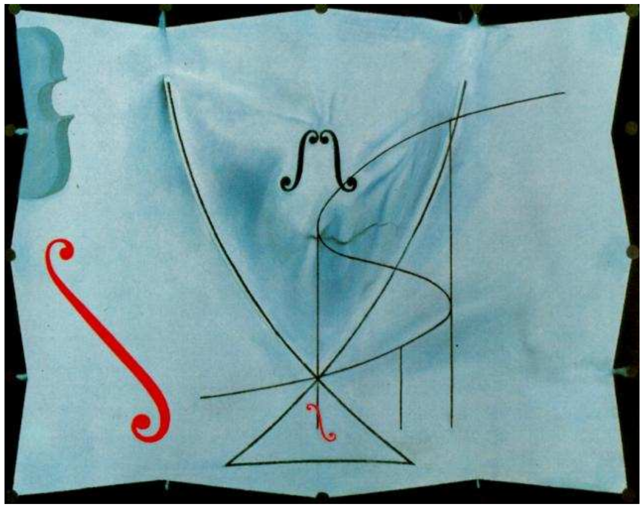

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Curves of Constant Width, Envelopes and Duals of Plane and Space Curves

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Abstract:

We begin by looking at curves of constant width. We study some of their properties such as length, vertices and the conditions necessary to avoid singular points. We then study some envelopes and verify to what extent they are the limit of intersections of nearby curves. We then take a look at duals of plane and space curves, comparing what features on the original curve correspond to singularities in the dual space.

Contents

1 Introduction

This project is divided into three sections; Curves of Constant Width, Envelopes and Duals of Plane and Space Curves. Although there are common themes throughout, each of the chapters can be read as seperate entities. This project takes material from the area of curves and singularities and tries to expand on some ideas. Relatively little prior knowledge is assumed and although most of the material is not technically that difficult, the project includes some quite lengthy calculations at times.

Singularity theory is a relatively new subject which was established by the American mathematician Hassler Whitney in 1955. The subject was then expanded by the French mathematician René Thom in the 1960s and '70s. It was Thom who originally came up with the idea of versal unfoldings which is central to this project. It is Thom's idea of versal unfoldings that enable us to tell whether certain singularities are isomorphic to cusps or swallowtails. The work of Thom greatly impressed the Spanish surrealist painter Salvador Dalí who described his theory of catastrophes as 'the most beautiful aesthetic theory in the world' [2]. The image on the front cover is a work by Dalí called 'The Swallow's Tail' painted in 1983 as a tribute to René Thom. The shape of Dalí's Swallow's Tail is taken directly from Thom's 4-dimensional graph of the same title, combined with a second catastrophe graph, the s-curve that Thom dubbed, 'the cusp' [3].

In the first chapter we study curves of constant width. That is a curve with the property that the distance between any two parallel tangents to the curve is constant. The simplest example is the circle but as we shall see there are some very peculiar shapes that also satisfy this property. By way of motivation there are many applications for curves of constant width such as car engines, drills that produce (almost) square holes and the design of money (think of the 50 pence coin). We study some of the properties of these curves and we try to find the necessary conditions to avoid the occurence of singular points. In the second chapter we look at a variety of envelopes. We look at the envelopes of tangent lines to curves and look at how the envelope can be thought of as the limit of intersections of nearby curves. We then try to extend this to envelopes of circles of curvature. Then we look at envelopes created by a process known as embroidery and use Thom's versal unfolding idea to show which singularities are cusps. Keeping on the theme of tangent lines, in the final chapter we examine duals of curves. In 2-dimensions the dual of a curve is a way of representing all of its tangent lines. Then we move on to three dimensions where things get a bit more interesting. The dual space then becomes a way of representing the tangent planes to a space curve, so consequently the dual is a surface. We look at three different space curves and study their properties and discuss how they correspond to various features of the dual surface.

The real attraction for this project stems from the fact that most people could gain something from its reading. This is because alongside each theorem there are many examples and pictures which should help in understanding. As a result it should, I hope, be possible to see the general concept without necessarily getting caught up with the intricacy of the calculations.

2 Curves of Constant Width

A curve of constant width (CCW) is a convex planar shape with the property that the distance between any two parallel tangents of the curve is constant. That is, every tangent to the curve has the same distance to its parallel tangent. The simplest case is a circle. Later on we shall see that the idea can be extended to certain non-convex curves containing cusps.

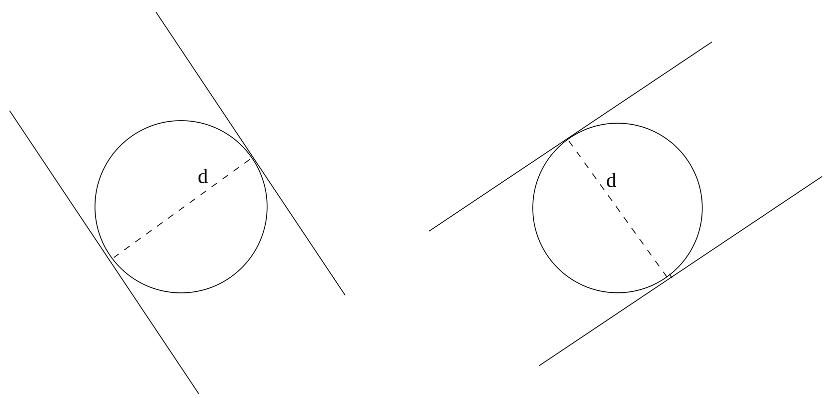


Figure 1: Circles as curves of constant width

As way of motivation consider the British fifty pence piece. The edges are not straight but in fact they are slightly curved. The edges of a fifty pence (or indeed the twenty pence) piece form a curve of constant width. This is so the coin can roll freely and so its diameter can be measured by vending machines.

Figure 2: The 50p piece has a constant diameter

Of course a circle would also suffice for the coin, being a CCW, but then maybe it would not be so distictive. Perhaps further motivation for the study of CCWs could be found in the Wankel rotary engine. The Wankel rotary engine is a type of internal combustion engine invented and developed by Felix Wankel in 1950s. The engine consists of a three sided shape of constant width rotating in a chamber see Figure 3.

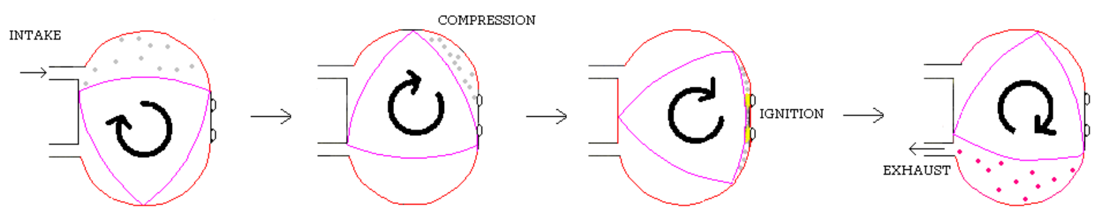


Figure 3: The Wankel engine

The engine is light-weight and reliable because of its relative simplicity. Over the years it has been used in everything form motor cycles, cars and aeroplanes to snowmobiles. There are also many other applications including cams and drills that cut square holes. Now that we are sufficiently motivated, let us take a look at thier construction.

2.1 Contructing Curves of Constant Width

For each angle t, the function $h(t)$ gives the distance to a tangent to the curve from the origin. The line l which is to be the tangent to the curve, is perpendicular to the line of angle t at the point $(h\cos t, h\sin t)$. Now let us

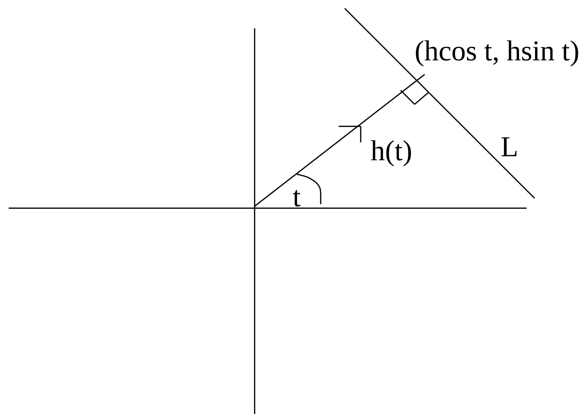


Figure 4: Support function

take a look at the equation for the line l which is to be the equation for the tangents to the curve.

Equation of line l

$$((x, y) - (h\cos t, h\sin t)) \cdot (\cos t, \sin t) = 0$$

$$(x - h\cos t)\cos t + (y - h\sin t)\sin t = 0$$

$$x\cos t + y\sin t = h(t)$$

So we define

$$F(t, x, y) = x\cos t + y\sin t - h(t)$$

We are now going to find the envelope of the tangent lines. This gives us the equation for the general curve tangent to the family of lines l. Only then can we impose the conditions necessary to be a curve of constant width. The envelope is constructed in the usual way, as $F(t, x, y) = \frac{\partial F}{\partial t} = 0$ where

$$F(t, x, y) = x\cos t + y\sin t - h(t)$$

$$\frac{\partial F}{\partial t} = -x\sin t + y\cos t - h'(t)$$

If we multiply F by $\cos t$ and multiply $\frac{\partial F}{\partial t}$ by minus $\sin t$ we obtain simultaneous equations

$$x\cos t\cos t + y\sin t\cos t - h(t)\cos t = 0$$

$$x \sin t \sin t - y \cos t \sin t + h'(t) \sin t = 0.$$

Then if we add the two equations we obtain the equation for x. Similarly we can calculate the equation for y.

Proposition 2.1 : *The parametrization of the curve given by the support function h is*

$$x = h \cos t - h' \sin t$$

$$y = h \sin t + h' \cos t$$

These are the basic equations but there are certain conditions which we must impose for constant width. Firstly for the sake of simplicity we will asume $h > 0$ for all t.

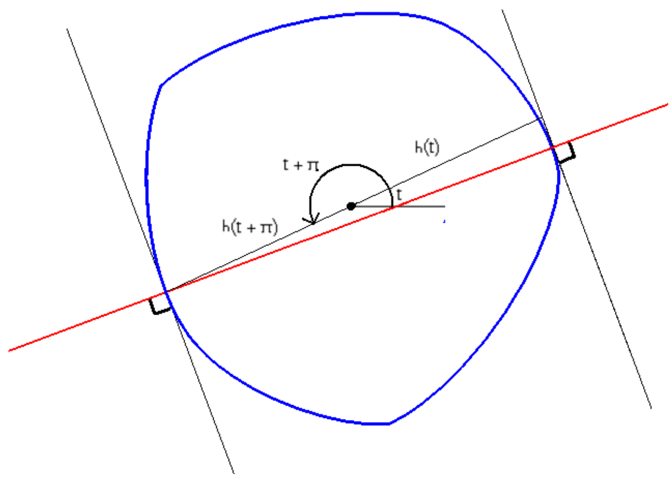


Figure 5: Conditions for a CCW

Note that the origin is not equidistant between the two tangent lines but from the diagram it is clear that for constant width we need to have $h(t)+h(t+\pi) =$ constant. For example we can use terms such as $\sin t$ and $\cos t$. If $h(t) = \sin t$, then $h(t + \pi) = -\sin t$ and if $h(t) = \cos t$ then $h(t + \pi) = -\cos t$.
e.g. If we take $h(t) = 2 + \sin t$, then
$h(t + \pi) = 2 - \sin t$, so then we get that

$$h(t) + h(t + \pi) = 2 + \sin t + 2 - \sin t = 4.$$

So the CCW would have a diameter of 4.

For terms involving $\sin(nt)$ or $\cos(nt)$ we need n to be an odd integer. Say if $n = 3$, $h(t) = a + \sin 3t$, $h(t + \pi) = a + \sin(3t + 3\pi) = a - \sin 3t$, then we get

$$h(t) + h(t + \pi) = 2a.$$

If we used $n = 2$, $h(t) = a + \sin 2t$ and $h(t + \pi) = a + \sin(2t + 2\pi) = a + \sin 2t$, so

$$h(t) + h(t + \pi) = 2a + 2\sin 2t$$

which is not constant width.

Let us take a look at some examples of CCWs and their support functions. The case $h(t) = 8 + \sin 3t$ is shown in Figure 6

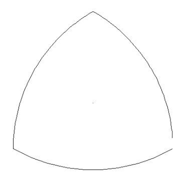


Figure 6: $h(t) = 8 + \sin 3t$

Note that the number n gives the number of 'sides' to the CCW. Depending on the constants involved, sometimes the curve can be singular.

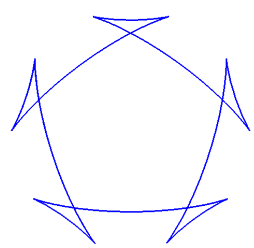


Figure 7: $h(t) = 7 + \sin 5t$.

The CCW in Figure 7 is an example of a curve which contains singular points causing the shape to be non-covex. Later on we shall look at the necessary conditions for singular points to arise. Now let us look at some interesting properties of these CCWs.

2.2 Circumference of a CCW

We know that for the simplest of CCWs, the circle, the circumference is equal to the diameter (or width) multiplied by π. What is the relation between area and width for a general CCW?

If we take our curve of constant width $\gamma = (x\cos t - h'\sin t, h\sin t + h'\cos t)$ We need to calculate the integral

$$\int_0^{2\pi} ds, \quad \text{where} \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So

$$\int_0^{2\pi} ds = \int_0^{2\pi} ||\gamma'||\ dt$$

$$\begin{aligned} \text{where } \gamma' &= (-h\sin t - h''\sin t, h\cos t + h''cost) \\ &= (h + h'')(-\sin t, \cos t) \end{aligned}$$

therefore

$$||\gamma'|| = |h + h''|$$

Suppose that $h + h'' > 0$ for all t

$$\text{Length} = \int_0^{2\pi} h(t) + h''(t)dt$$

$$= \int_0^{\pi} h(t) + h''(t)dt + \int_{\pi}^{2\pi} h(t) + h''(t)dt$$

$$= \int_0^{\pi} h(t) + h''(t) + h(t+\pi) + h''(t+\pi)dt$$

Since for a curve of constant width w we have $h(t+\pi) = w - h(t)$,

$$\frac{d}{dt}(h(t+\pi)) = -\frac{d}{dt}(h(t)), \quad h''(t+\pi) = -h''(t)$$

Hence

$$\text{Length} = \int_0^{\pi} (w + 0)dt = [wt]_0^{\pi} = w\pi$$

and we have shown the following.

Proposition 2.2 *The circumference of a curve of constant width w is given by $w\pi$.*

Remark 2.3 For a curve with singular points the curve traces backwards and the sign then changes for $h+h''$. This counts as 'negative distance' when summing the perimeter. Singular curves are discussed in more depth later on.

So we have shown that the same is true for any curve of constant width, the perimeter is equal to the width or diameter multiplied by π. This is known as Barbier's theorem, named after the French mathematician Joseph Emille Barbier(1839-89).

2.3 Curvature

The next question we ask is what is going on with the curvature of a CCW? It seems a reasonable assumption that there is something going on between the curvature and the curvature of the two points with parallel tangents.

$$\text{The curvature is given by } \kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}$$

and

$$\begin{aligned}
x &= h\cos t - h'\sin t \\
x' &= -h\sin t - h\cos t + h'\cos t - h''\sin t \\
&= -(h + h'')\sin t \\
x'' &= -(h + h'')\cos t - (h' + h''')\sin t \\
y &= h\sin t + h'\cos t \\
y' &= h\cos t + -h'\sin t + h'\sin t + h''\cos t \\
&= (h + h'')\cos t \\
y'' &= -(h + h'')\sin t + (h' + h''')\cos t \\
\text{Therefore} & \\
x'y'' - x''y' &= (h + h'')\sin t(h + h'')\sin t - (h + h'')\sin t(h' + h''')\cos t + \\
& \quad (h + h'')\cos t(h + h'')\cos t + (h + h'')\cos t(h' + h''')\sin t \\
&= (h + h'')^2.
\end{aligned}$$

$x'^2 + y'^2 = (h + h'')^2$. So therefore

$$\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}} = \frac{(h + h'')^2}{(h + h'')^3} = \frac{1}{h + h''}, \text{provided } h + h'' > 0$$

We know that for a CCW $h(t)+h(t+\pi) = w$. By differentiating with respect to t we get that
$h'(t) + h'(t + \pi) = 0$ and $h''(t) + h''(t + \pi) = 0$
Write $\rho = \frac{1}{\kappa} = h + h''$, then

$$\rho(t) + \rho(t + \pi) = h(t) + h''(t) + h(t + \pi) + h''(t + \pi) = h(t) + h(t + \pi) = w,$$

the width of the CCW.

