Recognition of centre symmetry set singularities

P.J.Giblin, V.M.Zakalyukin [1]

1 Introduction

Let M be a smooth hypersuface embedded in an affine space; consider pairs of points of M at which the tangent hyperplanes are parallel, and in particular the envelope of the straight lines joining these pairs of points. This set, which is intrinsically determined by M, is the 'Centre Symmetry Set' (CSS) of M. For a convex M centrally symmetric about a point $\mathbf{p}$ the CSS reduces to the point $\mathbf{p}$ itself, so that in general the CSS measures the extent to which M possesses central symmetry. For a curve M in the affine plane the CSS is generically a curve with cusps, and also endpoints in the inflexions of M.

The CSS was first defined in a different way in the curve case by Janeczko [7] and in the way stated above by Giblin and Holtom in [2, 6]. Recently the authors presented in [3, 4] a general method for analyzing the local structure of the CSS based on the theory of Lagrange and Legendre singularities.

For a plane curve we can expect the envelope of chords always to exist; however for a surface we obtain a 2-parameter family of lines in 3-space which may or may not have a real envelope. In this paper we consider in detail the case of a smooth (C^∞) surface M in $\mathbf{R}^3$. We describe the results of [4] in a geometrical way and provide explicit algorithms for recognizing the different types of singularities which occur. All our constructions and results in this paper are local, so that we actually consider two surface germs M and N and look for a surface—the CSS—tangent locally to all the straight lines joining parallel tangent pairs. Note that the CSS construction generalises both the euclidean focal set (envelope of common normals of two parallel surface germs) and the affine focal set (envelope of affine normals). In a subsequent paper we shall investigate some global properties of the CSS which generalise the result that the CSS of a generic convex plane curve has an odd number of cusps.

Consider then two surface germs (M, a_0) and (N, b_0), assuming that the tangent planes to M and N at the two base points a_0 and b_0 are parallel. We consider straight lines ('chords') l passing through points a, b close to a_0, b_0, called *parallel pairs*, such that the tangent planes T_a, T_b at a and b are parallel. Let l be such a chord; the points of l are $q = \lambda a + \mu b$ where λ and μ are real numbers (barycentric coordinates of q) such that $\lambda + \mu = 1$. If we fix λ (and hence μ) and let a and b vary among parallel pairs close to a_0, b_0, then the point q traces out an *affine equidistant* of the pair M, N. The union of all affine equidistants, considered as a subset of $\mathbf{R} \times \mathbf{R}^3$, forms a set $W(M, N)$ called the *family of affine equidistants*[2]. Thus the points $(\lambda, q) \in W(M, N)$ for a fixed chord l lie on a straight line in $W(M, N)$—the *lift* of l—and $W(M, N)$ is the union of the lifts of all chords.

[2]We also used the term *affine extended wavefront* in [4].

Let π be the projection from $\mathbf{R} \times \mathbf{R}^3$ to the second factor, $\pi(\lambda, q) = q$. The model we use for the CSS is then as follows: it provides us with a (singular) surface which has the chords for its tangent lines. (See [4, §2] for details.)

Definition 1.0.1 *The centre symmetry set (CSS) of the pair M, N is the set of critical values of the projection $\pi : W(M, N) \to \mathbf{R}^3$.*

The CSS is therefore defined on a neighbourhood of the base chord l_0 joining a_0 and b_0, and the singularities of the CSS depend on whether the base points are distinct and on whether the chord is transversal to the tangent plane to M at a_0. In the nondegenerate case any chord l will be tangent to the CSS (or a limit of tangents to the CSS at smooth points) in two points, one point or no points, but the chord can also be entirely contained in the CSS.

The principal method of [4] was to represent the CSS as the bifurcation set of an explicit generating family of functions $F(\mathbf{u}, t, \mathbf{q})$ of variables $\mathbf{u} \in \mathbf{R}^k$ depending on the 'time' parameter $t \in \mathbf{R}$ and 'space' parameters $\mathbf{q} \in \mathbf{R}^3$. Such a family determines an *extended wavefront* $W_F \subset \mathbf{R} \times \mathbf{R}^3$:

$$W_F = \left\{ (t, \mathbf{q}) \mid \exists \mathbf{u}, \ F(\mathbf{u}, t, \mathbf{q}) = 0, \ \frac{\partial F}{\partial \mathbf{u}}(\mathbf{u}, t, \mathbf{q}) = 0 \right\}.$$

The bifurcation set B_F of W_F is the set of critical values of the projection $W_F \to \mathbf{R}^3$. It consists of two components: $B_F = \Delta_F \cup \Sigma_F$ where

$$\Delta_F = \left\{ \mathbf{q} \mid \exists (\mathbf{u}, t), \ F(\mathbf{u}, t, \mathbf{q}) = 0, \ \frac{\partial F}{\partial \mathbf{u}} = 0, \ \frac{\partial F}{\partial t} = 0 \right\} \tag{1}$$

is the *criminant* of F and

$$\Sigma_F = \left\{ \mathbf{q} \mid \exists (\mathbf{u}, t), \ F(\mathbf{u}, t, \mathbf{q}) = 0, \ \frac{\partial F}{\partial \mathbf{u}} = 0, \ \det\left(\frac{\partial^2 F}{\partial \mathbf{u}^2}\right) = 0 \right\} \tag{2}$$

is the *caustic* of F.

In [4] the families $F(\mathbf{u}, t, \mathbf{q})$ were considered up to the following equivalence relation. Two families $F_i(\mathbf{u}, t, \mathbf{q})$ $i = 1, 2$ are v-equivalent if there exist a non-zero smooth function $\phi(\mathbf{u}, t, \mathbf{q})$ and a diffeomorphism $\theta : \mathbf{R}^k \times \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}^k \times \mathbf{R} \times \mathbf{R}^3$, of the form $\theta : (\mathbf{u}, t, \mathbf{q}) \mapsto (\mathbf{U}(\mathbf{u}, t, \mathbf{q}), T(t, \mathbf{q}), \mathbf{Q}(\mathbf{q}))$ such that $\phi F_1 = F_2 \circ \theta$. The extended wavefronts and the bifurcation sets of v-equivalent families are diffeomorphic. They remain diffeomorphic after a stabilization, that is the addition of non-degenerate quadratic forms in extra variables $\widetilde{\mathbf{u}}$ to the families.

