12;
DD:=(p+r)*(3*cos(phi)^2-1)+k+4*cos(phi)*sin(phi)*(p2+r2)+sin(phi)^2*(p22+r22);
```
So our matrix $-S^{-1}$ is called M here.
```
M:=matrix(2,2,[A,C,B,DD]);
```
Remember that the determinant of a matrix is equal to the product of its eigenvalues, which in our case equal the principal curvatures and therefore, the determinant equals the Gaussian curvature.
```
dM:=simplify(det(M));
```
Let $\phi = 0$, so that we are looking along a meridian (θ fixed) and the Gaussian curvature should then not depend on θ if our surface is smooth.
```
dM0:=simplify(subs(sin(phi)=0,cos(phi)=1,dM));
```
Remember that the trace of a matrix is equal to the sum of its eigenvalues, which in our case equal the principal curvatures and therefore, the trace equals (twice) the mean curvature.
```
tM:=trace(M);
```
Let $\phi = 0$, so that we are looking along a meridian and the mean curvature should then not depend on theta if our surface is smooth.
```
tM0:=simplify(subs(sin(phi)=0,cos(phi)=1,tM));
```

9.4 Appendix 4

First Part: This programme relates to the proof that our constant $k > 2a$ (when $b = 0$) and $k \geq 2a + 2b$ if we are to avoid singularities.

```
restart;
with(plots):with(plottools):with(student):with(PDEtools):with(linalg):
```

Use the following series of commands to save on typing.

```
declare(h(theta,phi),p(theta,phi),expr(theta,phi),r(phi),expr2(theta,phi),
expr3(theta,phi),h_sub(theta,phi),h_sub2(theta,phi),A(theta,phi),B(theta,phi),
C(theta,phi),DD(theta,phi));
```

Consider functions p and r in our support function.

```
p(theta,phi):=a*cos(theta);r(phi):=b*cos(phi);
```

Our support function.

```
h(theta,phi):=sin(phi)^3*p(theta,phi)+sin(phi)^2*r(phi)+k;
```

Our shape operator has entries A, B, C and DD.

```
A(theta,phi):=simplify(h(theta,phi)+diff(h(theta,phi),theta,theta)/sin(phi)^2
+diff(h(theta,phi),phi)*cos(phi)/sin(phi));
B(theta,phi):=simplify(-diff(h(theta,phi),theta)*cos(phi)/sin(phi)
+diff(h(theta,phi),theta,phi));
C(theta,phi):=simplify(diff(h(theta,phi),theta,phi)/sin(phi)^2
-cos(phi)/sin(phi)^3*diff(h(theta,phi),theta));
DD(theta,phi):=simplify(h(theta,phi)+diff(h(theta,phi),phi,phi));
```

Condition for nonsingular SCW is that this should never be zero for $0 < \phi < \pi$.

```
expr(theta,phi):=simplify(A(theta,phi)*DD(theta,phi)-B(theta,phi)*C(theta,phi));
```

Consider $AD - BC$ as a quadratic in k.

```
kquad2:=series(expr(theta,phi),k);
```

Express this as a Taylor series in a and b when $k = 2a + 2b$.

```
ktayl2:=mtaylor(subs(k=2*a+2*b,kquad2),[a,b],3);
```

Second Part: Applies to $k > 2a$ stuff only. Consider the more trivial case where $b = 0$.

```
expr2(theta,phi):=simplify(eval(subs(b=0,expr(theta,phi))));
```

Possible values for constants a and k.

```
a_val:=1;k_val:=3;
```

Could this be zero?

```
expr3a:=subs(a=a_val,k=k_val,expr2(theta,phi));
```

Consider $AD - BC$ as a quadratic in k, when $b = 0$.

```
kquad:=series(expr2(theta,phi),k);
```

Express this as a Taylor series in a when $k = 2a$.

```
ktayl:=mtaylor(subs(k=2*a,kquad),[a,b],3);
```

Notice this is the same as the first term in *ktayl*2 above. Dividing by a^2 and letting $e = \frac{k}{a}$ we find *equad*.