So we have shown that there is a relation between the curvatures of the diametrical opposite sides.

Proposition 2.4 *The sum of the two radii of curvatures at parallel tangent points is actually the diameter of the CCW itself.*
$\rho(t) + \rho(t + \pi) = w$.

2.4 Vertices

We have already seen that CCW's can have different (odd) numbers of sides according to the number n in the function $h(t)$. A vertex is a point on the curve where the first derivative of curvature is zero.

We now ask the question, how many vertices does a CCW have?
$\kappa = \frac{1}{h+h''} = (h + h'')^{-1}$
$\kappa' = -(h' + h''')(h + h'')^{-2}$
So $\kappa = 0$ when $h' + h''' = 0$. If we use for example $h = 2 + \sin 3t$, then $h' = 3\cos 3t$, $h'' = -9\sin 3t$, $h''' = -27\cos 3t$.
Hence

$$h + h'' = 0 \text{for} 3t = \frac{\pi}{2} + m\pi, m \in \mathbb{Z}.$$

So we have six vertices between between zero and 2π, namely at $\frac{\pi}{6} + \frac{m\pi}{3}$ for $0 \leq m \leq 5$.
For a CCW with $h = a+\sin bt$ where b is odd $h' = b\cos bt$, and $h''' = b^3 \cos bt$.
So $h + h'' = 0$ for $bt = \frac{\pi}{2} + m\pi$ that is $t = \frac{\pi}{2b} + \frac{m\pi}{b}$.

So the CCW has $2b$ vertices between zero and 2π.

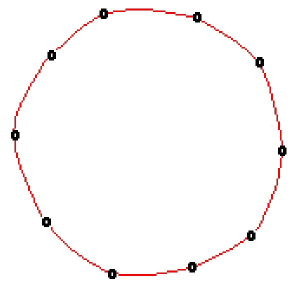


Figure 8: A 5 sided CCW has 10 vertices

What can we say about whether these vertices are maxima or minima?

$$\kappa' = \frac{-(h' + h''')}{(h + h'')^2}$$

$$\kappa'' = \frac{-(h'' + h^{(4)})}{(h + h'')^2}.$$

For a curve of constant width $h(t) + h(t + \pi) = const = w$.
Differentiating gives $h^{(n)}(t) = -h^{(n)}(t + \pi)$. Provided $h'' + h^{(4)}$ is never zero, $\kappa''(t) = -\kappa''(t + \pi)$. That is they have opposite signs, so we have shown that

Proposition 2.5 *Provided $h'' + h^{(4)}$ is never zero, then the vertices at t and $t + \pi$ have one a maximum, and the other a minimum of curvature.*

2.5 Singular Points

A singular point on a parametrised curve $\gamma(t) = (x(t), y(t))$ is a point where $\gamma'(t) = (0, 0)$. Our curve is constructed as an envelope of a family of tangent line so both F and $\frac{\partial F}{\partial t} = 0$.

In the general case a singular point occurs on an envelope when the extra condition that $\frac{\partial F^2}{\partial t^2} = 0$ holds. Let us look at our case for a curve of constant width.

$$F(t, x, y) = x \cos t + y \sin t - h(t)$$
$$\frac{\partial F}{\partial t} = -x \sin t + y \cos t - h(t)$$
$$\frac{\partial^2 F}{\partial t^2} = -x \cos t - y \sin t - h''(t)$$

Now we substitute our values for x and y with their respective parametric equations in to the equation for the second derivative giving us

$$\frac{\partial^2 F}{\partial t^2} = -(h\cos t - h'\sin t)\cos t - (h\sin t + h'\cos t)\sin t - h''$$

$$= -h\cos^2 t + h'\sin t\cos t - h\sin^2 t - h\sin t\cos t - h''$$

$$= -h - h''.$$

So singular points only occur if and only if $h + h'' = 0$. This seems to agree with the calculations for curvature that we did earlier. We had that $\kappa = \frac{1}{h+h''}$. So at a singular point where $h + h'' = 0$ we get that the curvature is infinite.

Remark 2.6 Cusps at the origin given by (t^2, t^3) and (t^3, t^4) both have limiting curvature of infinity and these are typical singular points on a CCW, the latter typically appearing when two ordinary cusps are born or come into coincidence and disappear.

Example 2.7 Now we shall look at a particular case for a curve of constant width, say where $h = 2 + k\sin 3t$.

$$h = 2 + k\sin 3t$$

$$h'' = -9k\sin 3t$$

So the condition for singular points $h + h'' = 2 - 8k\sin 3t$ is equal to zero if and only if $k = \frac{2}{8\sin 3t}$ for some t. So a curve will have singular points when $\sin 3t = \frac{1}{4k}$ for some t. The function $\sin 3t$ varies between -1 and 1 so it is possible for us to chose a value k such that we can avoid singular points if we so wish. For $-1 < \frac{1}{4k} < 1$ we will have singular points. If we choose $|k| < 1/4$ our curve will be free from singular points.

When k is equal to $\frac{1}{4}$ we get what is known as the transition moment, when singular points are 'born' (compare Remark 2.6), but more on this later.

Similarily if we are given the term k we can calculate the value of a constant term, a say, needed to be added on to h to eliminate singular points occurring on our curve.

$$h = a + k\sin 3t$$

$$a > 8k \sin 3t,$$

so we need $a > 8k$ in general if we are give $k = 5$ for instance

$$a > 8 \times 5 \times \sin 3t = 40 \sin 3t, \quad \text{for all } t,$$

which is true provided

$$a > 40$$

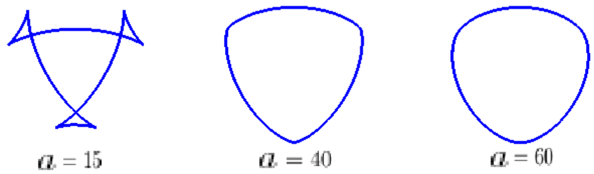


Figure 9: CCWs with different constants a.

2.6 Sums of Two Sines

Now let's look at some slightly more complicated examples. What about if our function $h(t)$ contains two terms involving sine or cosine? What is the condition on a to to avoid singular points if we have

$$h = a + k_1 \sin n_1 t + k_2 \sin n_2 t, \quad n_1, n_2 \text{ odd integers} > 1 \text{ and } k_1, k_2 \text{ real numbers} > 0$$

$$h'' = 0 - n_1^2 k_1 \sin n_1 t - n_2^2 k_2 \sin n_2 t$$
$$h + h'' = a + k_1(1 - n_1^2) \sin n_1 t + k_2(1 - n_2^2) \sin n_2 t$$

Sine functions vary between -1 and 1 so it should be obvious that choosing a such that $a > k_1(n_1^2 - 1) + k_2(n_2^2 - 1)$ we will get $h + h''$ non zero and hence avoid singular points. However is this the minimal condition on a the so called transition moment? This would only be the case if $\sin n_1 t$ and $\sin n_2 t$ are both equal to 1 (or indeed -1) for some t. This poses quite an interesting question so lets take a moment to find when and if this is possible.

Theorem 2.8 *For p, q given odd integers, there exists t such that $\sin pt = \sin qt = -1$ if and only if $p \equiv q \bmod 4$.*

That is $\sin pt$ and $\sin qt$ can both be equal to -1 simultaneously if and only if p and q differ by a mulitple of 4.

Proof

$$\sin pt = -1, \quad \sin qt = -1, \qquad \text{where } p \text{ and } q \text{ are given and odd integers}$$

We require for the same t,

$$\begin{aligned} t &= -\frac{\pi}{2p} + \frac{2k\pi}{p} \\ t &= -\frac{\pi}{2q} + \frac{2l\pi}{q} \end{aligned}$$

subtracting the second equation from the first gives

$$\begin{aligned} 0 &= -\frac{\pi}{2p} + \frac{2k\pi}{p} + \frac{\pi}{2q} - \frac{2l\pi}{q} \\ 0 &= -q + p + 4kq - 4lp \\ 4lp - 4kq &= p - q \end{aligned}$$

So a solution exists if and only if $4(p,q) \mid p - q$ since p, q are odd. This is equivalent to $p \equiv q \bmod 4$. $\square$

Similar calculations show that the same conditions apply in order for both be equal to 1 simultaneously. So if our equation has $p \equiv q \bmod 4$ we must take $a > k_1(n_1^2 - 1) + k_2(n_2^2 - 1)$ to avoid singular points. Note that when one of the variables n is equal to 1, the case is trivial since the the term involving it becomes zero.

Example 2.9 Now lets look at an example of a nontrivial case where $n_1 \equiv n_2 + 2 \bmod 4$. We take $n_1 = 5,\ n_2 = 3$, that is consider a support function $h = a + k_1 \sin n_1 t + k_2 \sin n_2 t$,

$$h(t) + h''(t) = a + k_1(1 - 5^2)\sin 5t + k_2(1 - 3^2)\sin 3t$$

We want a such that $h(t) + h''(t)$ is > 0 for all t

For simplicity let us write $b = k_1(5^2 - 1)$ and $c = k_2(3^2 - 1)$.

We shall call

$$f(t) = b \sin 5t + c \sin 3t$$

Now the problem is to find the maximum value of f in terms of $b, c > 0$. Let $s = \sin t$. Then with the help of Maple, using half and double angle formulas, we get as a function of s

$$f(s) = 16bs^5 - (20b + 4c)s^3 + (5b + 3c)s, \qquad |s| \le 1$$

$$f'(s) = 80bs^4 - 3(20b + 4c)s^2 + (5b + 3c)$$

The equation becomes much simpler if we substitute $b = \frac{B}{10}, \quad c = \frac{2C-B}{6}$ (i.e. $B = 10b,\ C = 3c + 5b$ so if $b,\ c > 0$ then $C > \frac{B}{2} > 0$.) The equation now becomes

$$f(s) = \frac{8B}{5}s^5 - \frac{4}{3}(B + C)s^3 + Cs$$

Differentiating w.r.t s gives

$$f'(s) = 8Bs^4 - (B + C)s^2 + C.$$

Which is a quadratic in s^2 with solutions $s^2 = \frac{B+C\pm\sqrt{B^2+C^2}}{4B}$.

We need to find the largest value of $f(s)$ at any of these points and also possibly $f(1)$ and $f(-1)$ to determine the maximum of f on $-1 \le s \le 1$. Note that $f(1) = \frac{1}{3}(4B - C) \quad (= b - c)$
and $f(-1) = -\frac{1}{3}(4B - C) \quad (= -b + c)$
It is also be helpful to write $B = R\cos u$ and $C = R\sin u$, for $R > 0$ and $\arctan\frac{1}{2} < u < \frac{\pi}{2}$.

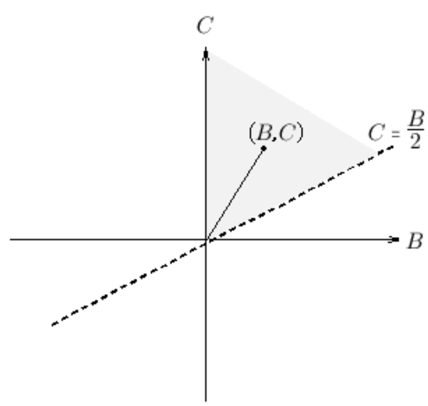


Figure 10: (B, C) lies in the first quadrant above $C = \frac{B}{2}$.

So now

$$s^2 = \frac{\cos u + \sin u \pm 1}{4 \cos u} \tag{1}$$

Since s stands for sine, in order for the equation to be valid the right hand side should lie between -1 and 1. Since the LHS is a square the RHS should also be positive. It is easy to see that for an acute angle u, both $\pm$ give the RHS of (1) greater than zero.

For the RHS of (1) to be ≤ 1 we require

$$\cos u + \sin u \pm 1 \leq 4 \cos u$$

$$\sin u \pm 1 \leq 3 \cos u$$

$$3 \cos u - \sin u \mp 1 \geq 0$$

This is considered over the range of u (for both signs).

For the plus sign it is equal to zero at $u = \frac{\pi}{2}$ so greater than or equal to zero over the range of u. For the minus sign it is -2 at $u = \frac{\pi}{2}$ and 2 at $u = 0$ so has one value where it is 0

$$3 \cos u - \sin u = 1$$

has solution $u = \cos^{-1} \frac{3}{5} = \tan^{-1} \frac{4}{3}$.

For equation (1) when using the upper sign we gives two solutions for s that is when $\frac{\pi}{2} > u > \tan^{-1} \frac{4}{3}$. From using both signs, we get 4 solutions for s between $\tan^{-1} \frac{4}{3} > u > \tan^{-1} \frac{1}{2}$. See Figure 11.

$$f(s) = s \left(\frac{8B}{5} s^4 - \frac{4}{3}(B + C)s^2 + C \right)$$

$$\frac{1}{B} f(s) = s \left(\frac{8}{5} s^4 - \frac{4}{3}(1 + \tan u)s^2 + \tan u \right)$$

From equation (1) we get

$$\frac{1}{B} f(s) = \frac{2s}{15} \left(\frac{4 \cos u \sin u \mp \cos u \mp \sin u - 1}{\cos^2 u} \right)$$

$$\frac{15 \cos^2 u}{2B} f(s) = s(4 \cos u \sin u \mp \cos u \mp \sin u - 1) \tag{2}$$

We want the largest value for the RHS for the two or four values of s given by (1).

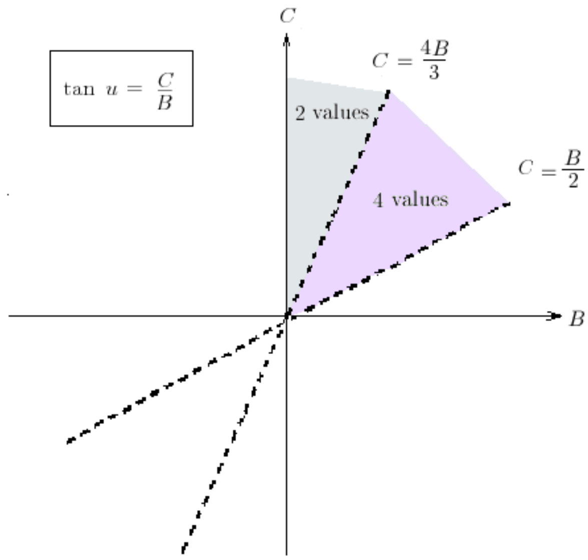


Figure 11: Number of solutions to s^2.

With the upper (minus) sign the bracket on the RHS of (2) is greater than zero for all acute u. With the lower (plus) sign the bracket on the RHS of (2) is less than zero for all acute u. So the relevant numbers to compare are

$$\frac{15\cos^2 u}{B} f(s) = -\sqrt{\cos u + \sin u + 1}(4\cos u \sin u - \cos u - \sin u - 1) = P$$
$$\frac{15\cos^2 u}{B} f(s) = \sqrt{\cos u + \sin u + 1}(4\cos u \sin u + \cos u + \sin u - 1) = Q$$

So for the two or four values of u given by (1) (also possibly the end points $s = -1, 1$)

It turns out that $P = Q$ when $\sin 2u = \frac{4}{5}$ which has two solutions for $u_1 < u_2$ both acute.

So Q is greater than P when $\frac{1}{2} < \tan u < 2$. However when $\tan u > 2$ only Q is relevant by equation (1) as only the lower sign gives values of $s^2 \leq 1$. See Figure 11 and Figure 12. So rather surprisingly the only value we need consider is Q. We should also have a greater than the end points given when $s = \pm 1$ that is $b - c$ and $c - b$. $a > |24k_1 - 8k_2| = 8|3k_1 - k_2|$. We have shown that

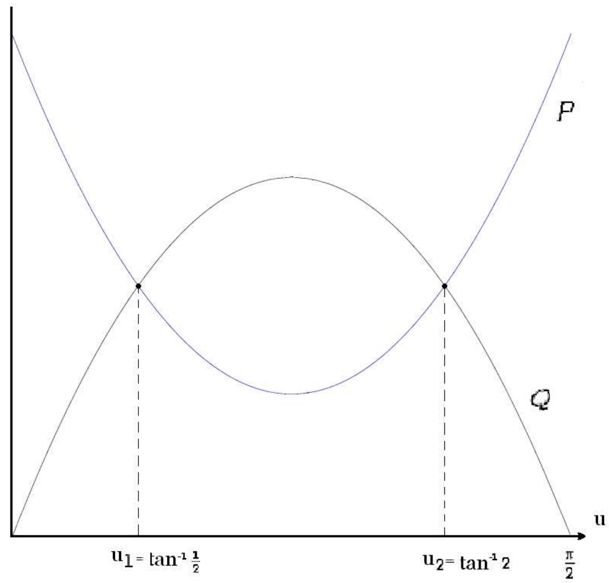


Figure 12: Graphs of P and Q in terms of u.