We recall that the low codimensions singularities of generic families with respect to this equivalence coincide with the versal deformations of singularities of projections of hypersurfaces onto a line, classified by V.Goryunov [5]. The low codimension singularities, which correspond to projections of a regular hypersurface are essentially the Thom-Arnold A_1, A_2, A_3, A_4, and D_4 classes. The respective normal form is the following family germ (at the origin)

$$A_k : \pm u^{k+1} + t + \sum_{i=1}^{k-1} u^i q_i,$$

$$D_4^{\pm} : u_1^3 + u_1 u_2^2 + u_1^2 q_3 + u_2 q_2 + u_1 q_1 + t$$

The classification of v-orbits corresponding to the projections of singular surfaces starts with Arnold's simple boundary classes [5, 1] ($u \in \mathbf{R}$) :

$$B_k : \ \pm u^2 + t^k + q_{k-2} t^{k-2} + \cdots + q_0, \ C_k : u^k + tu + q_{k-2} u^{k-2} + \cdots + q_2 u^2 + q_1 t + q_0, \ k = 2, 3, 4;$$

$$F_4 : u^3 + t^2 + q_2 ut + q_1 u + q_0.$$

In each of the cases below we shall give an appropriate F, such that the sets $W(M, N)$ and W_F coincide (see [4, Prop.2.2]). We can then take B_F as our model for the CSS except that, in the cases considered below, B_F contains redundant components which are the two surface germs M and N themselves. We shall ignore these components and therefore take B_F, minus these components, as an alternative construction for the CSS.

In §2 we cover the case where the parallel tangent planes to M and N are not in fact the same plane: the 'transversal' case where the base chord is transverse to M and N. Only the caustic Σ_F plays a role here. In particular, in §2.8 we consider the interesting case where the chord is 'special', that is the chord has only one point of tangency with the CSS. In the present situation this can occur at isolated points but also along *curves* on the CSS, in contrast to the euclidean case of coincident focal points which occur *only* at isolated umbilics. In §3 we turn to the case where the base chord lies in a common tangent plane to M and N. In that case both the caustic and the criminant are part of the CSS, and singularities of types B_3, B_4, C_3, C_4, F_4 and a variant of C_4 occur.

For a single surface M, and a parabolic point $\mathbf{p}$ of M, there are parallel pairs of tangent planes where both contact points are close to $\mathbf{p}$. This purely 'local' case presents features of special interest and we shall cover it in detail elsewhere. See also [4, §5].

2 The transversal case

Assume that the base points a_0, b_0 are distinct, and the base chord l_0 joining them is transversal to the parallel tangent planes to M and N at these points. In this case the criminant ΔF is always empty.

Theorems 3.3 and 3.7 of [4] state that in this case for generic surfaces the germ of the CSS at a point of a chord close to l_0 is diffeomorphic to the germ of the standard caustics of one of the types $A_2, A_3, A_4, D_4^{\pm}$. Thus besides regular points the CSS can have singularities of the form cuspidal edge, swallowtail, pyramid or purse.

2.1 The pencil of quadratic forms

To distinguish these cases introduce an affine coordinate system $Oxyz$ with the origin at the midpoint of the segment $[a_0, b_0]$, the z-axis along l_0 and the tangent planes at a_0 and b_0 being parallel to the planes $z =$ constant. We also assume that $a_0 = (0, 0, \frac{1}{2})$ and $b_0 = (0, 0, -\frac{1}{2})$. Let

$$M : \ z = \tfrac{1}{2} + f(x, y), \ N : \ z = -\tfrac{1}{2} + g(x, y), \tag{3}$$

The functions f and g are determined by M and N up to a linear transformation in the x, y coordinates. Write $f^{(i)}(x, y)$ for the degree i homogeneous form of the Taylor expansion of f at $x = y = 0$, and similarly for g; thus $f^{(0)} = f^{(1)} = g^{(0)} = g^{(1)} = 0$.

Lemma 3.1 of [4] states (in our current notation) that the family

$$F = -z + \lambda(\tfrac{1}{2} + f(x + \mu u, y + \mu v)) + \mu(-\tfrac{1}{2} + g(x - \lambda u, y - \lambda v)) \qquad (4)$$

of functions in variables $\mathbf{u} = (u, v) \in \mathbf{R}^2$ with affine time parameter $t = \lambda$ (and $\mu = 1 - \lambda$) and space parameters $\mathbf{q} = (x, y, z)$ is a generating family in the present transversal case.

Calculating W_F and B_F, the point $q_0 = (0, 0, \tfrac{1}{2}(\lambda_0 - \mu_0))$ of the base chord l_0 belongs to the CSS if and only if the quadratic form

$$Q_2(u, v) = Q_{2, \lambda_0}(u, v) = \mu_0 f^{(2)}(u, v) + \lambda_0 g^{(2)}(u, v) \qquad (5)$$

is degenerate.

In fact we can use the same criterion to calculate the CSS point on nearby chords: the conditions $\partial F / \partial u = \partial F / \partial v = 0$ in (2) state that the tangent planes at points with parameters $X = x + \mu u, Y = y + \mu v$ on M and $X^* = x - \lambda u, Y^* = y - \lambda v$ on N are parallel and the determinant condition then states that the pencil $Q_{2,\lambda}$ of quadratic forms is degenerate for the given $\lambda, \mu = 1 - \lambda$. This is exploited in drawing the CSS; see §2.3. Note also that $\partial F / \partial \lambda = 1$ when u, v, x, y, z are all 0, confirming that locally the criminant Δ in (1) is empty.