```
equad:=e^2+2*e*cos(theta)*sin(phi)*(5*cos(phi)^2-1)+4*cos(phi)^2*
```

```
sin(phi)^2*(4*cos(theta)^2*cos(phi)^2-1);
```
Find the disriminant of *equad*.
```
alpha:=1;beta:=2*cos(theta)*sin(phi)*(5*cos(phi)^2-1);
delta:=4*cos(phi)^2*sin(phi)^2*(4*cos(theta)^2*cos(phi)^2-1);
discrim1:=simplify(beta^2-4*alpha*delta,trig);
```
Simplify the discriminant here.
```
discrim2:=discrim1/(4*sin(phi)^2);
discrim3:=subs(cos(phi)^2=1-sin(phi)^2,cos(phi)^4=(1-sin(phi)^2)^2,discrim2);
```

This part doesn't appear to continue on from the previous, but see section 6 for further details. Plot of boundaries of y, z (variables of function H).
```
implicitplot([y^2+z=1,z=0],y=-1..1,z=0..1,scaling=constrained);
```
Actual plot of $H(y, z)$.
```
implicitplot(z^2*(4*y^2+1)+z*(5*y-1)+1-y,y=-1..1,z=0..1,
grid=[100,100],scaling=constrained);
```
Consider value of $H(y, z)$ for boundary $z = 1 - y^2$.
```
H:=z^2*(4*y^2+1)+z*(5*y-1)+1-y;
H1:=factor(simplify(subs(z=1-y^2,H)));
```
Multiply first and second brackets by -1 to give $H2$.
```
H2:=(1-y)*(1+y-y^3)*(1+2*y)^2;
```
Not obvious whether second bracket is greater than or equal to 0. Consider graph of it.
```
f:=1+y-y^3;
plot(f,y=-1..1,scaling=constrained);
subs(y=-5/9,f);
```
We want partial derivatives of H w.r.t. y, z.
```
Hz:=diff(H,z);Hy:=diff(H,y);
```
We want to solve $Hy = Hz = 0$ for y, z.
```
solve({Hz=0,Hy=0},{y,z});
```
We find the following polynomial for z, which we wish to solve.
```
poly1:=16*z^4-8*z^3-5*z+1;
fs:=fsolve(poly1=0,z);
```
The results are as follows.
```
z_1:=0.1929424625;z_2:=0.8384735937;
```
Check boundary condition.
```
y_1:=z_1-2*z_1^2;y_2:=z_2-2*z_2^2;
```
Find the corresponding value of H.
```
H_1:=subs(y=y_1,z=z_1,H);H_2:=subs(y=y_2,z=z_2,H);
```
Do these satisfy boundary condition?
```
bound1:=y_1^2+z_1;bound2:=y_2^2+z_2;
```

9.5 Appendix 5

This picks up directly from the end of the first part of Appendix 3.

It is now enough to put $b = 1$. Note this is equivalent to dividing through by b and then replacing $\frac{a}{b}$ by a. We hope that, if $a \geq 0$ and θ, ϕ are arbitrary, with $0 \leq \phi \leq \pi$ then $Y > 0$. So try to find the minimum.

```
Y:=subs(b=1,ktayl2);
```

There are three terms in Y. The coefficient of a^2 has already been shown to be ≥ 0 for all θ and ϕ (this was the case $b = 0$ done earlier). So we'll concentrate on the other two terms (coefficients of a and constant term), and find that these are > 0 for all values of θ and phi. First the constant term:

```
Y0:=subs(a=0,Y)/4;
```

So the zeros are at $\cos\theta = -1$ (double root) and $\cos\theta = \frac{1}{2}$ (double root) only since $c^2 - c + 1 = 0$ has no real solutions. So $Y0$ is certainly ≥ 0 on $[0, \pi]$, and $= 0$ if and only if $\theta = \frac{\pi}{3}, \pi$.

```
factor(subs(cos(phi)=c,Y0));
plot(Y0,phi=0..Pi);
```

Now the coefficient of a. The first bracket is ≥ 0.

```
Ya:=factor(-16*sin(phi)*cos(theta)*cos(phi)^3-4*cos(theta)*sin(phi)
+20*cos(phi)^2*sin(phi)*cos(theta)+8-12*cos(phi)+20*cos(phi)^3+32*sin(phi)
*cos(theta)*cos(phi)^5);
```

Remove part we know to be ≥ 0 to get $Ya2$.

```
Ya2:=op(3,Ya);
```

Looks as if $Ya2$ is also > 0 for all θ and ϕ; if we can prove this then it makes $ktayl2 > 0$ for all $a > 0$ and all θ, ϕ

```
plot3d(Ya2,theta=0..2*Pi,phi=0..Pi);
```

We'll find the turning points of $Ya2$ and show that it is > 0 at all of them. So all turning points of $Ya2$ are given by $\theta = 0, \pi$ or $\phi = 0, \pi, \frac{\pi}{3}$, and one other value of ϕ for which the last bracket is 0. The calculations below just work through the turning points of $Ya2$ and check that at each one the value of $Ya2$ is > 0.