Proposition 2.10 *For the case where* $h(t) = a + k_1 \sin 5t + k_2 \sin 3t$ *the minimum value for a we must take in order to avoid singlular points is*
$a > \frac{B}{15\cos^2 u}\sqrt{\cos u + \sin u + 1}(4\cos u \sin u + \cos u + \sin u - 1), \quad 8|3k_1 - k_2|,$
where $B = 10k_1$ *and u is the acute angle with tangent* $\frac{k_2+5k_1}{10k_1}$.

Using maple with a range of different constants seems to validate that Q is indeed the minimal value for a we need to avoid singular points. (see Appendix A)

Of course one could always search for the turning points of the function manually and then compute which has the greatest value.

As we have seen deriving the general formula for the minimum value of a needed to avoid singular points in the example

$$h(t) = a + k_1 \sin 5t + k_2 \sin 3t$$

is rather complicated. Luckily for us with this example the derivative of the function for the turning points was a quadratic in s^2. For the next non-trivial case dfdfd dfdf the general solution would be much more complicated. I am unsure if such solutions exist, it would be interesting to know if there was indeed a more general formula.

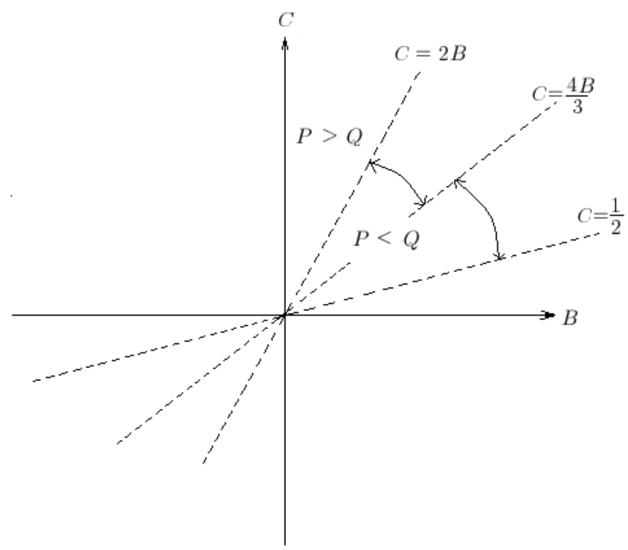


Figure 13: Comparison of P and Q for acute u.

Using cosine instead of sine in the equations has the effect of simply rotating the CCW about the origin. If we use both sine and cosine then we get some quite interesting shapes.

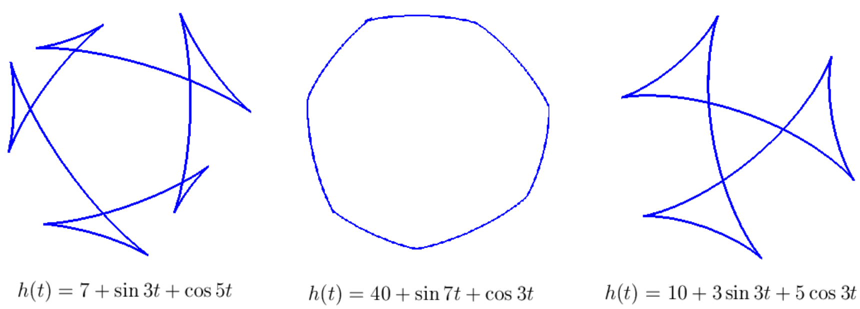


Figure 14: Some irregular CCWs.

The number of sides of the CCW seems to be equal to the greatest of n_1 and n_2.

If we have n_1 and n_2 the same the question of minimal condition is quite trivial. For $h = a + k_1 \sin(nt) + k_2 \cos(nt)$ Then $|a| > \frac{k_1}{k_2} \tan(nt)$

The question of the minimal condition on a when n_1 and n_2 are different is more tricky and I leave this as an exercise for the reader.

2.7 Other Methods of Constructing CCWs

The method that I have shown for constructing CCW is by no means the only one. I feel that I could not write a chapter on curves of constant width without at least mentioning this alternative way of constructing them. Probably the most famous CCW (after the circle) is known as the Reuleaux triangle, named after Franz Reuleaux who taught in Berlin during the late nineteenth century. It Reuleaux be constructed by starting with an equilateral triangle. You then proceed by replacing each side by a circular arc with the other two sides as radii.

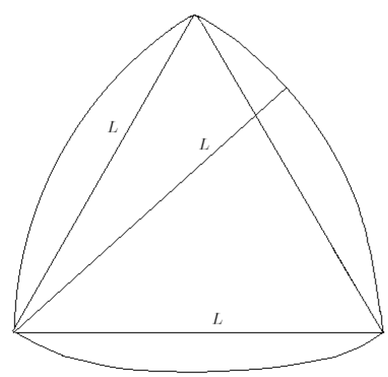


Figure 15: Reuleaux triangle

The vertices of the Reuleaux triangle are actually corners as the tangents do not have the same limit when you approach from opposite sides. This is because at the vertices the tangents are perpendicular to the triangle's sides (L).

The corners of the Reuleaux triangle are infact the sharpest possible for any CCW. Using a similar piecewise method you construct a CCW with round corners. This is done by extending the lines in all directions, then using a larger circle to create the edges, then connecting them using smaller circles to fill in the gaps. See figure 17. Similar constructions can be done using any regular polygon with an odd number of sides. Another clever way of creating CCWs is the crossed-lines method. It works as follows: Draw as many straight lines as you like all mutually intersecting. Select a point on

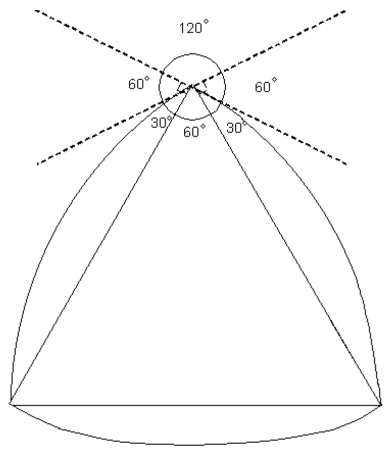


Figure 16: Limits of the tangents are different

one of the lines then using a compass draw an arc to the next line with the compass point where the two lines intersect. See Figure 18.

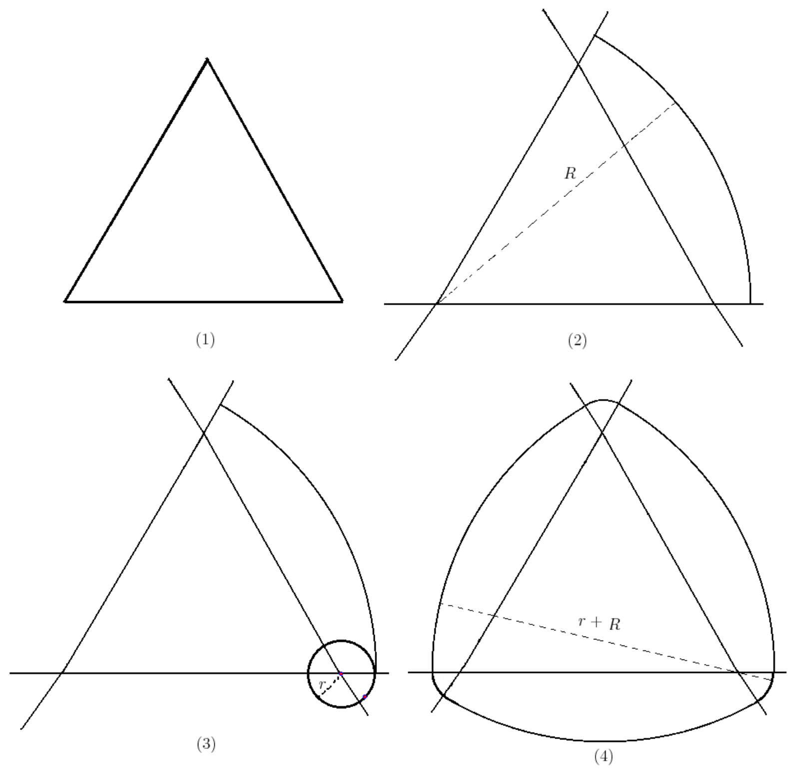


Figure 17: Piecewise construction of 3-sided CCW with width $R + r$

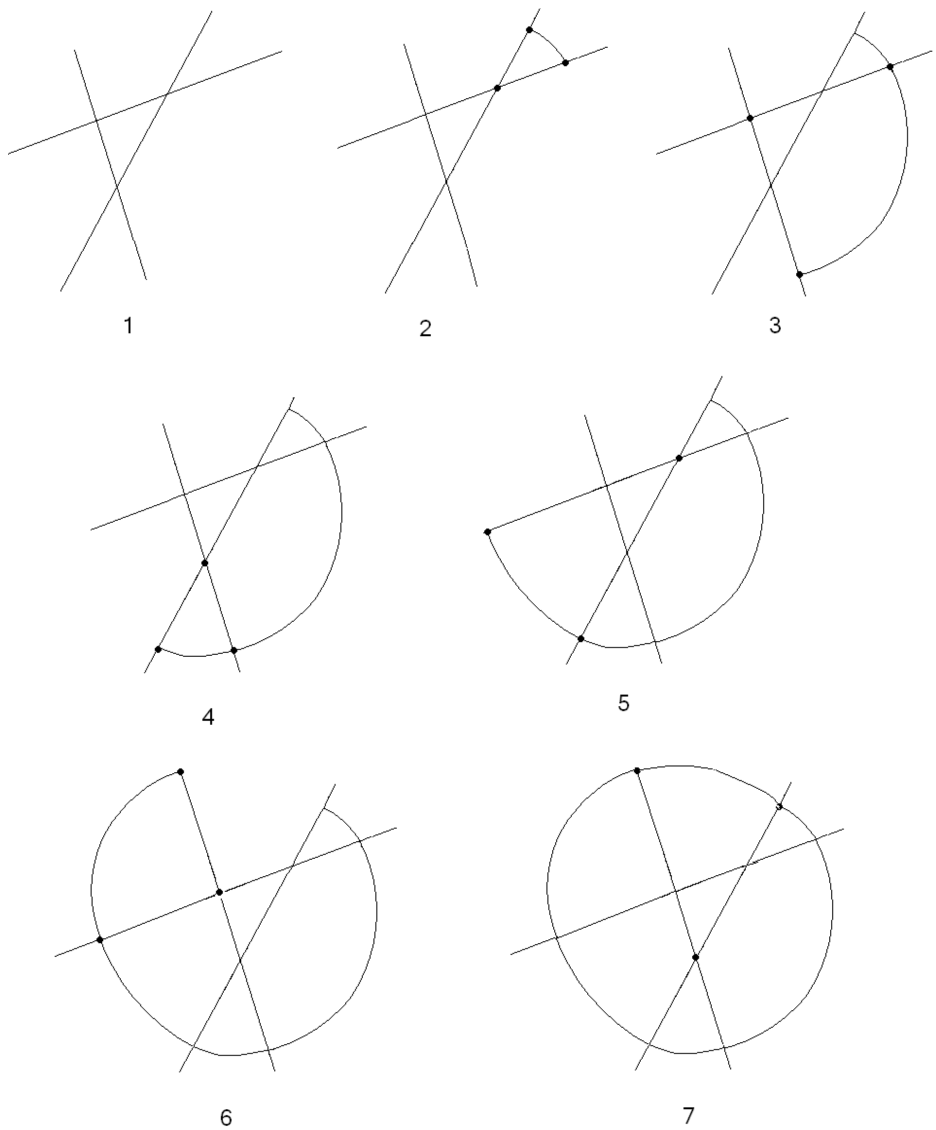


Figure 18: Crossed lines method of constructing ccw

Now if I may go off at a slight tangent and take a look at some envelopes of tangent lines to a curve.

3 Envelopes

We begin by looking at the envelopes of tangent lines to a curve. We examine the curve $y = x^3$ and show that its evelope fits with the alternative definition that it is the limit of nearby curves. We then study evelopes of circles of curvature and try to establish the same claim. Finally we take a look at evelopes created by a process known as Embroidery.

3.1 Envelopes of Tangents to a Curve

Let us now take some time to look at Envelopes of the family of curves given by the tangents to a curve. Using the fact that at any point the tangent is perpendicular to the normal the curve α can be given by the equation

$$F(t, x) = (x - \alpha) \cdot N = 0.$$

We then Calculate

$$\begin{aligned} \frac{\partial F}{\partial t} &= (x - \alpha)' \cdot N + (x - \alpha) \cdot N' \\ &= -T \cdot N + (x - \alpha) \cdot \kappa T \\ &= (x - \alpha) \cdot \kappa T \end{aligned}$$

where of course the standard definition of envelope of family of curves is given by

$$D_F = \{x : F = \frac{\partial F}{\partial t} = 0\}.$$

So if $F = 0$ then either $x = \alpha$ or $(x - \alpha)$ is perpendicular to N. As $(x - \alpha)$ can not additionally be perpendicular to T, the extra condition that $\frac{\partial F}{\partial t} = 0$ implies that either $x = \alpha$ or $\kappa(t) = 0$.

Intuitively it makes sense that $x = \alpha$ should be the envelope of tangent lines, since if we ask the question what is the curve that is tangent to all the tangent lines of a curve? clearly it must be the curve itself.

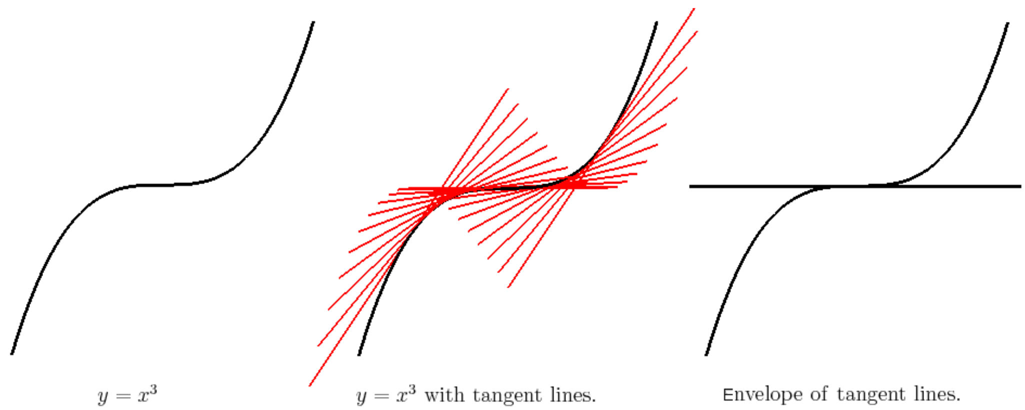


Figure 19: $y = x^3$, tangent lines of $y = x^3$ and envelope of tangent lines

So if we have an inflexion, that is a point where the curvature, $\kappa(t)$ is zero, we additionally get the the tangent line at this point included in the envelope. So for example the curve $y = x^3$ has an inflexion at the origin, so the envelope includes both the curve and the tangent line at this point.

From the second picture of Figure 19 it looks reasonable that the envelope would include the tangent line at the origin. The tangent lines of graph seem to 'bunch up' to the line $y = 0$. There is another definition of envelopes that is as follows.

Theorem 3.1 *[1, Theorem 5.8] The envelope $E_1 \subset D_F$ is the limit of intersections of nearby curves C_t.*

Proof Let $E_1 \subset \mathbb{R}^2$ be the set of x for which there exist the following sequences: $x_n = (x_{1n}, x_{2n})$ in $\mathbb{R}^2$, (t_n) and (t'_n) in $\mathbb{R}$ where $t_n \neq t'_n$ for all n. We also need for all n that $F(t_n, x_n) = F(t'_n, x_n) = 0$, such that $x_n \in C_{t_n} \cap C_{t'_n}$. We also require that as $n \to \infty$ that $t_n, t'_n \to t$ and $x_n \to x$ where (t, x) is in the domain of F.

As $n \to \infty$ we get $F(t_n, x_n), F(t'_n, x_n) \to F(t, x) = 0$. Let $f(t) = F(t, x_n)$ for sufficiently large n. Then we get that $f(t_n) = f(t'_n) = 0$.