The pencil of quadratic forms Q_2 thus plays a key role in the CSS, determining the points of any chord which lie on the CSS. (It is analogous to the pencil of the second and first fundamental forms in euclidean geometry.) Let us represent Q_2 by a straight line in the 3-space of all quadratic forms in two variables u, v. For generic surfaces we may assume that the line does not lie in the cone of degenerate forms, since this is equivalent to saying that the base points a_0 and b_0 are not *both* parabolic points with the same asymptotic direction. The cone can meet the line in up to two real points:

(i) If the points are real and distinct, then Q_2 intersects the region of (positive or negative) definite forms, and using the standard normal form theorem both $f^{(2)}$ and $g^{(2)}$ can be reduced to diagonal form by a linear transformation of u, v coordinates. See §2.2.

(ii) If Q_2 passes through the zero form, then $f^{(2)}$ and $g^{(2)}$ are proportional. In this case they are also simultaneously diagonalizable. See §2.5.

(iii) Suppose Q_2 is tangent to the cone at a nonsingular point. This is equivalent to the statement that the quadratic forms $f^{(2)}$ and $g^{(2)}$ have exactly one common linear factor, which we may take to be u by a linear transformation. (Note that both surfaces must therefore be hyperbolic or parabolic at the base-points.) At least one of the quadratic forms is not a multiple of u^2 so that using a further linear transformation we can reduce it to uv and the other to $u(\pm u + \alpha v)$ for a constant α. In §2.8 we take these as $g^{(2)}$ and $f^{(2)}$ respectively.

(iv) If Q_2 does not meet the cone in real points then there are no real CSS points so we ignore this case.

.

4

2.2 The diagonalizable case

We consider cases (i) and (ii) of §2.1. Until §2.7 we shall assume that neither a_0 nor b_0 is parabolic, that is neither $f^{(2)}$ nor $g^{(2)}$ is degenerate. Then we can choose coordinates (principal coordinates in fact) so that $f^{(2)} = \alpha u^2 + \beta v^2$ and $g^{(2)} = \varepsilon_1 u^2 + \varepsilon_2 v^2$ where ε_1 and ε_2 are independently ± 1. It follows that the two CSS points on l_0 are respectively given by the solutions of the equations

$$\alpha \mu_0 + \varepsilon_1 \lambda_0 = 0, \quad \beta \mu_* + \varepsilon_2 \lambda_* = 0. \tag{6}$$

Remark 2.2.1 Recalling that $\lambda_0 + \mu_0 = 1$ and $\lambda_* + \mu_* = 1$ there will be no finite solution to (6) in the respective cases $\alpha = \varepsilon_1$, $\beta = \varepsilon_2$. This means that the CSS point has gone to infinity, in the same way that the euclidean focal set goes to infinity near the normal to a surface at a parabolic point. Generically $f^{(2)}$ and $g^{(2)}$ will not be identical, so both CSS points will not go to infinity at the same time.

Assume now that $\alpha \mu_0 + \varepsilon_1 \lambda_0 = 0$, $\alpha \neq \varepsilon_1$, that $f^{(2)}$ and $g^{(2)}$ are not proportional, and that $\alpha \neq 0$. The latter condition guarantees $\lambda_0 \neq 0$; we also have $\mu_0 \neq 0$ by the choice of g above.

The proof of the following statement consists of direct calculations of normal forms of versal families of functions. In the principal coordinates let the higher order forms of f and g be

$$f^{(n)}(u, v) = \sum_{i+j=n} f_{i,j} u^i v^j, \quad g^{(n)}(u, v) = \sum_{i+j=n} g_{i,j} u^i v^j.$$

Theorem 2.2.2 *With the above assumptions and notation, we have the following.*

(i) A_2 *case: Suppose that $A \neq 0$ where $A = \mu_0^2 f_{3,0} - \lambda_0^2 g_{3,0}$. Then the point $q_0 = \lambda_0 a_0 + \mu_0 b_0$ is a regular point of the CSS, the tangent plane to the CSS at q_0 being the yz coordinate plane. The corresponding germ of the generating family is the germ of a versal deformation of the function $u^3 + v^2$. The versality condition is $\mu_0 \lambda_0 (\alpha - \varepsilon_1) \neq 0$, which holds automatically.*

(ii) A_3 *case: Suppose that $A = 0$; generically we can expect this condition to hold along a line in the family if chords, that is for points along curves in the two surface germs M, N. Suppose also that the two CSS points do not coincide and that $B \neq 0$ where*

$$B = f_{4,0} \mu_0^3 + g_{4,0} \lambda_0^3 - \frac{(\mu_0^2 f_{2,1} - \lambda_0^2 g_{2,1})^2}{4(\mu_0 \beta + \varepsilon_2 \lambda_0)}.$$

Then the CSS at q_0 has a cuspidal edge. (The denominator is nonzero provided the two CSS points are not coincident.) The versality condition of the generating family in this case holds provided $\mu_0 \lambda_0 (\alpha - \varepsilon_1) \neq 0$, which is again automatic.