```
Ya2t:=factor(diff(Ya2,theta));
factor(subs(cos(phi)=c,op(5,Ya2t)));
plot(op(5,Ya2t),phi=0..Pi);
fsolve(op(5,Ya2t)=0,phi=1.9..2.1);
Ya2p:=factor(diff(Ya2,phi));
```

Try $\theta = 0$ in $Ya2p$ first. So $\theta = 0$ gives values of ϕ which are rather complicated!

```
Ya2p0:=simplify(subs(sin(theta)=0,cos(theta)=1,Ya2p));
```

So plot the graph of the function, there seems to be 3 solutions.

```
plot(Ya2p0,phi=0..Pi);
sol1:=fsolve(Ya2p0=0,phi=0.2..0.5);
sol2:=fsolve(Ya2p0=0,phi=1..1.5);
```

```
sol3:=fsolve(Ya2p0=0,phi=2.5..3);
```
Then put $\theta = 0$, plus solutions of $Ya2p0$ ($Ya2p$ where $\theta = 0$) in $Ya2$. So $Ya2$ is certainly well clear of 0 at these turning points.
```
evalf(subs(phi=sol1,theta=0,Ya2));
evalf(subs(phi=sol2,theta=0,Ya2));evalf(subs(phi=sol3,theta=0,Ya2));
```
Secondly, put $\theta = \pi$ into $Ya2p$.
```
Ya2p1:=simplify(subs(sin(theta)=0,cos(theta)=-1,Ya2p));
```
Seem to be 3 solutions from graph.
```
plot(Ya2p1,phi=0..Pi);
sol4:=fsolve(Ya2p1=0,phi=0.5..1);
sol5:=fsolve(Ya2p1=0,phi=2..2.1);
sol6:=fsolve(Ya2p1=0,phi=2.4..2.5);
```
Put $\theta = \pi$, plus solutions of $Ya2p1$ ($Ya2p$ where $\theta = \pi$) in $Ya2$.
```
evalf(subs(phi=sol4,theta=Pi,Ya2));
evalf(subs(phi=sol5,theta=Pi,Ya2));
evalf(subs(phi=sol6,theta=Pi,Ya2));
```
Thirdly, try $\phi = 0$ in $Ya2p$, which only allows $\theta = \pm\frac{\pi}{2}$ for it to equal 0.
```
simplify(subs(sin(phi)=0,cos(phi)=1,Ya2p));
```
Put $\phi = 0$, plus solutions of $Ya2p$ (where $\phi = 0$) in $Ya2$.
```
evalf(subs(phi=0,theta=Pi/2,Ya2));
evalf(subs(phi=0,theta=-Pi/2,Ya2));
```
Forthly, put $\phi = \pi$ into $Ya2p$. This only allows $\theta = \pm\frac{\pi}{2}$ as solutions to $Ya2p = 0$.
```
simplify(subs(sin(phi)=0,cos(phi)=-1,Ya2p));
```
Put $\phi = \pi$, plus solutions of $Ya2p$ (where $\phi = \pi$) in $Ya2$.
```
evalf(subs(phi=Pi,theta=Pi/2,Ya2));
evalf(subs(phi=Pi,theta=-Pi/2,Ya2));
```
Fifthly, put $\phi = \frac{\pi}{3}$ into $Ya2p$. This allows solutions of only $\theta = \pm\frac{\pi}{2}$.
```
simplify(subs(sin(phi)=sqrt(3)/2,cos(phi)=1/2,Ya2p));
```
Put $\phi = \frac{\pi}{3}$, plus solutions of $Ya2p$, where $\phi = \frac{\pi}{3}$, in $Ya2$.
```
evalf(subs(phi=Pi/3,theta=Pi/2,Ya2));evalf(subs(phi=Pi/3,theta=-Pi/2,Ya2));
```
Finally, put $\phi = 1.988068135$ into $Ya2p$, where,
```
sin(1.988068135);cos(1.988068135);
Ya2p2:=evalf(subs(phi=1.988068135,Ya2p));
plot(Ya2p2,theta=0..2*Pi);
```
This has no real roots and therefore does not satisfy the turning point condition $Ya2t = Ya2p = 0$.

9.6 Appendix 6

This is a Maple programme designed to calculate the conditions on our chosen support function to give f (see below) to have type A_2 and A_3.
```
restart;
```

Our chosen support function in terms of θ, ψ is h.

```
h:=a*cos(psi)^3*cos(theta)+b*cos(psi)^2*sin(psi)+k;
```

These give $h(0,0) = x = a + k, h_\theta(0,0) = y = 0, h_\psi(0,0) = z = b$.

```
eval(subs(theta=0,psi=0,h));
eval(subs(theta=0,psi=0,diff(h,theta)));
eval(subs(theta=0,psi=0,diff(h,psi)));
```

For this to be singular with value 0 we need $x = a + k, y = 0$ and $z = b$ as above. This can be seen by writing f as a Taylor series around the base point $(0,0)$.