Rolle's theorem states that any smooth function that reaches the same value at two points must have a stationary point somewhere between them. See Figure 21.

So if $f(t_n) = f(t'_n) = 0$ then there must exist τ_n between t_n and t'_n with $f'(\tau_n) = 0$. Hence $\frac{\partial F}{\partial t}(\tau_n, x_n) = 0$, and letting $n \to \infty$, $\frac{\partial F}{\partial t}(t_n, x_n) = 0$. Hence:

As $n \to \infty$,

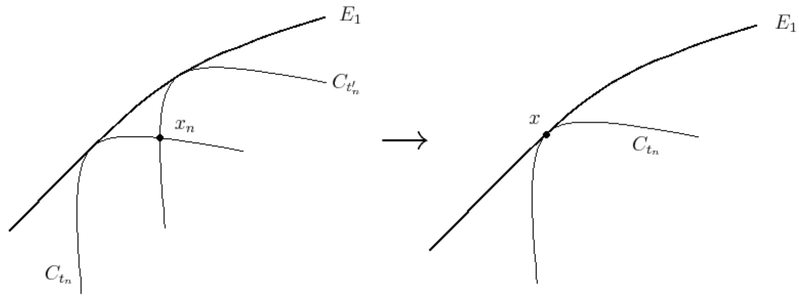


Figure 20: E_1 as the Limit of Intersections of Nearby Curves C_t

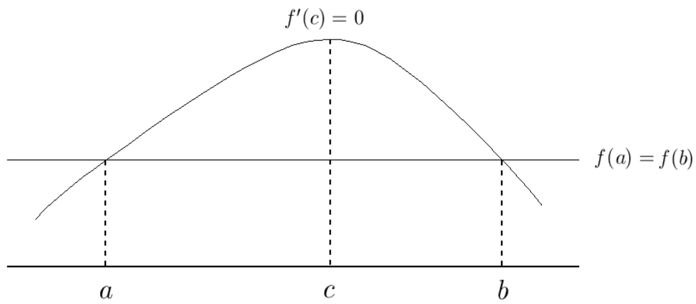


Figure 21: Rolle's Theorem

$E_1 \subset D_F \qquad \square$

Let's attempt to prove that the envelope of the family of tangent lines to the curve $y = x^3$ is indeed the limit of intersection of nearby lines of the family. It should be fairly obvious that points the curve itself are limits of intersections of tangent lines. Perhaps it is less obvious that the tangent line at zero is alos the limit of nearby tangent lines.

Theorem 3.2 *For $y = x^3$, every point on the tangent line at zero is the limit of intersection of nearby tangent lines.*

Proof Tangent lines to $y = x^3$ are given by

$$y - t^3 = 3t^2(x - t)$$

$$F = y - 3t^2 x + 2t^3 = 0$$

$$\frac{\partial f}{\partial t} = -6tx + 6t^2 = 0$$

So $t = y = 0$ or $x = t$ and $y = t^3$. Which as we would expect is the curve itself and the tangent line at the origin.

We want for any given point on the positive x axis, say $(a, 0)$ where a > 0, to choose two tangents at $t = u, v$ so that the two tangents cross at (a, b) for some b. We choose a to be postive but similar arguments apply for negative a. Now we need to find a sequence so that when $u, v \to 0$ then $b \to 0$ and a remains the same.

The two equations for tangents at u and v are

$$y - 3u^2x + 2u^3 = 0$$

$$y - 3v^2x + 2v^3 = 0$$

Subtracting the equations gives

$$3(u^2 - v^2\ )x = 2(u^3 - v^3)$$

and dividing by $(u - v)$ which in non-zero gives

$$3(u + v)x = 2(u^2 + uv + v^2).$$

We want u, v to satisfy

$$3(u + v)a = 2(u^2 + uv + v^2) \tag{3}$$

so that the tangents meet on the line $x = a$.

We need both

$$u \to 0$$

$$v \to 0$$

so that equation (3) is satisfied.

We want also $y = 3u^2a - 2u^3 \to 0$ which it inevitably will if $u \to 0$.

We can rearrange equation (3) to make a quadratic equation in v.

$$2v^2 + v(2u - 3a) + (2u^2 - 3ua) = 0$$

Then applying the quadratic formula gives

$$v = \frac{3a - 2u - \sqrt{(2u - 3a)^2 - 8(2u^2 - 3ua)}}{4}$$

Now we need to take a sequence of u's $\to 0$ Let's try $u_n = \frac{1}{n}$.

then $v_n = \frac{3a - 2\frac{1}{n} - \sqrt{(2\frac{1}{n} - 3a)^2 - 8(2(\frac{1}{n})^2 - 3\frac{1}{n}a)}}{4}$

$$v_n = 3a - \frac{2}{n} - \sqrt{\frac{4}{n^2} + 9a^2 - \frac{12}{n}a - 8\left(2\frac{1}{n^2} - 3\frac{1}{n}a\right)}$$

as $n \to \infty, v_n \to 3a - 0 - \sqrt{0 + 9a^2 - 0}$

$$= 3a - 3a = 0, \quad \text{since } a > 0$$

as $u \to 0$ also $v, y \to 0$ $\square$

So in this case the discriminant really is the limit of intersection of nearby tangent lines.

3.2 Envelope of Circles of Curvature

In the last example we saw that the envelope of tangent lines for the curve $y = x^3$ is the limit or nearby curves. Is this true of other types of envelope? We shall now take a look at the envelope of a family of circles [1, pg106]. We shall try to ascertain whether the envelopes are, as in the last example, the limit of intersections of nearby curves.

Let $\gamma : I \to \mathbb{R}^2$ be unit speed, where the curvature κ is never zero. Then the centre of curvature at $\gamma(t)$ is a distance $\frac{1}{\kappa}$ in the normal direction from $\gamma(t)$.

$$\text{Centre of curvature} = \gamma(t) + \frac{1}{\kappa(t)}N(t)$$

Now we need the equations for the families of circles of curvature. That is the circles centered at the centre of cuvature and radius $\frac{1}{\kappa(t)}$. Dropping t from the notation we have

$$||(x - \gamma) - \frac{1}{\kappa}N||^2 = \frac{1}{\kappa^2}$$

$$((x - \gamma) - \frac{1}{\kappa}N) \cdot ((x - \gamma) - \frac{1}{\kappa}N) = \frac{1}{\kappa^2}$$

$$(x - \gamma) \cdot (x - \gamma) + \frac{1}{\kappa^2} - \frac{2}{\kappa}(x - \gamma) \cdot N = \frac{1}{\kappa^2}$$

$$(x-\gamma)\cdot(x-\gamma)-\frac{2}{\kappa}(x-\gamma)\cdot N=0$$

We shall denote this

$$F(x,t)=(x-\gamma)\cdot(x-\gamma)-\frac{2}{\kappa}(x-\gamma)\cdot N$$

We now go about obtaining the envelope in the usual way, that is we calculate when both $F(x,t)$ and $\frac{\partial F}{\partial t}(x,t)$ are equal to zero.

$$\begin{aligned}\frac{\partial F}{\partial t}&=2(x-\gamma)'\cdot(x-\gamma)+\frac{2\kappa'}{\kappa^2}(x-\gamma)\cdot N-\frac{2}{\kappa}(x-\gamma)'\cdot N-\frac{2}{\kappa}(x-\gamma)\cdot N'\\&=-2T\cdot(x-\gamma)+\frac{2\kappa'}{\kappa^2}(x-\gamma)\cdot N+\frac{2}{\kappa}T\cdot N-2(x-\gamma)\cdot T\\&=\frac{2\kappa'}{\kappa^2}(x-\gamma)\cdot N\end{aligned}$$

This is equal to zero if and only if $\kappa'=0$ or if $(x-\gamma)\cdot N=0$. In the latter case for F to equal zero as well, $(x-\gamma)\cdot(x-\gamma)$ must be equal to zero also. As in the last example with tangents to a cubic we get the curve itself as as part of the envelope. We also have $\frac{\partial F}{\partial t}=0$ when $\kappa'=0$. So at a vertex, that is where we a zero of κ', we additionally have the whole circle of curvature as part of the envelope. Figure 22 shows an ellipse next to its respective envelope of circles of curvature complete with the circles of curvature at the vertices.

Let's take a look and see what happens when additionally $\frac{\partial^2 F}{\partial t^2}=0$

$$\frac{\partial F}{\partial t}=\frac{2\kappa'}{\kappa}(x-\gamma)\cdot N$$

$$\begin{aligned}\frac{\partial^2 F}{\partial t^2}&=\frac{2\kappa\kappa''-2\kappa'^2}{\kappa^2}(x-\gamma)\cdot N-\frac{2\kappa'}{\kappa}T\cdot N+2\kappa'(x-\gamma)\cdot T\\&=2\left(\frac{\kappa\kappa''-2\kappa'^2}{\kappa^2}\right)(x-\gamma)\cdot N+2\kappa'(x-\gamma)\cdot T\end{aligned}$$

This is again equal to zero if $x=\gamma$. If $\kappa'=0$ then we get

$$\frac{\partial^2 F}{\partial t^2}=\frac{2\kappa''}{\kappa}(x-\gamma)\cdot N$$

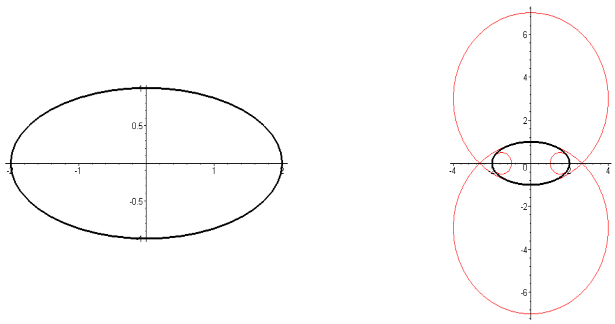


Figure 22: Ellipse and envelope of circles of curvature

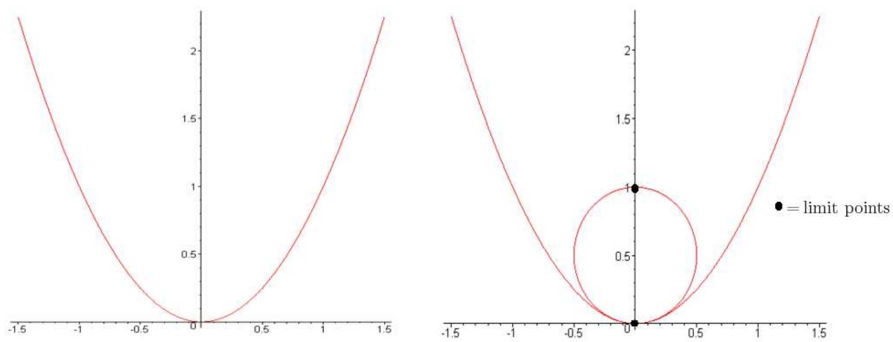


Figure 23: Parabola and envelope of circles of curvature

which equals zero if $\kappa'' = 0$.

Rather interestingly we find that every point on the original curve is a point of regression. If the vertex is ordinary ($\kappa' = 0, \kappa'' \neq 0$) then points of the circle are not points of regression except for the contact point with γ (where in fact the envelope is smooth). If on the other hand we have a higher vertex, that is it has at least five point contact with the curve, we also get that every point on the circle of curvture is also a point of regression.

3.3 Intersection of Nearby Circles

We now take a look to see if the envelope is also the limit of intersection of nearby circles. The whole proof is rather long and complicated to include here, so instead I shall state the result and just a brief sketch of the proof. The complete proof in Maple can be found in Appendix B. We want to know whether all the points on the circle of curvature at the origin are limits of the intersection of two nearby circles of curvature that approach the origin from opposite directions. The circle must approach the origin from opposite directions because of the well known theorem in curves and singularities, that circles do not intersect unless on other sides of a vertex. This is because the are actually nested in one another, in our case getting bigger as they get futhe away from the origin.

Sketch of Method for Determining Limit Points

For this we look at different general equations with vertices and ascertain what the limit points are. So for example we could use $y = x^2 + ax^4$.

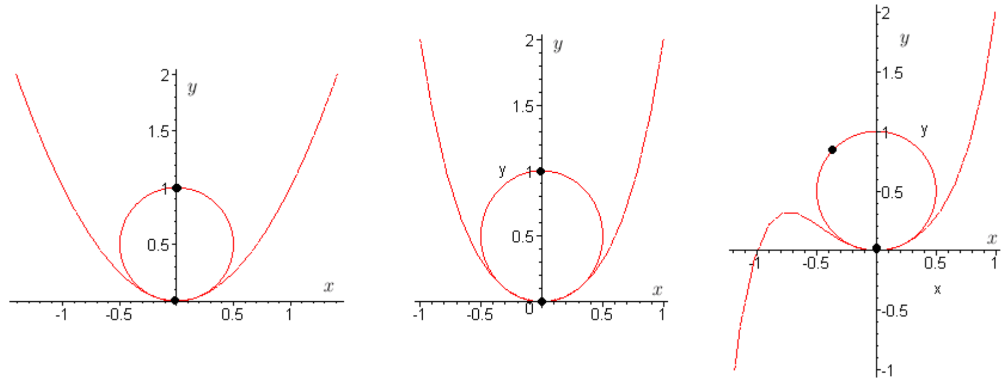
We first find the eqautions for the circles of curvature at two points u, v on the curve. Subtracting the equations and taking out the trivial factor $(u - v)$ gives us the equation of the line, in X and Y say, connecting the two intersections. So we now have a a linear function $X = f(Y, u, v)$. Substituting back in to the equation gives us the solutions for Y, call them Y_1 and Y_2.

The trick is to convert to polar co-ordinates, $u = r\cos t, v = r\sin t$.

Then you take the limit of Y_i as $r \to 0$, then find the limit of X from $f(Y, u, v)$

Unlike where we had intersections of nearby tangent lines, it seems as though we always get two limit points of intersections on the envelope. For the case $y = x^2$ and $y = x^2 + ax^4$, the two limit points are (0, 0) and (1, 0). For the equation $y = x^2 + bx^5$ we still two limit points, one at the origin but this time the other is found at $\left(\frac{20|b|}{16+25b^2}, \frac{16}{16+25b^2}\right)$, see Figure **??**

So we have found that in these cases that the points on the envelope of circles of curvature are not all the limit of intersections of nearby curves of the family. We found that only two points on the envelope satisfy this property. However this still fits in with out alternative definition of the envelope $E1$, see theorem 3.1. Before we proved that envelope E_1, considered as the limit of intersections of nearby curves, lies inside the discriminant D_F which is still true.



$y = x^2$ $\qquad$ $y = x^2 + ax^4$ $\qquad$ $y = x^2 + bx^5$

Figure 24: Curves wtih vertices and limit points $\bullet$.

3.4 Embroidery

We are now going to study another type of envelope. This time we shall look at ones created by a process known as embroidery.

The process works by taking a parametrised (closed) curve. In our case we shall take the unit circle given by $(\cos t, \sin t)$. We then take the envelope of the family of lines connecting $(\cos t, \sin t)$ to $(\cos mt, \sin mt)$ for some positive integer m.

The family of curves is given by

$$F = x(\sin mt - \sin t) - y(\cos mt - \cos t) - \sin(m-1)t = 0.$$

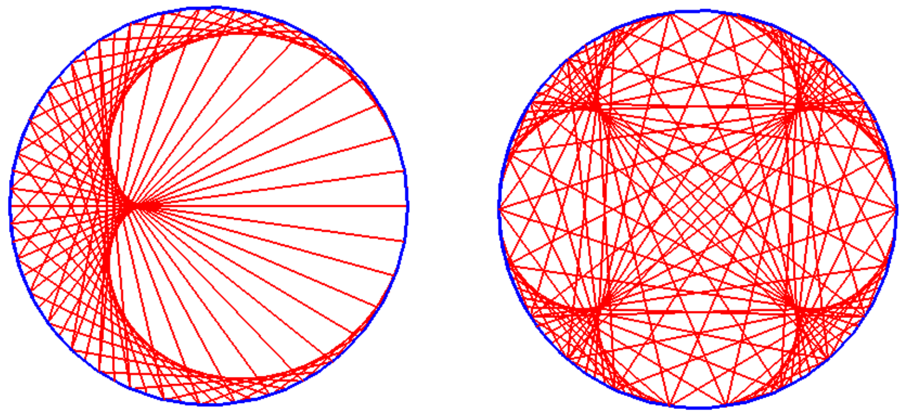
Differentiating gives

$$\frac{\partial F}{\partial t} = x(m\cos mt - \cos t) - y(\sin t - m\sin mt) - (m-1)\cos(m-1)t = 0$$

We need to find the equations for x and y that satisfy both $F = \frac{\partial F}{\partial t} = 0$. If we simplify the coefficients in the expressions for a moment it will make the method clearer.