(iii) A_4 *case: Finally at isolated points on M, N it can happen that $A = B = 0$ but $C \neq 0$ where*

$$\begin{aligned} C &= f_{5,0} \mu_0^4 - g_{5,0} \lambda_0^4 - \frac{(\mu_0^2 f_{2,1} - \lambda_0^2 g_{2,1})(\mu_0^3 f_{3,1} + \lambda_0^3 g_{3,1})}{2(\mu_0 \beta + \varepsilon_2 \lambda_0)} \\ &+ \frac{(\mu_0^2 f_{2,1} - \lambda_0^2 g_{2,1})^2 (\mu_0^2 f_{1,2} - \lambda_0^2 g_{1,2})}{4(\mu_0 \beta + \varepsilon_2 \lambda_0)^2}. \end{aligned}$$

In this case q_0 is a swallowtail point. The versality condition is the automatically satisfied $\mu_0\lambda_0(\alpha - \varepsilon_1) \neq 0$ together with $D \neq 0$ where

$$
\begin{aligned}
D &= (f_{3,0}\mu_0^2(\mu_0 - 3\lambda_0) + g_{3,0}\lambda_0^2(\lambda_0 - 3\mu_0))\left(\frac{\beta - \varepsilon_2}{2(\mu_0\beta + \lambda_0\varepsilon_2)} + \mu_0 f_{2,1} + \lambda_0 g_{2,1}\right) \\
&+ \alpha\mu_0\left[f_{3,1}\mu_0^2 - g_{3,1}\lambda_0^2 - \frac{(\mu_0^2 f_{2,1} - \lambda_0^2 g_{2,1})(\mu_0 f_{1,2} + \lambda_0 g_{1,2})}{2(\mu_0\beta + \lambda_0\varepsilon_2)}\right] \\
&+ \frac{\alpha\mu_0(\beta - \varepsilon_2)}{4(\mu_0\beta + \lambda_0\varepsilon_2)^2}\left[(\mu_0^2 f_{2,1} - \lambda_0^2 g_{2,1})(\mu_0^2 f_{1,2} - \lambda_0^2 g_{1,2})\right. \\
&\qquad\qquad\qquad \left. - 2(\mu_0\beta + \lambda_0\varepsilon_2)(\mu_0^3 f_{3,1} + \lambda_0^3 g_{3,1}).\right]
\end{aligned}
$$

Remark 2.2.3 As stated in the theorem, the versality conditions hold automatically for the singularities A_2 and A_3 whose strata in the CSS have positive dimensions. Only in the A_4 case is an extra condition $D \neq 0$ required. This leads to an interesting feature: the metamorphosis of singularities in families of CSS depending on a parameter. Such a metamorphosis can happen only for strata of codimension 3 singularities. Notice also that the open condition depends only on the 3-jet of the family at the reference point.

Similarly to the classical Euclidean case, the versality condition for codimension 3 singularites unfortunately does not have an explicit geometrical interpretation.

2.3   The pre-CSS and drawing the CSS

Consider two surfaces M and N in $\mathbb{R}^3$ whose tangent planes at $a_0 \in M$ and $b_0 \in N$ are parallel. Suppose M is locally parametrized by u_1, v_1 and N by u_2, v_2, with a_0, b_0 given by zero values of the parameters. We need to find the pairs of points a of M and b of N close to these two for which the tangent planes are parallel. The set of these pairs, as a subset of a neighbourhood of $(a_0, b_0) \in M \times N$ or of a neighbourhood of $\mathbf{0}$ in (u_1, v_1, u_2, v_2)-space, is called the *pre-CSS*. By the implicit function theorem the pre-CSS is smooth, parametrized by u_1 and v_1, provided N is not parabolic at b_0. Thus the pre-SS is smooth provided not both of a_0, b_0 are parabolic. See §2.7.

In the transversal case we can take M, N as in (3):

$$
M = \{(x, y, z) = (u_1, v_1, f(u_1, v_1) + \tfrac{1}{2})\}, \quad N = \{(x, y, z) = (u_2, v_2, g(u_2, v_2) - \tfrac{1}{2})\},
$$

where f, g and their first partial derivatives vanish at $(0,0)$, so that the tangent planes to M at $a_0 = (0, 0, \tfrac{1}{2})$ and N at $b_0 = (0, 0, -\tfrac{1}{2})$ are parallel to the plane $z = 0$. In the tangential case, where the tangent planes to M at a_0 and N at b_0 coincide, but $a_0 \neq b_0$, we can take M, N as graphs tangent to the x, y-plane at two points of the x-axis. See §3.

Of course actually finding say u_2 and v_2 as functions of u_1 and v_1 may be a tall order, but we can find examples of all the phenomena we discuss by choosing $g(u, v) = \varepsilon_1 u^2 + \varepsilon_2 v^2$, with no higher terms (the ε_i being independently ± 1). This makes the pre-CSS condition $u_2 = -\tfrac{1}{2}\varepsilon_1(\partial f/\partial u_1)$, $v_2 = -\tfrac{1}{2}\varepsilon_2(\partial f/\partial v_1)$.

We then calculate the zeros of $\det\mathcal{H}$ where

$$
\mathcal{H} = \begin{pmatrix} f_{uu} + \theta g_{uu} & f_{uv} + \theta g_{uv} \\ f_{vu} + \theta g_{vu} & f_{vv} + \theta g_{vv} \end{pmatrix},
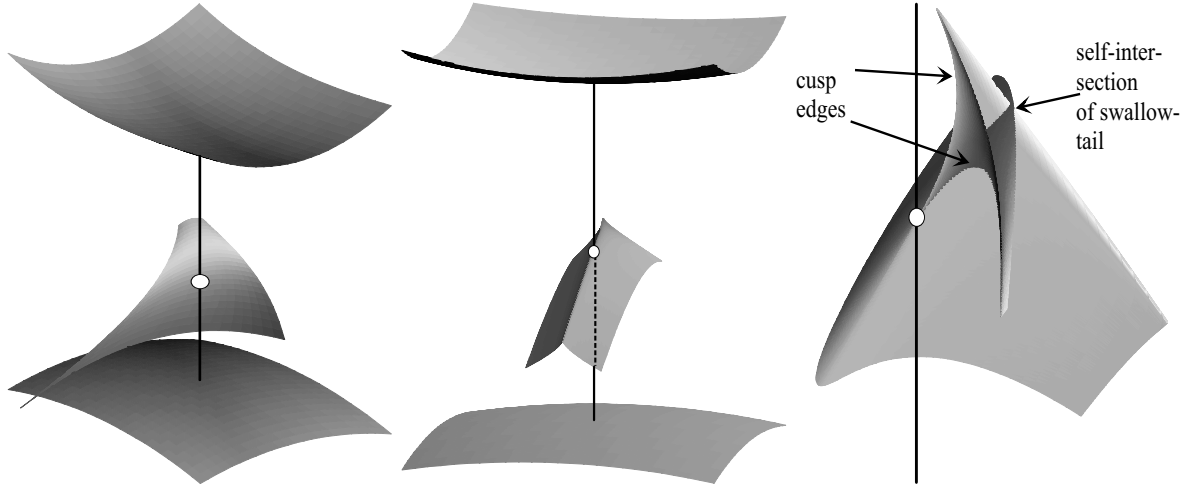$$