```
f:=mtaylor(x*cos(theta)*cos(psi)+y*sin(theta)*cos(psi)+z*sin(psi)-h,
[theta,psi],6);
```

Condition for this to have type A_2 at least is that $k = 0$ or $k = 2a$. When $k = 2a$ we obtain $x = 3a$ and we also need $b \neq 0$ for exactly A_2. If $k = 0$ then $x = a$ and we can never get A_2.

```
f2:=subs(y=0,z=b,f);
```

If $b = 0$ but $a \neq 0$ then this has type A_3 exactly. So for A_3 we have $x = 3a, y = 0, z = 0$ (since $b = 0$) and $k = 2a, b = 0, a \neq 0$.

```
f3:=subs(x=a+k,y=0,z=b,k=2*a,b=0,f2);
```

9.7 Appendix 7

My reference for this is [CG].

This can almost be considered an extra section, but due to time constraints, we cannot really go into the theory and it was left unfinished. However, this could certainly be looked at in the future. In the cited literature, it is claimed that the volume V of a smooth SCW, as found by a support function h, satisfies the following condition,

$$V \geq \left(\frac{2\pi}{3} - \frac{\sqrt{3}\pi}{4}\cos^{-1}\left(\frac{1}{3}\right)\right) w^3$$

where w is the width. We find that we can express the voulme in terms of a support function h,

$$V = \frac{1}{3}\int \mathbf{x}.\mathbf{u}.dS$$

where dS is an element of the area on our surface T, $\mathbf{x}$ is a point on T and $\mathbf{u}$ is the unit normal. Missing out the theory, this can be expressed more simply as,

$$V = \int_{\phi=0}^{\phi=\pi}\int_{\theta=0}^{\theta=2\pi} h\left(\theta,\phi\right)\left|\mathbf{x}_\theta \times \mathbf{x}_\phi\right|.d\theta.d\phi = \int_{\phi=0}^{\phi=\pi}\int_{\theta=0}^{\theta=2\pi} h\left(\theta,\phi\right)\sin\phi(AD - BC).d\theta.d\phi$$

and so we want to show that this is greater than, or equal to $\left(\frac{2\pi}{3} - \frac{\sqrt{3}\pi}{4}\cos^{-1}\left(\frac{1}{3}\right)\right)w^3$. What follows is the maple programme where we are able to prove the writers' claims when in our support function $h = a\cos\theta\sin^3\phi + b\cos\phi\sin^2\phi + k$ we had $a > 0, b = 0$ and $k = 2a$.

```
restart;with(student):with(plots):
```
Define our support function, where $t = \theta$ and $p = \phi$.
```
h:=(sin(p))^2*(q(t,p)+r(p))+k;
```
Consider functions q and r.
```
q:=a*cos(t)*sin(p);r:=b*cos(p);
h1:=subs(q(t,p)=q,r(p)=r,h);
```
Consider entries for shape operator.
```
A:=simplify(h1+diff(diff(h1,t),t)/(sin(p))^2+diff(h1,p)*cos(p)/sin(p));
B:=diff(diff(h1,t),p)-diff(h1,t)*cos(p)/sin(p);
C:=diff(diff(h1,t),p)/(sin(p))^2-diff(h1,t)*cos(p)/(sin(p))^3;
DD:=simplify(h1+diff(diff(h1,p),p));
```
Consider $(AD - BC)$, we want this to be > 0 for all values of θ and ϕ in order for the SCW to be nonsingular.
```
product1:=simplify(A*DD-B*C);
```
We find (from our own calculations) that our volume V equals the double integral of our support function, multiplied by $\sin\phi(AD - BC)$.
```
product3:=simplify(h1*product1*sin(p));
V:=1/3*value(Doubleint(product3,t = 0..2*Pi, p = 0..Pi));
```
In theory, $V \geq K$ multiplied by w^3, where w equals the width of our surface.
```
K:=evalf(2*Pi/3-Pi*sqrt(3)/4*arccos(1/3));
```
Consider some values which should maintain smoothness and therefore our condition.
```
a_val:='a';b_val:=0;k_val:=2*(a_val+b_val);
V_ex1:=evalf(subs(k=k_val,a=a_val,b=b_val,V));
```
This gives $V_{ex1} = 31.59544613a^3$ and note that the width of our surface with a support function as chosen will always be $2k$ so w^3 here equals $8k^3$. V should be greater than or equal to this for a smooth surface.
```
evalf(K*8*k_val^3);
```
This turns out to be $26.87104294a^3$ and this is less than or equal to V_{ex1} for $a > 0$. ∎

10 Bibliography

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