We shall denote them

$$\begin{aligned} F &= ax + by = -p \\ \frac{\partial F}{\partial t} &= cx + dy = -q \end{aligned}$$



$m = 2$ $\qquad$ $m = 5$

Figure 25: Embroidery for $m = 2$ and $m = 3$.

We then multiply F by d and $\frac{\partial F}{\partial t}$ by b. Then if we subtract the equations we get

$$(ad - bc)x + (bd - db)y = pd - qc$$

If we assume $ad - bc \neq 0$ then we get $x = \frac{pd-qc}{ad-bc}$. Similarly we get $y = \frac{aq-pc}{ad-bc}$ assuming that $ad - bc = (m+1)(1 - \cos(m-1)t) \neq 0$. Now when we put back the coefficients we get

$$x = \frac{(m \sin mt - \sin t)\sin(m-1)t - (m-1)(\cos t - \cos mt)\cos(m-1)t}{(\sin mt - \sin t)(m \sin mt - \sin t) + (\cos mt - \cos t)(m \cos mt - \cos t)}$$

$$= \frac{m \sin mt \sin(m-1)t - \sin t \sin(m-1)t - (m \cos t - m \cos mt + \cos mt - \cos t)\cos(m-1)t}{(m+1)(1 - \cos(m-1)t}$$

$$= \frac{m \cos t + \cos mt - (m \cos t + \cos mt)\cos(m-1)t}{(m+1)(1 - \cos(m-1)t)}$$

$$= \frac{(m \cos t + \cos mt)(1 - \cos(m-1)t)}{(m+1)(1 - \cos(m-1)t)}$$

$$= \frac{m \cos t + \cos mt}{m+1}$$

Similarly the expression for y can be calculated as

$$y = \frac{m \sin t + \sin mt}{m+1}$$

If on the other hand $(m+1)(1-\cos(m-1)t) = 0$, that is where $\cos(m-1)t = 1$, we get the whole line included as part of the envelope. For example if $m = 2$ and $t = 0$ we have

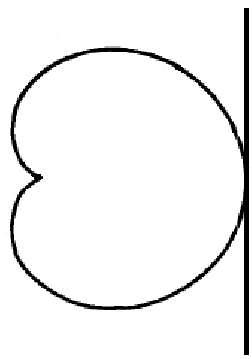


Figure 26: Embroidery Envelope for $m = 2$.

$$\begin{aligned} F &= x(\sin 2t - \sin t) - y(\cos 2t - \cos t) - \sin t \\ \frac{\partial F}{\partial t} &= x(2\cos 2t - \cos t) - y(\sin t - 2\sin mt) - \cos t \end{aligned}$$

then $F = \frac{\partial F}{\partial t} = 0$ has solution $x = 1$ and y is arbitrary.

First lets take a look at where where the singular points occur on the envelope.

$$\frac{\partial^2 F}{\partial t^2} = x(\sin t - m^2 \sin mt) - y(\cos t - m^2 \cos mt) + (m-1)^2 \sin(m-1)t$$

Now we substitute in the parametric equations for x and y and get

$$\begin{aligned} &= \left(\frac{m\cos t + \cos mt}{m+1}\right)(\sin t - m^2 \sin mt) - \left(\frac{m \sin t + \sin mt}{m+1}\right)(\cos t - m^2 \cos mt) \\ &\quad + (m-1)^2 \sin(m-1)t \\ &= \frac{1}{m+1}(m \cos t \sin t - m^3 \cos t \sin mt + \cos mt \sin t - m^2 \cos mt \sin mt - m \sin t \cos t \\ &\quad + m^3 \sin t \cos mt - \sin mt \cos t + m^2 \sin mt \cos mt) + (m-1)^2 \sin(m-1)t \end{aligned}$$

$$= \frac{1}{m+1}((m^3+1)\sin(1-m)t) + (m-1)^2\sin(m-1)t$$
$$= \frac{-1}{m+1}(m^3+1)\sin(m-1)t + (m^2-2m+1)\sin(m-1)t$$
$$= \frac{-1}{m+1}(m^3+1)\sin(m-1)t + \frac{m+1}{m+1}(m^2-2m+1)\sin(m-1)t$$
$$= \frac{-m^3-1+m^3+m^2-2m^2-2m+m+1}{m+1}\sin(m-1)t$$
$$= \frac{-m^2-m}{m+1}\sin(m-1)t$$
$$= -m\sin(m-1)t$$

Which is equal to zero if and only if $t = \frac{n\pi}{m-1}$. So we appear to get $2(m-1)$ singular points on the curve, whereas for the example cusps appear to be only $m-1$ of them.

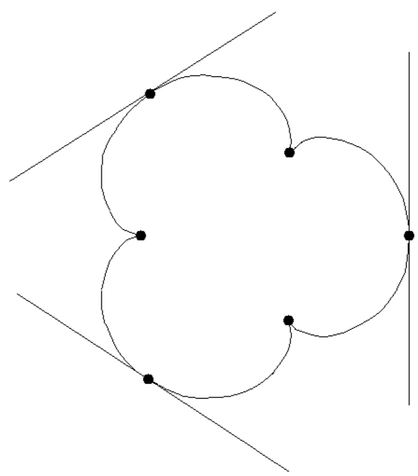


Figure 27: Embroidery Envelope for $m = 4$ has 6 Singular Points.

On our curve we have two different types of singular points. Some which look like cusps (we shall prove that they are in a moment) and some where we get the tangent line included in the curve as well. The latter occurs when $(\cos t, \sin t)$ is equal to $(\cos mt, \sin mt)$. So in effect when we draw the line connecting the two we are actually connecting the point with itself.

For which singular points does this occur?

$$t = 0, \frac{\pi}{m-1}, \frac{2\pi}{m-1}, \frac{3\pi}{m-1}, \frac{4m\pi}{m-1}, \cdots, \frac{n\pi}{m-1}, \cdots$$

$$mt = 0, \frac{m\pi}{m-1}, \frac{2m\pi}{m-1}, \frac{3m\pi}{m-1}, \frac{4m\pi}{m-1}, \cdots, \frac{nm\pi}{m-1}, \cdots$$

Where $n = 0, 1, ..., 2m-3$

$$\frac{mn\pi}{m-1} - \frac{n\pi}{m-1} = n\pi$$

So we get the same point when n is even. This means that the line in the family joins itself to itself and hence appears in the envelope. We have shown that

Proposition 3.3 *For even n when $t = \frac{n\pi}{m-1}t$ we get the whole line $F(t, x, y)$ appearing as part of the envelope.*

We can use the versal unfolding condition to see what we can say about these singular points. When is F a versal unfolding of an A_2 singularity? First we must check when $\frac{\partial^3 F}{\partial t^3} = 0$.

$$\frac{\partial^2 F}{\partial t^2} = x(\sin t - m^2 \sin mt) - y(\cos t - m^2 \cos mt) + (m-1)^2 \sin(m-1)t$$

Differentiating gives

$$\frac{\partial^3 F}{\partial t^3} = x(-m^3 \cos mt + \cos t) - y(m^3 \sin mt - \sin t) + (m-1)^3 \cos(m-1)t$$

$$= \frac{m \cos t + \cos mt}{m+1}(-m^3 \cos mt + \cos t) - \frac{m \sin t + \sin mt}{m+1}(m^3 \sin mt - sint) + (m-1)^3 \cos(m-1)t$$

$$= \frac{1}{m+1}(m \cos t \cos t - m^4 \cos t \cos mt - m^3 \cos^2 mt + \cos mt \cos t - m^4 \sin t \sin mt$$
$$+ m \sin^2 t - m^3 \sin^2 mt + \sin mt \sin t) + (m-1)^3 \cos(m-1)t$$

$$= \frac{1}{m+1}(m - m^4 \cos(m-1)t - m^3 + \cos(m-1)t) + (m-1)^3 \cos(m-1)t$$

$$= \frac{1}{m+1}(m - m^3 + (-2m^3 - 3m^2 + 2m) cos(m-1)t$$

Remember that we are looking for the extra condition that $\frac{\partial^3 F}{\partial t^3}$ equals zero as well as $\frac{\partial^2 F}{\partial t^2}$ being zero. So we can substitute $t = \frac{n\pi}{m-1}$.

$$\frac{\partial^3 F}{\partial t^3} = \frac{1}{m+1}(m - m^3 + (-2m^3 - 3m^2 + 2m)\cos\left(m - 1\frac{n\pi}{m-1}\right)$$
$$= \frac{1}{m+1}(m - m^3 + (-2m^3 - 3m^2 + 2m)\cos n\pi$$
$$= \frac{1}{m+1}(m - m^3 \pm (-2m^3 - 3m^2 + 2m)$$

Which equals either $\frac{1}{m+1}(-3m^3 - 3m^2 + 3m)$ or $\frac{1}{m+1}(m^3 + 3m^2 - m)$. Neither of which contain rational roots except for $m = 0$. So we get $\frac{\partial^3 F}{\partial t^3} \neq 0$.

Remark 3.4 If we allow m to be an irrational number we get a very different, though probably interesting situation. The curve in this case would not be closed and would continue to loop around the origin indefinately.

We now know that the third partial derivative is non-zero, so we now know that we have A_2 singularities. Can they be versally unfolded though? Let's look at an example with $m = 2$.

$$F = x(\sin 2t - \sin t) + y(\cos t - \cos 2t) - \sin t$$
$$\frac{\partial F}{\partial x} = \sin 2t - \sin t$$
$$\frac{\partial F}{\partial y} = \cos t - \cos 2t$$

The corresponding Jet matrix is

$$\begin{pmatrix} \sin 2t - \sin t & \cos t - \cos 2t \\ 2\cos 2t - \cos t & -\sin t + 2\sin 2t \end{pmatrix}$$

$$\det = (\sin 2t - \sin t)(-\sin t + 2\sin 2t) - (\cos 2t - \cos 2t)(2\cos 2t - \cos t)$$
$$= 2\sin^2 2t + +\sin^2 t - 3\sin t \sin 2t - 3\cos t \cos 2t + \cos^2 t + 2\cos^2 2t$$
$$= 3 - 3(\sin 2t \sin t + \cos 2t \cos t)$$
$$= 3 - 3(\cos t)$$

For $t = \pi$ this is non zero so can be versally unfolded and hence is a cusp. For $t = 0$ we have no versal unfolding so can say nothing about the type of singularity. If we do this for a general integer m we find that we get cusps where $t = \frac{n\pi}{m-1}$ when n is odd. As we might have expected the points where n is even and which have the line included in the envelope are not isomorphic to a cusp.

It now seems an obvious question to ask whether, like in the last example, the lines included in the envelope where n is odd, are the limit of intersection of nearby lines. However, strictly speaking this makes no sense to ask this as it does not fit entirely with the defintion of an envelope. The problem is that

$$F(t, x, y) = x(\sin 2t - \sin t) + y(\cos t - \cos 2t) - \sin t$$

has

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \text{ when } t = 0.$$

In this sense, $t = 0$ gives the whole xy-plane since $F(0, x, y) = 0$ for all x, y. So for this example, perhaps dissapointingly it does not seem to make sense having limits of intersection of nearby lines being the whole plane.

For the example where $m = 2$ the extra solution for $t = 0$ can be removed with clever use of double and half angle formulae.

$$F(t, x, y) = x(\sin 2t - \sin t) + y(\cos t - \cos 2t) - \sin t$$

By replacing $\sin 2t$ with $2 \sin t \cos t$ and $\cos 2t$ by $2\cos^2 t - 1$ we get

$$F(t, x, y) = x(2 \sin t \cos t - \sin t) + y(\cos t - 2\cos^2 t - 1) - \sin t$$

Now we subsitute $t = 2u$ and we get

$$F(t, x, y) = x \sin 2u(2 \cos 2u - 1) - y(2 \cos 2u + 1)(2 \cos 2u - 1) - \sin 2u$$

Now we replace $\sin 2u$ with $2 \sin u \cos u$ and $\cos 2u$ with $1 - 2 \sin u$ and we get

$$F(t, x, y) = 2x \sin u \cos u(1 - 4 \sin 2u) + 4y \sin^2 u(4 \cos^2 u - 1) - 2 \cos u \sin u$$

then cancelling $2 \sin u$ gives

$$F(t, x, y) = x \cos u(1 - 4 \sin^2 u) + y \sin u(4 \cos^2 u - 1) - \cos u$$

This still gives the same family of curves, only now $F(t,x,y) = \frac{\partial F}{\partial t} = 0$ when $u = 0$ gives only the point $(1,0)$, which is not a point of regression. The only point of regression now is at $t = \pi, u = \frac{\pi}{2}$. Presumably this sort of approach could be used for higher values of m but of course it would be a lot more complicated.

4 Duals of Plane Curves

In two dimension the dual space is a way of representing all the tangent lines to a curve. Each tangent line corresponds to a point in the dual space. Consequently the whole family of tagent lines to curve correspond to another curve in the dual space. We begin by looking at ways of representing lines in the plane. We then at what corresponds to singularities in the dual space and use Thom's idea of Versal unfoldings. Finally we take a look at the dual spaces of various space curves.

4.1 Representing lines in the plane

In two dimesnsions it is possible to represent the plane in several ways that we should be familiar with. There is the the general form; $Ax + by + C = 0$, then there is the y-axis formula; $y = mx + c$, and also the intercept form; $\frac{x}{a} + \frac{y}{b} = 1$.

Every straight line in the plane can be written in the general form $Ax + By + C = 0$, with A, B not both zero. Other ways which only use two constants such as the y-axis formula has no way of expressing vertical lines. The intercept form is even worse and cannot represent neither vertical nor horizontal lines.

We have a way of representing lines, l say, in the plane by two $(\underline{u}, v)$ where $\underline{u} \in \mathbb{S}^1$ is the direction of vector through the origin and perpendicular to l and $v \in \mathbb{R}$ is the perpendicular distance to l.

$(\underline{u}, v)$ actually gives us oriented lines, that is they come equiped with a direction. Using oriented lines is much simpler than working with un-oriented which require some additional structure. The only difference is we get two copies of everything because $(\underline{u}, v)$ bar orientation gives the same line as $(-\underline{u}, -v)$. See Figure 31.

Each oriented line can be represented by its corresponding point $(\underline{u}, v)$ in what we cal the dual space. The dual space can be visualised as the surface

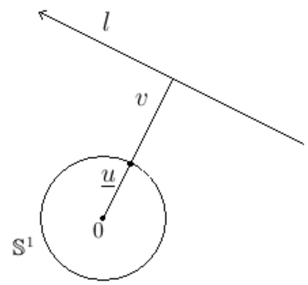


Figure 28: Representing the line l using u and v.

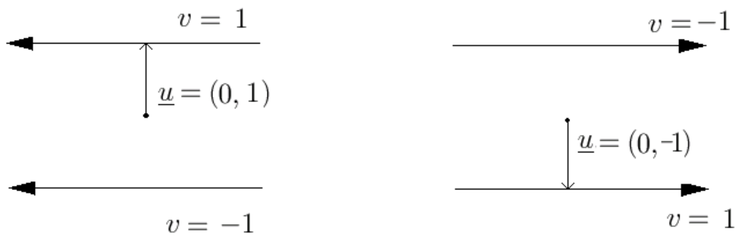


Figure 29: u and v produce oriented lines.

of a cylinder see Figure 30. Locally the surface of the cylinder looks like euclidean space where each point on the cylinder is associated with a line in the plane.

4.2 Representing the Dual of a Plane Curve as a Discriminant

If we use the (rabbit out of a hat) function

$$\begin{aligned} F(t, \underline{u}, v) = H(t, \underline{u}) &= \gamma(t) \cdot \underline{u} - v, \quad \underline{u} \in \mathbb{S}^1, v \in \mathbb{R} \\ &= \gamma(t) \cdot (\cos x, \sin x) - y, \quad \text{(say)} \end{aligned}$$

The discriminant is given by

$$F = \frac{\partial F}{\partial t} = 0$$

$$\frac{\partial F}{\partial t} = T \cdot (\cos x, \sin x) = 0$$

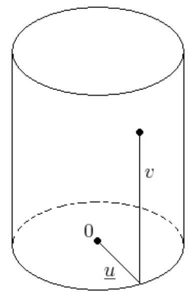


Figure 30: The dual space for 2-dimensions is the surface of a cylinder

$$T \cdot \underline{u} = 0 \ \text{ means that } T \text{ is perpendicular to } (\cos x, \sin x)$$

$$\underline{u} = \lambda N, \ \text{where } u \text{ is unit length so } \lambda = \pm 1$$

$$v = \gamma \cdot \underline{u}$$

So we get $(\underline{u}, v) = (\pm N, \pm\gamma \cdot N)$. As luck would have it this is precisely the dual of γ. This is particularly fortunate for us as we can now apply a host of tricks that we know from studying envelopes. We can use this property to see what features in the plane correspond to various singularities in the dual space.