Figure 1: One of the CSS sheets in the three transverse cases: A_2 (smooth), A_3 (cusp edge), A_4 (swallowtail), the last being in close-up with the surface pieces not shown. The base chord, joining $a_0 = (0, 0, \frac{1}{2})$ and $b_0 = (0, 0, -\frac{1}{2})$, is also shown.

regarded as a function of u_1, v_1. For each such (real) zero θ we use the values $\lambda = \theta/(1 + \theta)$, $\mu = 1/(1 + \theta)$ (thus $\theta = \lambda/\mu$ and $\lambda + \mu = 1$) to find the point $\lambda a + \mu b$ on the CSS. These sheets of the CSS will then be parametrized by u_1 and v_1. (If a_0 is parabolic then $\theta = 0$ and the corresponding CSS point is b_0; see §2.7.)

2.4 Examples

We shall take $g(u, v) = \varepsilon_1 u^2 + \varepsilon_2 v^2$ without higher terms, and we shall arrange things so that the CSS point of interest is at the origin, that is to say $\lambda_0 = \mu_0 = \frac{1}{2}$. In fact let us choose $\varepsilon_1 = -1$; then from (6) we require $\alpha = 1$. Further let us choose $\varepsilon_2 = -1$, and say $\beta = 2$. The expressions occurring in Theorem 2.2.2 are then given by
$4A = f_{3,0}$, and, assuming $f_{3,0} = 0$,
$16B = 4f_{4,0} - f_{2,1}^2$
$64C = 4f_{5,0} - 2f_{2,1}f_{3,1} + f_{2,1}^2 f_{1,2}$
$32D = -2f_{3,1} + f_{2,1}f_{1,2}$
    As examples, we can take
(i) Smooth CSS: $f = u^2 + 2v^2 + u^3$,
(ii) Cusp edge CSS: $f = u^2 + 2v^2 + 2u^2 v$,
(iii) Swallowtail CSS: $f = u^2 + 2v^2 + 2u^2 v + uv^2 + u^4$. (An example with the upper surface hyperbolic is $f = u^2 - 2v^2 + 6u^2 v + uv^2 - 3u^4$.)
See Figure 1 for these examples[3].
    We can also choose $\varepsilon_2 = 1$, making the lower surface hyperbolic, but the expressions for A, B, C, D are then slightly different, namely
$4A = f_{3,0}$, and, assuming $f_{3,0} = 0$,
$16B = 12f_{4,0} - f_{2,1}^2$,

---

[3]The figures were drawn using MAPLE.

$$64C = 36f_{5,0} - 6f_{2,1}f_{3,1} + f_{2,1}^2 f_{1,2},$$
$$288D = 30f_{3,1} - 5f_{2,1}f_{1,2}.$$

Remark 2.4.1 We are concerned here with *one* of the sheets of the CSS only. In example (i) above, the other sheet is highly degenerate, in fact 1-dimensional. We can make both sheets nonsingular by adding v^3 to f.

2.5 Quadratic forms $f^{(2)}$ and $g^{(2)}$ proportional

This is case (ii) of §2.1. Generically the forms $f^{(2)}$ and $g^{(2)}$ are proportional only for isolated basic chords. We may therefore assume that the forms are non-degenerate and equal to

$$f^{(2)}(u,v) = \alpha g^{(2)}(u,v), \quad g^{(2)}(u,v) = \varepsilon_1 u^2 + \varepsilon_2 v^2, \quad \alpha \neq 0, \quad \alpha \neq 1.$$

The CSS points coincide since $(\lambda_0, \mu_0) = (\lambda_*, \mu_*) = (\alpha/(\alpha-1), 1/(1-\alpha))$.

The genericity conditions consist of non-degeneracy of the cubic form

$$r = \lambda_0 \mu_0 Q_3 \text{ where } Q_3(u,v) = \mu_0^2 f^{(3)}(u,v) - \lambda_0^2 g^{(3)}(u,v)$$

and the versality conditions consist of (i) $\mu_0^2 a(3a-1) \neq 0$ and (ii) the quadratic forms $\varepsilon_1 u^2 + \varepsilon_2 v^2$, $\partial r/\partial u$, $\partial r/\partial v$ span the space of all quadratic forms. This is a $D_4^{\pm}$ singularity of the CSS.

As examples let us consider the case where $g^{(3)} = 0$ and write $f^{(3)} = b_0 u^3 + b_1 u^2 v + b_2 u v^2 + b_3 v^3$. Then the condition (ii) above becomes

$$\varepsilon_1(3b_1 b_3 - b_2^2) + \varepsilon_2(3b_0 b_2 - b_1^2) \neq 0. \tag{7}$$

If the left hand side of this is < 0 then the resulting D_4 is of type D_4^+ with one cuspidal edge ('purse', hyperbolic umbilic) and if the left hand side is > 0 it is of type D_4^- with three cuspidal edges ('pyramid', elliptic umbilic).

Note that we can make the surfaces M and N both elliptic ($\varepsilon_1 \varepsilon_2 = 1$) or both hyperbolic ($\varepsilon_1 \varepsilon_2 = -1$). The geometry of these two situations is quite different. In the elliptic case, all points close to the base-points a_0, b_0 contribute chords to the CSS, but in the hyperbolic case there are two smooth curves on M through a_0 and on N through b_0 which separate neighbourhoods of these base points into four regions, only two of which contribute to the CSS. This is illustrated for the 'pyramid' case in Figure 2.

2.6 Equations for the strata

It is not difficult to use the generating family F in (4) to provide explicit equations for the strata of the CSS. Let us write, as above, $X = x + \mu u, Y = y + \mu v, X^* = x - \lambda u, Y^* = y - \lambda v$, and write for instance f_X to mean the derivative of f with respect to its first variable, evaluated at (X, Y). Then the equations of the CSS, that is the caustic, are $F = 0$, $f_X = g_{X^*}$, $f_Y = g_{Y^*}$ and $H = 0$ where

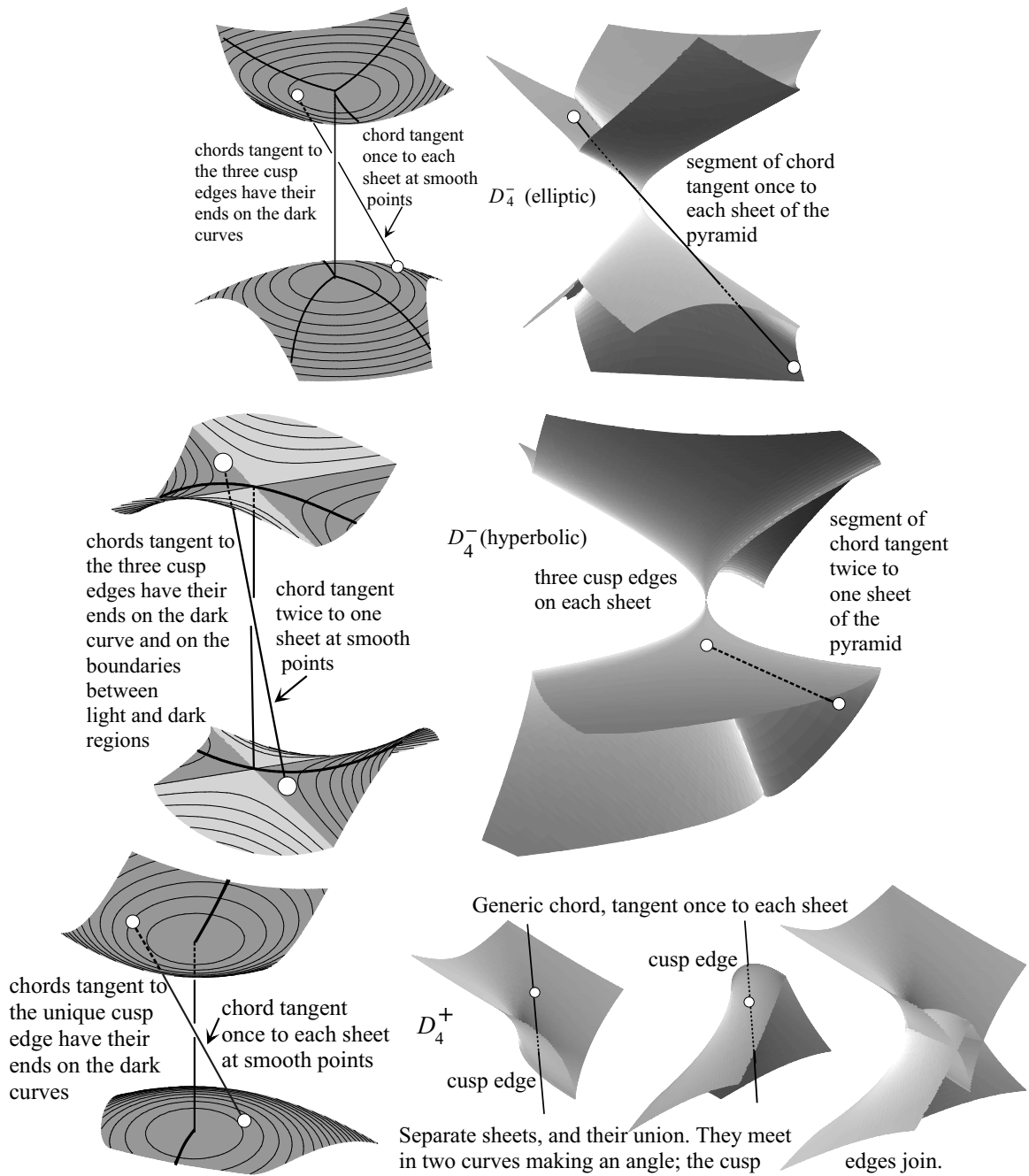$$H = (\mu f_{XX} + \lambda g_{X^*X^*})(\mu f_{YY} + \lambda g_{Y^*Y^*}) - (\mu f_{XY} + \lambda g_{X^*Y^*})^2.$$

chords tangent to
the three cusp
edges have their
ends on the dark
curves

chord tangent
once to each
sheet at smooth
points

D_4^- (elliptic)

segment of chord
tangent once to
each sheet of the
pyramid

chords tangent to
the three cusp
edges have their
ends on the dark
curve and on the
boundaries
between
light and dark
regions

chord tangent
twice to one
sheet at smooth
points

D_4^- (hyperbolic)

three cusp edges
on each sheet

segment of
chord tangent
twice to
one sheet
of the
pyramid

Generic chord, tangent once to each sheet

cusp edge

chords tangent to
the unique cusp
edge have their
ends on the dark
curves

chord tangent
once to each sheet
at smooth points

D_4^+

cusp edge

Separate sheets, and their union. They meet
in two curves making an angle; the cusp

edges join.