Since we are dealing woth 2-dimensions we need only consider A_1 and A_2 singularities. Now F has an A_1 singularity at t if and only if $u = \pm N(t)$

$$\frac{\partial^2 F}{\partial t^2} = \kappa N \cdot u = \kappa\lambda$$

So the condition for A_1 singularities is that $\kappa \neq 0$

$$\begin{aligned} \frac{\partial^3 F}{\partial t^3} &= \kappa' \cdot u + \kappa T \cdot u \\ &= \kappa' \lambda \end{aligned}$$

The condition for A_2 singularity is therefore $\kappa = 0$ and $\kappa' \neq 0$. So we find that where we have an ordinary inflexion in the plane the corresponding curve in the dual space as A_2 singularity. We can now use the versality criterion to see if the singularities are cusps.

$$\frac{\partial F}{\partial x} = (x(t), y(t)) \cdot (-\sin x, \cos x)$$

$$\frac{\partial F}{\partial y} = -1$$

At (t_0, x_0, y_0) where we have an A_2 singularity

$$\frac{\partial F}{\partial x} = \gamma \cdot (-T)$$

$$\frac{\partial F}{\partial y} = -1$$

$$\det \begin{pmatrix} -\gamma \cdot T & -1 \\ -T.T & 0 \end{pmatrix} \neq 0, \quad \text{hence versal unfolding is automatic}$$

So an inflexion in the curve corresponds to a cusp in the dual space. See Figure dfdfdfdaASA.

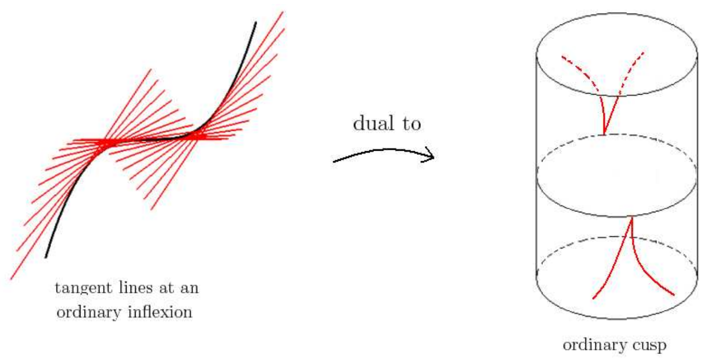


Figure 31: The dual of an inflexion is a cusp.

Remark 4.1 *Note we get two copies in the dual space because we are looking at oriented lines. The two cusps are on opposite sides of the dual space (u and $-u$) so this does not affect us if we are looking at the local structure.*

The next logical steps is to look at space curves

4.3 Duals of Space Curves

In 2-dimensions we had a method for setting up a one-to-one correspondence between oriented tangent lines in the plane and points $\mathbb{S}^1 \times \mathbb{R}$ in the dual space. Similarly to when we had 2 dimensions, in 3 dimensions we can set up a one-to-one correspondence between oriented planes and points $\mathbb{S}^2 \times \mathbb{R}$ in the dual space.

This time we use an extended version of the height function,

$$F : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$$

$$F(t, u, v) = \gamma(t) \cdot u - v$$

If we differntiate with respect to t

$$\frac{\partial F(t, u, v)}{\partial t} = T \cdot u$$

If we take the discriminant set $F = \frac{\partial F(t,u,v)}{\partial t} = 0$ then we get the exact set of oriented tangent planes which are given by $T \cdot u = 0, \gamma(t) \cdot u = v$.

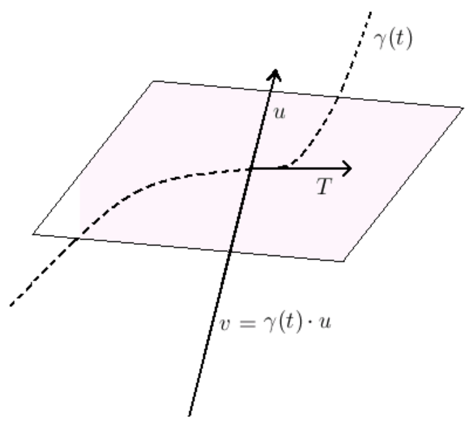


Figure 32: The tangent plane as a discriminant of F.

Remark 4.2 *The planes are oriented by this method of construction. Looking at the plane in the direction of u and consider clockwise rotation about the normal u gives us the orientation. This means that* (u, v) *and* $(-u, -v)$ *look the same but they have different orientation and are different in the dual* *space. As with the 2-dimensional case it is easier than adding the additional structure $(u, v) = (-u, -v)$. It makes no difference to us as we shall be looking at local strucure whereas (u, v) and $(-u, -v)$ are far apart.*

We shall now look at the structure of the dual space. Each tangent plane to the curve corresponds to a point in the dual space. For each point $\gamma(t)$ there are an infinite number of tangent planes (namely all the planes containing the tangent line at $\gamma(t)$). The tangent plane has two degrees of freedom as the plane can move along or rotate about the curve so the dual to a space curve is infact a surface.

At $\gamma(t)$ we we can specify an oriented tangent plane by a unit (normal) vector u perpendicular to $T(t)$. The plane then has the equation $x \cdot u = \gamma(t) \cdot u$. The set of oriented tangent planes to γ, the dual, is then identified with (u, v) where $v = \gamma(t) \cdot u, T(t) \cdot u = 0, \quad$ where $u \in \mathbb{S}^2, v \in \mathbb{R}$. Which is equal to the discrimant set of F given by

$$F(t, u, v) = \gamma(t) \cdot u - v, \quad u \in \mathbb{S}^2, v \in \mathbb{R}$$
$$\frac{\partial F}{\partial t} = T.u = 0 \text{ implies } u \text{ is perpendicular to } T$$

so $u = \lambda N + \mu B$.
$F = 0$ implies that $V = \gamma(t) \cdot u$

So we really do get the dual space. We shall now look at when singular points occur on the dual space and whether or not they can be versally unfolded. We assume $\kappa \neq 0$.

$$\frac{\partial^2 F}{\partial t^2} = \kappa N \cdot u = \kappa \lambda$$

As we assumed $\kappa \neq 0, \quad \kappa\lambda = 0$ if and only if $\lambda = 0$. So we get an A_1 singularity at t if and only if $\lambda \neq 0$.

$$\begin{aligned} \frac{\partial^3 F}{\partial t^3} &= \kappa' N \cdot u + \kappa(\tau B - \kappa T) \cdot u \\ &= \kappa' \lambda + \kappa \tau \mu \end{aligned}$$

If $\lambda = 0$ then $\frac{\partial^3 F}{\partial t^3} = 0$ if and only if $\tau = 0$, since $\kappa \neq 0$ and u is a unit vector. So we get an A_2 singularity if and only if $\lambda = 0, \tau \neq 0$.

$$\frac{\partial^4 F}{\partial t^4} = \kappa'' N \cdot u + \kappa'(-\kappa T + \tau B) \cdot u - (k^2)' T \cdot u - \kappa^3 N \cdot u + \kappa \tau' B \cdot u + -\tau^2 N \cdot u$$

$$= \tau' \mu \text{ (at } \lambda = \tau = 0)$$

Which equals zero if and only if $\tau' = 0$. So we get an A_3 singularity if and only if $\tau = \lambda = 0, \tau' \neq 0$. Like with the 2-dimensional case we shall now look to see if theses singularities can be versally unfolded and hence cusps. Without loss of generality we can put our curve in to standard postition,

$$(t - \frac{1}{6}\kappa^2 t^3, \frac{1}{2}\kappa t^2 + \frac{1}{6}\kappa' t^3, \frac{1}{6}\kappa\tau t^3)$$

where $\gamma(t_0) = 0, T(t_0) = (1,0,0), N(t_0) = (0,1,0)$ and $B(t_0) = (0,0,1)$.

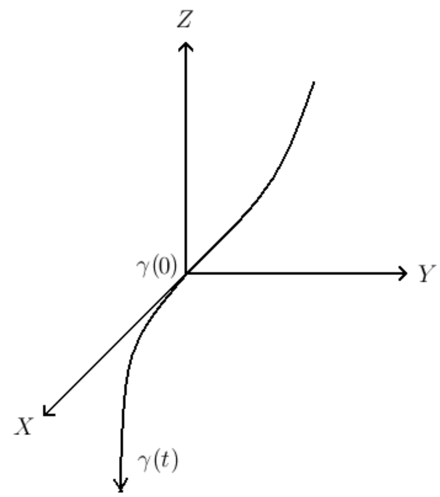


Figure 33: The tangent plane as a discriminant of F.

We are interested in the structure of the dual for points of γ near to $(0,0,0)$. Since A_2 and A_3 singularities occur only on osculating planes we need a parametrisation of $\mathbb{S}^2$ which works near $(0,0,1)$. The function

$$(x_1, x_2) \to (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$$

is an immersion parametrising all unit vectors close to $(0,0,1)$ which is the binormal at $t = 0$. So we write

$$F(t, x, y) = \gamma(t) \cdot (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) - y, \text{ where } \gamma(t) = (X(t), Y(t), Z(t))$$

dropping t form the notation gives

$$\begin{aligned}
F &= Xx_1 + Yx_2 + Z\sqrt{1 - x_1^2 - x_2^2} - x_3 \\
\frac{\partial F}{\partial x_1} &= X + Z(.......)(x_1) = X \ (\text{at } x_1 = x_2 = 0) \\
\frac{\partial F}{\partial x_2} &= Y \\
\frac{\partial F}{\partial x_3} &= -1
\end{aligned}$$

$$\begin{aligned}
F &= Xx_1 + Yx_2 + Z\sqrt{1 - x_1^2 - x_2^2} - x_3 \\
\frac{\partial F}{\partial x_1} &= X + Z(.......)(x_1) = X \ (\text{at } x_1 = x_2 = 0) \\
\frac{\partial F}{\partial x_2} &= Y \\
\frac{\partial F}{\partial x_3} &= -1
\end{aligned}$$

So the jet matrix is given by

$$\begin{pmatrix} X(0) & Y(0) & -1 \\ X'(0) & Y'(0) & 0 \\ X''(0) & Y''(0) & 0 \end{pmatrix}$$

$$\begin{aligned}
\gamma &= (X, Y, Z) \\
\gamma' &= (X', Y', Z') = T, \quad T(0) = (1, 0, 0) \\
\gamma'' &= T' = \kappa N, \quad N(0) = (0, 1, 0)
\end{aligned}$$

$$M = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & \kappa(0) & 0 \end{pmatrix}$$

So long as $\kappa \neq 0$ the versal unfolding is automatic. Therefore the osculating planes correspond to a cuspidal edge and the osculating plane at a zero of torsion corresponds to a swallowtail. See Figure 34.

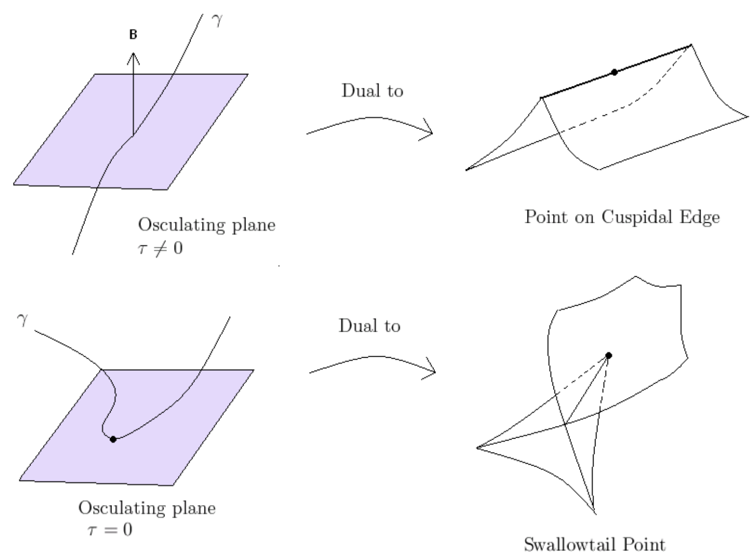


Figure 34: The tangent plane as a discriminant of F.

Swallowtails consist of an A_3 singularity which corresponds to the point on the space curve with $\tau = 0, \tau' \neq 0$. It has A_2 singularities or cusp edges corresponding to the osculating planes of the curve. The swallowtail also has a self intersection line but what does this correspond to on the original curve? Well each point on the dual surface corresponds to a tangent plane on the original curve. So where we have two equal points on different sheets of the dual surface there must be two points on the original curve with the same tangent plane. We shall refer to such planes which are tangent to the curve at two distinct points as bitangent planes.

In the swallowtail we get a half-line of self intersection from the A_3 singularity. This suggests that for all points on the original curve close to where we have zero torsion there is point on the other side of the torison zero that shares the same bitangent plane.

4.4 Bitangent Planes Near Torsion Zeros

How do these bitangent planes behave and is there a relation between the two points on either side that share a bitangent plane? We shall first look at the curve $\gamma(t) = (t, t^2, t^4)$ which has a simple zero of torsion at the origin.

$$\begin{array}{rcll}\gamma(t) & = & (t, t^2, t^4), & \gamma(0) = (0,0,0) \\ \gamma'(t) & = & (1, 2t, 4t^3) & \gamma'(0) = (1,0,0) \\ \gamma''(t) & = & (0, 2, 12t^2) & \gamma''(0) = (0,2,0) \\ \gamma'''(t) & = & (0, 0, 24t) & \gamma'''(0) = (0,0,0)\end{array}$$

The torsion at the origin $\tau(0) = (\gamma(0)' \times \gamma(0)'') \cdot \gamma(0)''' = 0$.
The first derivative at the origin can be calculated as $\tau'(0) = 12 \neq 0$). So we really do have a simple zero of torsion st the origin. We start by looking for the equation for tangent planes of the form $ax + by + cz = d$. We substitute our values for x, y and z of γ.

$$at + bt^2 + ct^4 = d$$

We also wish the plane to be contain the tangent vector, perpendicular to the vector (a, b, c).

$$(a, b, c) \cdot (1, 2t, 4t^3) = 0$$
$$a + 2bt + 4ct^3 = 0$$

So for two points on the curve t_1, t_2, they share a bitangent plane when

$$\begin{array}{rcl} at_1 + bt_1^2 + ct_1^4 - d & = & 0 \\ at_2 + bt_2^2 + ct_2^4 - d & = & 0 \\ (a, b, c) \cdot (1, 2t, 4ct^3) & = & 0 \\ a + 2bt + 4ct^3 & = & 0 \end{array}$$

To solve for t_1 and t_2 we solve when the determinant

$$\begin{vmatrix} t_1 & t_1^2 & t_1^4 & -1 \\ t_2 & t_2^2 & t_2^4 & -1 \\ 1 & 2t_1 & 4t_1^3 & 0 \\ 1 & 2t_2 & 4t_2^3 & 0 \end{vmatrix} = 0$$

After several row operations we get

$$2(t_1 - t_2)^4 (t_1 + t_2) =$$

Discarding the trivial ($t_1 = t_2$), we are left with the solution $t_1 = -t_2$. So we have shown

Proposition 4.3 *For the space curve γ either side of the simple torision zero the points $\gamma(t)$ and $\gamma(-t)$ both share a common bitangent plane. See Figure 35.*

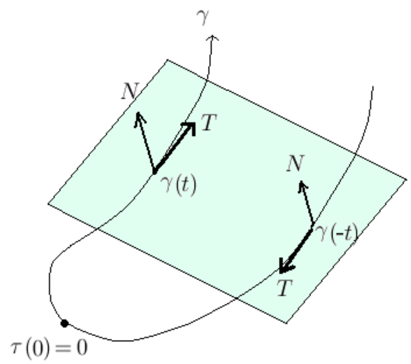


Figure 35: t and $-t$ share a bitangent plane.