Figure 2: Top row: a D_4^- (pyramid) CSS generated by two elliptic surface patches $f = u^2 + v^2 + u^2v - v^3$ and $g = -u^2 - v^2$; in the middle row by two hyperbolic surface patches $f = u^2 - v^2 - u^2v + v^3$ and $g = -u^2 + v^2$. The base chord and one generic chord are shown, and the segment of the generic chord between tangency points to the CSS is shown on the right. In the D_4^- hyperbolic case only the darker regions on each surface patch contribute to the CSS; the three cusp edges correspond to the boundaries of these regions and to another curve on each patch, drawn as a dark line. Bottom row: the D_4^+ 'purse' case generated by $f = u^2 + v^2 + u^2v + v^3$, $g = -u^2 - v^2$. On the surface patches there is just one curve, shown as a dark line, corresponding to the unique cusp edge on the CSS. The figure shows the two sheets separately as well as the CSS which is the union of the sheets.

9

These four equations in x, y, z, λ, u, v define a map $\widetilde{F} : \mathbf{R}^6 \to \mathbf{R}^4$, with the CSS obtained by projecting $\widetilde{F}^{-1}(0)$ to the $\mathbf{q} = (x, y, z)$ coordinates. Using the implicit function theorem it follows that the singular set of the CSS (the closure of the cuspidal edges) is given by imposing a further condition to make the matrix

$$\mathcal{A} = \begin{pmatrix} \mu f_{XX} + \lambda g_{X^*X^*} & \mu f_{XY} + \lambda g_{X^*Y^*} & \mu H_X - \lambda H_{X^*} \\ \mu f_{XY} + \lambda g_{X^*Y^*} & \mu f_{YY} + \lambda g_{Y^*Y^*} & \mu H_Y - \lambda H_{Y^*} \end{pmatrix}$$

of rank 1. The leftmost 2×2 minor is already zero since $H = 0$, and the other two minors will therefore differ only by a (generically nonzero) multiplicative constant. Let us choose one of these minors, H_1 say. Then the 0-dimensional strata (A_4 and D_4 singularities) are given by adding an extra column to the matrix $\mathcal{A}$ and still requiring rank 1. This extra column is obtained from H_1, differentiating with respect to u and v:

$$(\mu H_{1X} - \lambda H_{1X^*} \quad \mu H_{1Y} - \lambda H_{1Y^*})^\top.$$

Note that at D_4 points F has no quadratic terms so that the first two columns of $\mathcal{A}$ are zero. This distinguishes the D_4 points from the A_4 points.

2.7 Parabolic base point

If *both* base points a_0, b_0 are parabolic then (compare §2.3) we cannot use a neighbourhood of either of them to parametrize the parallel tangencies of M, N. Generically both base points are parabolic only for isolated chords. The corresponding CSS points are at a_0 and b_0 themselves and we need to ignore the redundant components of Σ_F which are equal to the two surface germs M and N through these points. By an affine transformation we can assume $f^{(2)} = \varepsilon_1 u^2$ and $g^{(2)} = \varepsilon_2 v^2$, where the ε_i are independently ± 1. Then, in the notation of §2.3, v_1 and u_2 can be used as local parameters for the pairs of points where the tangent planes are parallel and for the CSS. If $f_{3,0}$ and $g_{0,3}$ are nonzero (these hold generically) then the two sheets of the CSS are smooth near a_0 and b_0 and intersect M and N transversally there.

If a_0 is parabolic but b_0 is not then we can use a neighbourhood of a_0 to parametrize the parallel tangencies, as in §2.3. Assuming that the pencil Q_2 (5) of quadratic forms is not tangent to the cone of degenerate forms, $f^{(2)}$ and $g^{(2)}$ will not have a common factor and we can reduce to $f^{(2)} = \varepsilon_1 v^2$, $g^{(2)} = \varepsilon_2 u^2 + \varepsilon_3 v^2$. Then using [4, Prop.3.9] we find:
- If $f_{3,0} \neq 0$ the CSS at b_0 ($\lambda_0 = 0, \mu_0 = 1$) is smooth, transversally intersecting N,
- If $f_{3,0} = 0$ but $f_{2,1} \neq 0$ and $4f_{4,0} - f_{2,1}^2 \neq 0$ then the CSS has a cusp edge at b_0, which is transverse to N.

2.8 Special chords

Let us consider the case (iii) of §2.1 where the quadratic forms $f^{(2)}$ and $g^{(2)}$ can be reduced to $u(\pm u + \alpha v)$ and uv for a constant α. The pencil of quadratic forms is tangent to the cone of singular forms and this implies that, for a curve of points on M and N, there is exactly *one* CSS point: the equation giving the CSS points on the chord joining a and b along these curves has equal roots. Generically we obtain a smooth curve on M and
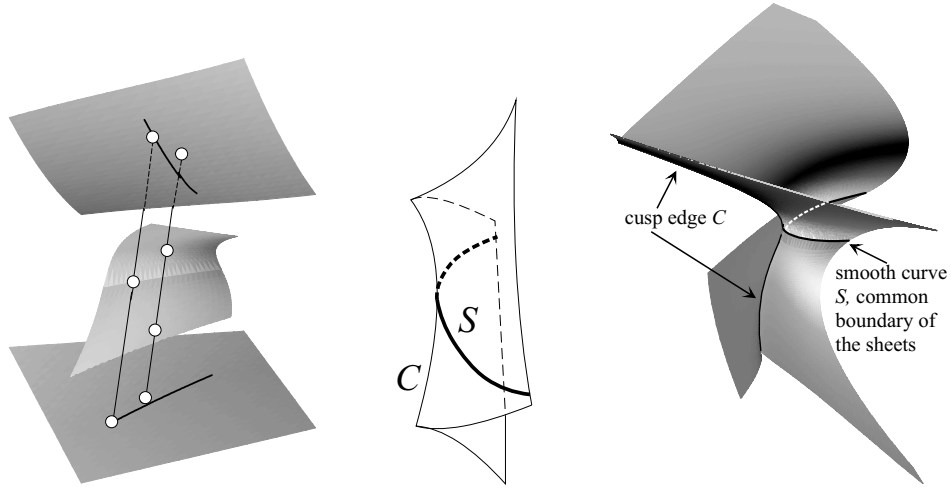
Figure 3: Special chords; see §2.8. Left: The upper and lower surfaces are M and N and their CSS is between them. (In this situation, both M and N must be hyperbolic.) The CSS consists of two smooth sheets, meeting along a smooth curve to form a single smooth surface which is the total CSS. The dark lines on M and N correspond to the common boundary of the two sheets. Thus points of M on one side only of this dark line contribute at all to the CSS, and similarly for N. The figure shows one *special chord* which has 4-point contact at the common boundary of the two sheets of the CSS; this has its endpoints *on* the dark lines on M and N. A generic second chord is shown whose endpoints are away from the dark lines; this is tangent once (with ordinary 2-point contact) to each sheet of the CSS. Centre and right: The cuspidal edge case of special chords. Centre: a schematic picture of the CSS which is a cuspidal edge surface, made up of two sheets which join along the special curve S. In a generic situation, S meets the cuspidal edge curve C in just one point and S and C are smooth as space curves. Right: an example, $f(u,v) = u(u-v) + u^3 - u^2v + uv^2 + v^4$ and $g = uv$.