4.5 Bitangent Planes of Curves with Non-Simple Zeros of Torsion

What happens when we have a non-simple zero of torsion, that is when $\tau = \tau' = 0$? We shall now look at the curve $\gamma_u(t) = (t, t^2, t^5 + ut^3), \quad u < 0$ which has two simple zeros of torsion which come together as $u \to 0$ to form a degenerate non-simple zero of torsion at the origin.

$$\begin{aligned}
\gamma_u(t) &= (t, t^2, t^5 + ut^3) \\
\gamma_u'(t) &= (1, 2t, 5t^4 + 3ut^2) \\
\gamma_u''(t) &= (0, 2, 20t^3 + 6ut) \\
\gamma_u'''(t) &= (0, 0, 50t^2 + 6u)
\end{aligned}$$

We shall start by looking for the bitangent planes. In the same way that we performed the calculations for the simple zero of torsion, we solve simluta-neous equations for the tangent planes at t_1 and t_2.

$$\begin{aligned}
at_1 + bt_1^2 + ct_1^4 &= d \\
at_2 + bt_2^2 + ct_2^4 &= d \\
a + 2bt_1 + c(5t_1^4 + 3ut_1^2) &= 0 \\
a + 2bt_2 + c(5t_2^4 + 3ut_2^2) &= 0
\end{aligned}$$

Which is equivalent to solving the determinant of

$$\begin{vmatrix} t_1 & t_1^2 & t_1^5 + ut_1^3 & -1 \\ t_2 & t_2^2 & t_2^5 + ut_2^3 & -1 \\ 1 & 2t_1 & 5t_1^4 + 3ut_1^2 & 0 \\ 1 & 2t_2 & 5t_2^4 + 3ut_2^2 & 0 \end{vmatrix} = 0$$

After several row operations we arrive at the equation

$$(t_1 - t_2)^4(3t_1^2 + 4t_1t_2 + 3t_2^2 + u) = 0$$

As before we are not interested in when t_1 is the same point as t_2, so we can ignore the terms $(t_1 - t_2)$. This leaves us with $(3t_1^2 + 4t_1t_2 + 3t_2^2 + u) = 0$ which looks like an ellipse when $u < 0$, otherwise it has no solutions. See Figure 36. The ellipse shrinks to a point as $u \to 0$. See Figure 37.

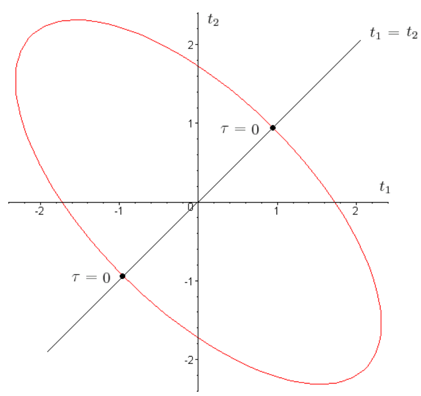


Figure 36: Ellipse showing where t_1 and t_2 share a bitangent plane.

Each point (t_1, t_2) on the ellipse represent an un-ordered pair or points that share a tangent plane on the space curve. Where do the points of zero torsion fit into the picture?

For a point of zero torsion we need γ', γ'', and γ''' linearly independent.

$$(\gamma' \times \gamma'') \cdot \gamma''' = 2(60t^2 + 6u) = 0$$

which implies for a given u that the zero of torsions appear at

$$t = \pm\sqrt{\frac{-u}{10}}.$$

So the torisons of zero on the space curve our represented on our ellipse by the points on the diagonal line. See Figure 36. Tracing along the ellipse gives an unordered pair of bitangent planes which give us the self intersection in the dual space. It does not matter which way you traverse around the ellipse since (t_1, t_2) are an unordered pair.

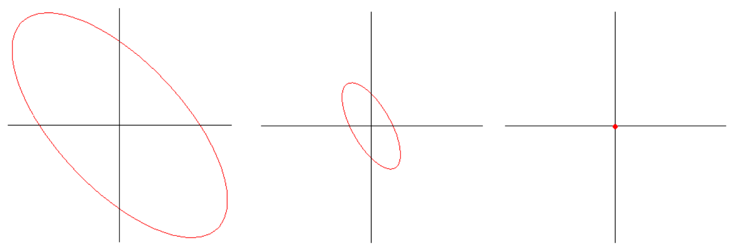


Figure 37: The ellipse shrinks as $u \to 0$.

How do the points on the curve t_1 and t_2 travel relative to each other? We start from the torision zero in the $t_1, t_2 > 0$ quadrant, follow the curve around the ellipse anti-clockwise. We can see that t_1 decreases whilst t_2 increases until it reaches a turning point for t_2. Then t_2 and t_1 both decrease for a moment before hitting the turning point in t_1 which then decreases until again both t_1 and t_2 are the same point on the curve, the other zero torsion point. See Figure 38.

What are the turning points on the ellipse, or in other words how far do the bitangent planes move away from the torsion zeros? We first calcluate the gradient function of the ellipse, say F, then

$$\nabla F = \left(\frac{\partial F}{\partial t_1}, \frac{\partial F}{\partial t_2}\right) = (6t_1 + 4t_2, 6t_2 + 4t_1)$$

So the turning points of t_1 and t_2 are $t_1 = -\frac{3}{2}t_2$ and $t_2 = -\frac{3}{2}$ respectively. Substituting in to the original equation we get the turning points as

$$(t_1, t_2) = \left(\pm\sqrt{\frac{-4u}{15}}, \mp\sqrt{\frac{-3u}{5}}\right) \text{ and } \left(\mp\sqrt{\frac{-3u}{5}}, \pm\sqrt{\frac{-4u}{15}}\right).$$

We have shown that tosion zeros on a space curve correspond to swallowtail points in the dual space. We have seen that the curve $\gamma_u(t) =$

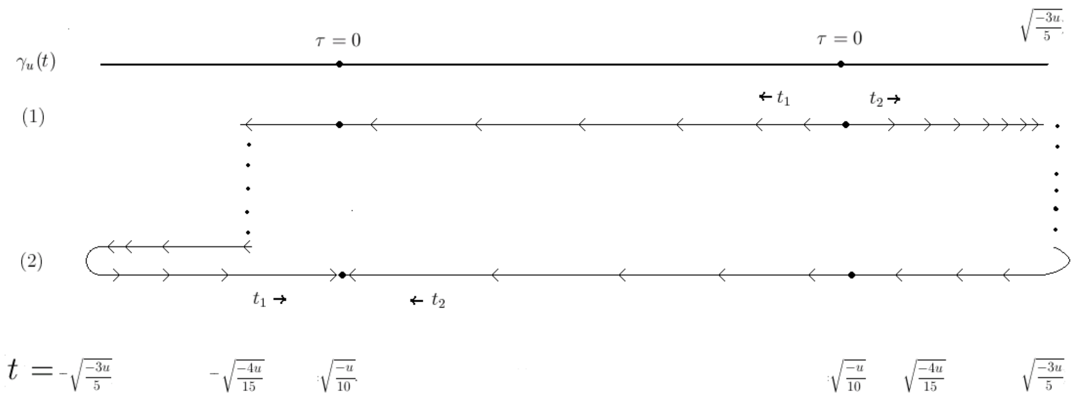


Figure 38: How the points t_1 and t_2 sharing a bitangent plane travel along the curve.

$(t, t^2, t^5 + ut^3)$ has two zero torsion points. We have also shown that the two torsion points are connected by a line of self intersection. The dual surface of $\gamma_u(t) = (t, t^2, t^5 + ut^3)$ can be seen in Figure 39.

From Figure 39 it does seem as though the self intersections turn around at two points between the two torsion zeros. It also looks as though they turn around at the cuspidal edges (which if you remeber corrsepond to oscualting planes on the space curve, see Chapter 4.3. We shall now prove that this is indeed the case.

Proposition 4.4 *The line of self intersection of the dual of the curve $\gamma_u(t) = (t, t^2, t^5 + ut^3), u < 0$ have turning points at the cuspidal edges.*
Which is equivlanent to:
The function of the points t_1 and t_2 which share a bitangent plane on the curve $\gamma_u(t) = (t, t^2, t^5 + ut^3), u < 0$ has a turning point when the bitangent plane is also the oscualting planes.

Proof 4.5

$$\gamma_u(t) = (t, t^2, t^5 ut^3)$$
$$T = \gamma_u'(t) = (1, 2t, 5t^4 3ut^2)$$
$$\kappa N = \gamma_u''(t) = (0, 2, 20t^3 6ut^2)$$
$$\tilde{B} = T \times \kappa N = (30t^4 + 6ut^2, -20t^3 - 6ut, 2)$$

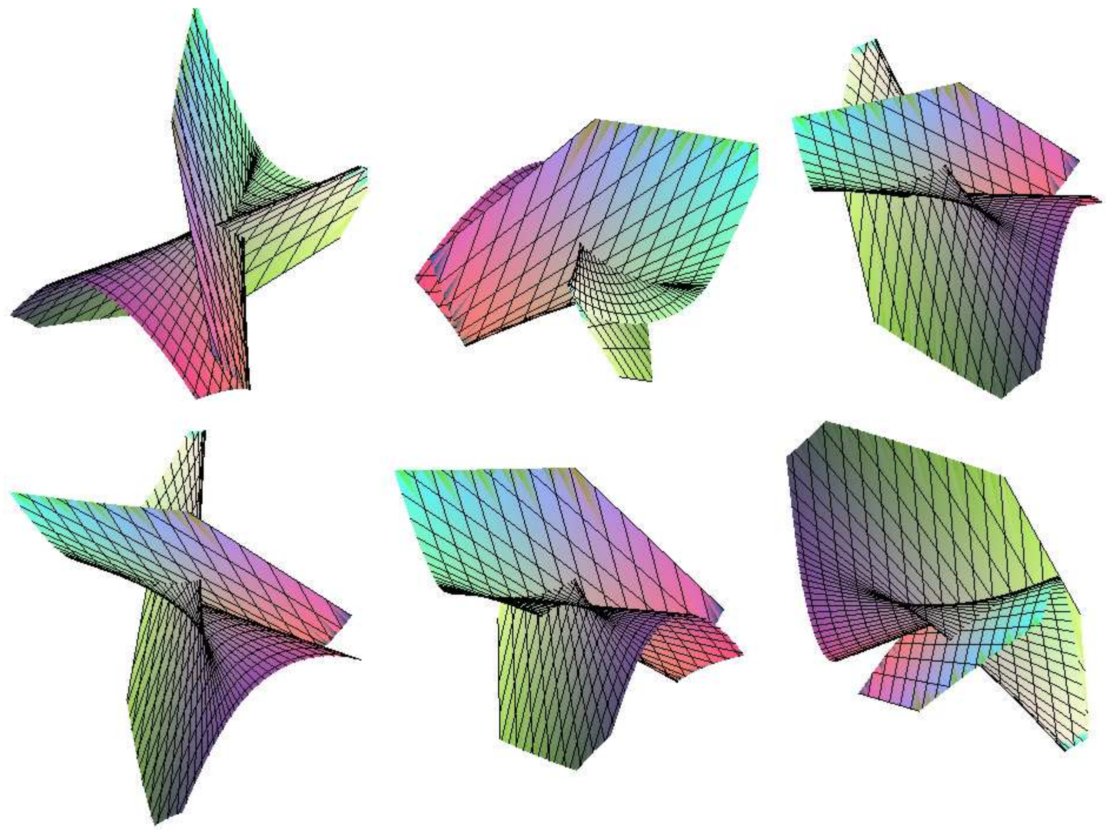


Figure 39: The dual surface of $\gamma_u(t) = (t, t^2, t^5 + ut^3)$

At a turning point, for example $\left(-\sqrt{\frac{-4u}{15}}, \sqrt{\frac{-3u}{5}}\right)$, *we evaluate* $\tilde{B}$ *and* T *for both* t_1 *and* t_2. *Denote* $u = -c^2$

$$\tilde{B}(t_1) = \left(\frac{8c^4}{15}, \frac{-4\sqrt{15}c^3}{45}, 2\right)$$

$$\tilde{B}(t_2) = \left(\frac{36c^4}{5}, -\frac{-6\sqrt{15}c^3}{5}, 2\right)$$

$$T(t_1) = \left(1, -\frac{4c\sqrt{15}}{15}, -\frac{4c^4}{9}\right)$$

$$T(t_2) = \left(1, \frac{2c\sqrt{15}}{5}, 0\right)$$

By definition $T(t_1)$ will be perpendicular ($\perp$) to $B(t_1)$ and $T(t_2)$ will be $\perp$ to $B(t_2)$. If B_1 it perp to T_2 however, then the bitangent plane to both t_1 and t_2 is also the oscualating plane at t_2. If on the other hand $B(t_2) \perp T(t_1)$ then the bitangent plane is also the oscualting plane at t_1. See Figure (below). Technically to complete the proof we need to show $T(t_1) \neq \lambda T(t_2), \lambda \in \mathbb{R}$. From the ellipse in Figure 36 it should be obvious that this does not occur for our particular family of curves. In the unlikely event that $T(t_1)$ is a multiple of $T(t_2)$, then we should also check that additonally either $N(t_1) \perp B(t_2)$ or $N(t_2) \perp B(t_2)$, in which case it is an osculating plane.

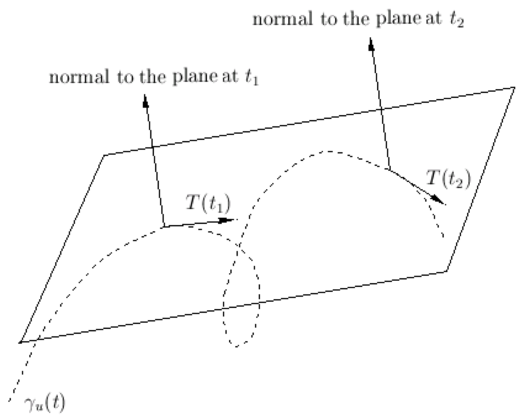


Figure 40: Bitangent plane for t_1 and t_2 of $\gamma_u(t)$

It turns out, that

$$T(t_1).B(t_2) = 100c^4/9$$
$$B(t_1).T(t_2) = 0 \qquad \square$$

So where we get a turning point in t_1 we get the osculating plane for t_2. Similar calculations using other turning points show that the turning point in t_2 corresponds to the osculating plane for t_1.

We have also seen that the curve $\gamma_u(t) = (t, t^2, t^5 + ut^3)$, has two torsion zeros which come into coincidence when u approaches zero. So the corresponding dual space has two swallowtails which come in to coincidence when u becomes zero. See Figure 42.

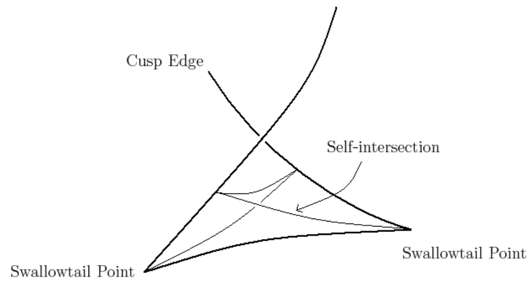


Figure 41: Bitangent plane for t_1 and t_2 of $\gamma_u(t)$

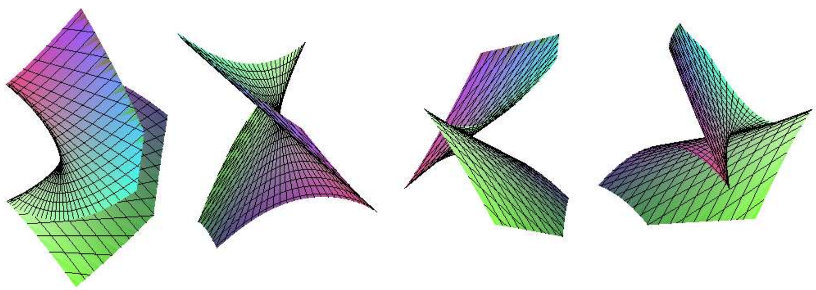


Figure 42: Dual surface of $\gamma_u(t) = (t, t^2, t^5 + ut^3)$ for $u = 0$.

4.6 Dual of Space Curve with a Zero of Curvature and Two Torsion Zeros

The next curve we shall look at is $\gamma_u(t) = (t, t^3, t^4 + ut^2)$. For $u \in \mathbb{R}$

$$\begin{aligned}
\gamma_u(t) &= (t, t^3, t^4 + ut^2) \\
\gamma_u'(t) &= (1, 3t^2, 4t^3 + 2ut) \\
\gamma_u''(t) &= (0, 6t, 12t^2 + 2u) \\
\gamma_u'''(t) &= (0, 6, 24t)
\end{aligned}$$

$$\tau = \frac{(72t^2 - 12u)}{((12t^4 - 6ut^2)^2 + (-12t^2 - 2u)^2 + 36t^2)}$$

So as in the last example we have two zeros of torsion, $\pm\sqrt{\frac{u}{6}}$, which come in to coincidence at $t = 0$ as $u \to 0$.

$$\kappa = \frac{\gamma' \times \gamma''}{||\gamma'||^3} = 2\frac{(36t^8 - 36t^6u + 9u^2t^4 + 36t^4 + 12ut^2 + u^2 + 9t^2)^{1/2}}{(1 + 9t^4 + 16t^6 + 16t^4u + 4u^2t^2)^{3/2}}$$

So the curve $\gamma_u(t)$ has zero curvature at $t = 0$ when $u = 0$.

As in the previous example we shall start with the bitangent planes. The bitangent planes are derived by

$$\begin{aligned} at_1 + bt_1^3 + c(t_1^4 + ut_1^2) &= d \\ at_2 + bt_2^3 + c(t_2^4 + ut_2^2) &= d \\ a + 3bt_1^2 + c(4t_1^3 + 2ut_1) &= 0 \\ a + 3bt_2^2 + c(4t_2^3 + 2ut_2) &= 0 \end{aligned}$$

So we solve the determinant of

$$\begin{vmatrix} t_1 & t_1^3 & t_1^4 + ut_1^2 & -1 \\ t_2 & t_2^3 & t_2^4 + ut_2^2 & -1 \\ 1 & 3t_1 & 4t_1^3 + 2ut_1^2 & 0 \\ 1 & 3t_2 & 4t_2^3 + 2ut_2^2 & 0 \end{vmatrix} = 0$$

Ignoring trivial solutions $t_1 = t_2$ we get the equation

$$t_1^2 + 4t_1t_2 + t_2^2 = u$$

Which is the equation for a hyperbola. See Figure 43.

So with the curve which has curvature equal zero we get a very different picture from the last example. This time the self intersection line in the dual space does not connect the two zeros of torsion.

In the example we saw in Chapter 4.5 the swallowtails opened out towards each other and therefore interacted with each other. For this curve with limiting zero of curvature the swallowtails open out away from each other and therefore have no interaction locally. See Figure 44.

For our curve as $u \to 0$ the angle between the two binormals at the torsion zeros tends to π. That is they are pointing in opposite directions. So the binormal plane rotates about the curve very quickly for a very small piece of curve. We shall now to prove this fact.

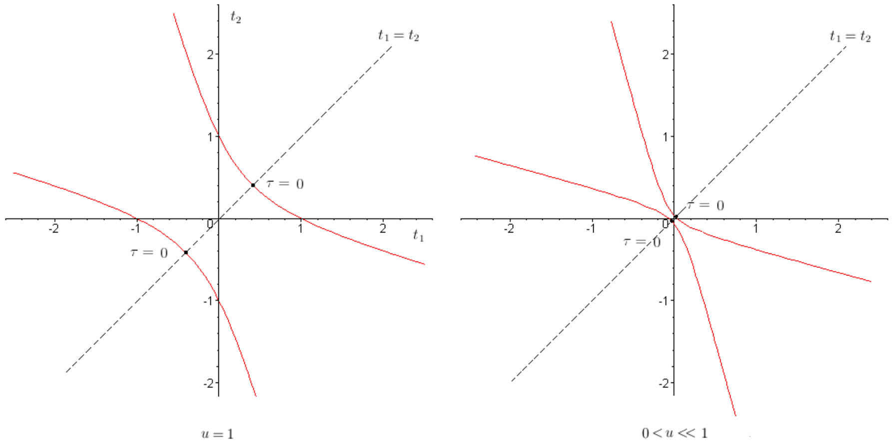


Figure 43: Un-ordered points (t_1, t_2) sharing a bitangent plane.

Proposition 4.6

$$\gamma(t) = (t, t^3, t^4 + ut^2)$$

$$\gamma' = Ts'$$

$$\gamma'' = Ts'' + \kappa N s'^2$$

$$\gamma' \times \gamma'' = \kappa s'^3 B, \qquad \text{where } \kappa s'^3 > 0$$

$$\gamma' = (1, 3t^2, 4t^3 + 2ut)$$

$$\gamma'' = (0, 6t, 12t^2 + 2u)$$

$$\gamma' \times \gamma'' = (12t^4 - 6ut^2, -12t^2 - 2u, 6t)$$

The torsion zeros occur at $t = \pm\sqrt{\frac{u}{6}}$.

$$\gamma' \times \gamma'' = \left(-\frac{2}{3}u^2, -4u, \pm\sqrt{\frac{u}{6}}\right)$$

$$\text{so } \gamma' \times \gamma'' \text{ is parallel to } \left(-\frac{2}{3}u^{\frac{3}{2}}, -4u^{\frac{1}{2}}, \pm\frac{1}{\sqrt{6}}\right)$$

We have B at torsion zero is unit vector in the direction

$$\left(-\frac{2}{3}u^{\frac{3}{2}}, -4u^{\frac{1}{2}}, \pm\frac{1}{\sqrt{6}}\right)$$

Let $u \to 0$. The limit is unit vectors $(0, 0, \pm 1)$. Hence B has no limit. $\square$

We have shown that as $u \to 0$ the binormals at the two torsion zeros face opposite directions and have no limit. Hence the osculating plane spins rapidly through π at this point. In the dual space this corresponds to the cuspidal edges. Though we are really only concerned with local structure, one can imagine how globally the cuspidal edges 'extend away' from each other and meet at 'infinity'.

As $u \to 0$ the swallowtails move further apart, they face open away from each other and therefore do not interact. See Figure 44 where the two swallowtails are shown side by side with the axes shown locating their relative positions.

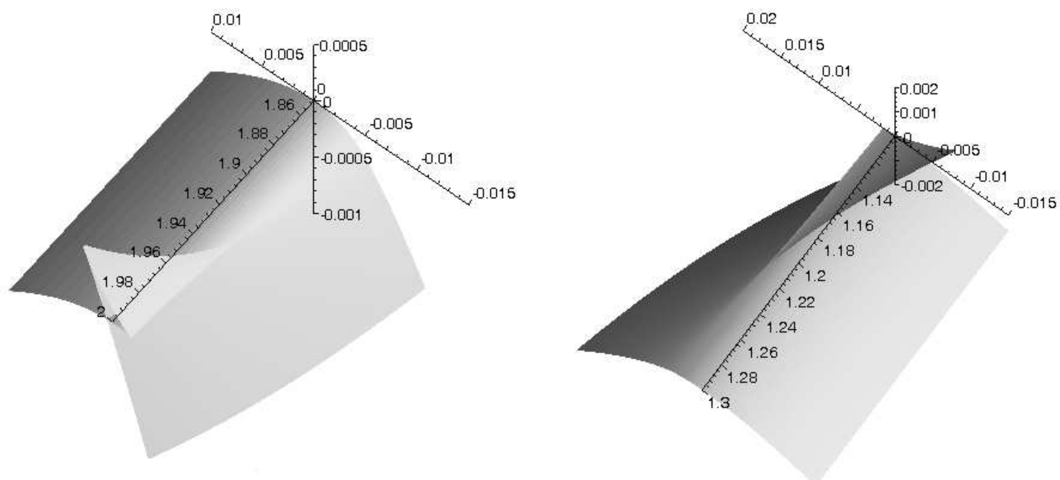


Figure 44: The two swallowtails of the dual of $\gamma_u(t) = (t, t^3, t^4 + ut^2)$.

4.7 The Darboux Vector

The Darboux vector, discovered by Jean Gaston Darboux (1842 - 1917), is defined as

$$D = \tau T + \kappa B.$$

The first derivative of the unit vectors of the Frenet-Serret frame; the tangent, normal and binormal give us the Frenet-Serret folmulas. Comparing this with the Darboux vector shows

$$T' = \kappa N, \quad \text{so therefore } T' \cdot D = 0$$

$$N' = -\kappa T + \tau B \text{ so } N' \cdot D = (-\kappa T + \tau B) \cdot (\tau T + \kappa B) = -\kappa\tau + \tau\kappa = 0$$

$$B' = -\tau \cdot N, \quad \text{so therefore } T' \cdot D = 0.$$

We find that the Darboux vector is perpendicular to all T', N' and B'. If you imagine a particle moving along the curve, the Darboux vector gives us the axis of instantaneous rotation of the particle. On a curve where we have two torsion zeros astride a point of zero curvature, we have three points where the Darboux vector is stationary in terms of direction. When the two torision zeros come into coincidence at the point of zero curvature, the three Darboux stationary points come together to leave just one point on the curve where the Darboux vector is stationary [4]. So let us now try to validate this property for our particular curve.

Proposition 4.7 *The Darboux vector has three stationary points seperated by the two torsion zeros. Where the torsion zeros come into coincidence as* $(u \to 0)$ *the Darboux stationary points also come together to form a single degenerate stationary point.*

First we calculate the first derivative of the Darboux vector.

$$\begin{aligned} D' &= (\tau T + \kappa B)' = \tau' T + \tau T' + \kappa' B + \kappa B' = \tau' T + \tau(\kappa N)\kappa' B - \kappa(\tau N) \\ &= \tau' T + \kappa' B \end{aligned}$$

We want to know when D' is parrallel to D. So we calculate when the cross product is zero.

$$\begin{aligned} D' \times D &= \tau' T + \kappa' B \times \tau T + \kappa B \\ &= -\tau' \kappa B + \kappa' \tau \end{aligned}$$

Which is zero when $\tau\kappa' = \tau'\kappa$, or equivalently using the quotient rule, this is when $\left(\frac{\tau}{\kappa}\right)' = 0$.

Using maple to Plot the graphs of $\tau(t)$ and the numerator of D' (since we are only interested in zeros) confirms Proposition 4.7. See Figure 45.

The first graph of Figure 45 shows us when $u > 0$ we have three solutions for $D' = 0$, that is the stationary points of D. The second graph shows when $u = 0$ we get a degenerate stationary point of D and a degenerate torson zero at the origin. The case where $u < 0$ has no solutions for $\tau(t) = 0$ but there is still a Darboux stationary point at the origin.

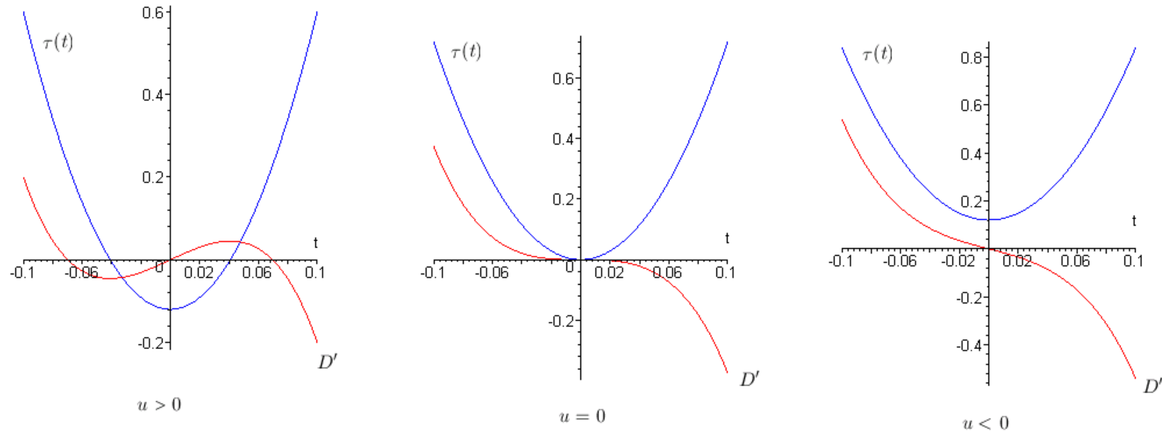


Figure 45: Graphs showing $D'(t)$ and $\tau(t)$.

5 Conclusion

I have enjoyed working on this project and I believe that it has been a worthwile exercise. Along the way we have studied some interesting propeties of curves.

In Chapter 2 we looked at curves of constant width. First of all we looked at how to construct their equations by using a support function for the tangent lines and then taking their envelope. Then we went on to look at various properties of these curves. We looked at the circumference and found how, as with the circle, that it is equal to π multiplied by the diameter. Then we looked at the relation between the curvature at points with parallel tangents. We then went on to look at when and where vertices occur on the curve. A fairly large part of this chapter was then devoted to analysing the criteria for avoiding singular points on the curve for particular support functions. Of course it is easy enough to find the condition manually; finding the turning points, plugging them back into the orriginal equation and observing which is the greatest. However we were searching for a more general formula. We analysed the particular case of two sine functions with $n_1 = 5, n_2 = 3$ in some depth. For this we managed to find a monstrous equation for the necessary and sufficient constant a to avoid the occurence of singular points. See Chapter 2.6.

If there had been more time available it might have been nice to look at equations with higher values of n_i. Or maybe we could have looked at three functions of sine, although perhaps this would not be so bad as it may first seem. Remember that we discussed how $\sin pt = \sin qt = -1$ for some t if and only $p \equiv q \bmod 4$. See Theorem 2.8. If we have three of more functions then there must be at least two of them such that $n_i \equiv n_j \bmod 4$. What made the case we looked at possible was that when you differentiate to find the turning points you get a quadratic in $\sin^2 t$. With higher values n_i this would not be so simple. Whether or not there is a more general result remains to be seen.

In Chapter 3 we took a look at some envelopes. First we observed how the envelope of tangent lines not only included the curve itself, but also the tangent line at points of inflexion. We then looked at how there is another type envelope, E_1, given by the limit of intersection of two two nearby curves of the family. We went on to prove how E_1 is a subset of the our definition of envelope (discriminant). See Theorem 3.1. We then proved that the tangent line at the inflexion in the envelope for the curve $y = x^3$ is indeed contained in E_1. See Theorem 3.2. We then went on to envelopes of circles of curvature and found that at a vertex we get the circle included as part of the envelope. Then, as with the inflexion in the last example, we tried to see whether this circle was the limit of intersection of nearby curves. By looking at some examples of curves with vertices we found that we only got two points in the E_1 envelope which lie on the envelope of circles of curvature.

If there had been more time it would have been nice to look at some more examples of envelopes of circles of curvature with higher order vertices. There are also other types of envelopes that are contained within the discriminant envelope. There is 'the envelope E_2 which is a curve tangent to the C_t' . There is also E_3, 'the boundary of the region filled by the curves C_t' [1]. If there had been more time it would have been interesting to study some specific examples of discriminant envelopes and see how they fit in with these alternative envelopes.

In the next section we considered envelopes created by a process known as embroidery. We used Thom's idea of versal unfolding to find out which singular points were isomorphic to cusps. It turned out that it was not possible to see if the extra tangent lines included at the points where $\cos(m-1)t = 1$ were the limits of intersections of nearby curves. This was because it did not fit with the definition of envelopes as we had $F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ when $t = 0$.

The final chapter was on duals of both plane and space curves. We started with how the dual space is a way or representing tangent lines or tangent planes, depending on how many dimensions we were working with. We saw that in two dimensions inflexions in the curve correspond to cusps in the dual space. With space curves we saw that the dual space is a surface with the oscualting plane corresponding to cusp edges and torsion zeros to swallowtail points. We looked at some specific curves with torsion zeros and how the corresponding swallowtails interacted with each other on the dual surface. If there had been more time it would have been nice to look at some more examples of space curves, perhaps where three torsion zeros come into coincidence such as $\gamma_u(t) = (t, t^2, t^6 + ut^4)$ for instance. Though it is a real shame that there was insufficient time to look at the dual of surfaces. With surfaces the dual sapce of a surface is another surface (whose dual is the original surface). Graph functions given by $z = f(x, y)$ have normals given by $(-f_x, -f_y, 1)$. The tangent planes close to the origin can be paramterised by

$$-f_x X - f_y Y + Z = -x f_x - y f_y + f(x, y)$$

It was shown by Whitney [5] that singularities on the dual surface corresponds precisely to parabolic points on the surface. These are given when the Hessian determinant

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = 0$$

It would have been nice to study look at some specific examples of surface with parabolic points and study their properties in the dual space, though this could easily be a complete project in itself. This fascinating topic is dicussed in depth in Banchoff Gaffney & McCroy [6].

Special thanks to Prof. Peter Giblin without whom this project would not have been possible.

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