another smooth curve on N ([4, Theorem 3.11]); points of M on one side only of the curve in M contribute to the CSS, and likewise for N. For the base chord l_0 there is just one CSS point, namely the one with $(\lambda_0, \mu_0) = (\alpha/(\alpha-1), 1/(1-\alpha))$.

Suppose that $\alpha \neq 0$. Provided $\lambda_0\mu_0(\mu_0^2 f_{0,3} - \lambda_0^2 g_{0,3}) \neq 0$ this is a smooth point (A_2) of the CSS. Otherwise it belongs to a cuspidal edge (A_3) on the CSS provided

$$4\mu_0 \left(\mu_0^3 f_{0,4} + \lambda_0^3 g_{0,4}\right) \neq \left(\mu_0^2 f_{1,2} - \lambda_0^2 g_{1,2}\right)^2. \tag{8}$$

If $\alpha = 0$ then M is parabolic at $a_0 = (0,0,\frac{1}{2})$. Generically $f_{0,3} \neq 0$ and the CSS is smooth and intersects N transversely at $b_0 = (0,0,-\frac{1}{2})$. In this case the ends of the special chords trace a curve in the closure of the hyperbolic region of M, this curve being smooth and *tangent* to the (smooth) parabolic curve in M.

Figure 3 shows the surface pieces $z = f(u,v) + \frac{1}{2} = u(u-v) + v^3 + \frac{1}{2}$ (top) and $z = g(u,v) - \frac{1}{2} = uv - \frac{1}{2}$ (bottom). We can solve for parallel tangent planes in the same way as §2.3; in fact parallel tangent planes are at points given by (u_1, v_1) on the first surface and $(u_2, v_2) = (-u_1 + 3v_1^2, 2u_1 - v_1)$ on the second. The two values of θ giving CSS points (again as in §2.3) are $1 \pm \sqrt{12v_1}$ so that $v_1 > 0$ for distinct solutions and the special chords are given by $v_1 = 0$. Of course we can smoothly parametrize the whole CSS

(locally) using $(u_1, \widetilde{v})$ where $v_1 = \widetilde{v}^2$, so that the values of θ are both given by $1 + \widetilde{v}\sqrt{12}$, allowing $\widetilde{v}$ to take both signs.

We can also realise the special cuspidal edge case of the CSS. However note that the realization of an A_3, versally unfolded by the family F, does not guarantee that the cusp edge is in general position with respect to the special curve on the CSS. This actually imposes new conditions, not arising directly[4] from the family F. To avoid confusion let us call the CSS itself a 'cuspidal edge surface' and the singular set on this (the A_3 set) the 'cuspidal edge curve'. Then the example $f(u,v) = uv$, $g(u,v) = u(u - v) + uv^2$ satisfies the above genericity condition (8) but the cuspidal edge curve *coincides* with the special curve. To make the special curve on the CSS meet the cuspidal edge curve generically we need an additional condition, which in fact reduces to $f_{04}(8f_{04} - 3\alpha f_{12}^2) \neq 0$ when $f = u(u + \alpha v) +$ h.o.t., $g = uv +$ h.o.t. In particular we need a v^4 term in f. See Figure 3, centre and right.

3   The tangential case

We suppose now that the basic chord lies in a bitangent plane, that is a plane tangent to both M and N. We may suppose this plane is $z = 0$ and that $a_0 = (x_0, 0, 0), b_0 = (x_*, 0, 0)$, so that the surfaces locally have the form $M : z = f(x - x_0, y)$, $N : z = g(x - x_*, y)$. We can also suppose that $q_0 = \lambda_0 a_0 + \mu_0 b_0$ is the origin, that is $\lambda_0 = x_*/(x_* - x_0)$.

From [4, Prop.4.1] the following family of functions in variables u, v with real parameters ϵ, x, y, z is a generating family for the germ of the CSS at the origin.

$$F = -z + \lambda f(x + \epsilon + \mu u, y + \mu v) + \mu g(x + \epsilon - \lambda u, y - \lambda v),$$

where $\mu = \mu_0 - \epsilon$, $\lambda = \lambda_0 + \epsilon$ and all parameters and variables are close to 0.

In a similar way to the transverse case (see (5)), the point q_0 belongs to the caustic Σ_F if and only if the quadratic form $Q_2(u,v) = \mu_0 f^{(2)}(u,v) + \lambda_0 g^{(2)}(u,v)$ is degenerate. However the criminant Δ_F now contains the whole line through a_0 and b_0: in fact Δ_F coincides with the ruled surface of all bitangent chords, so that according to our definition the CSS contains this ruled surface. The generic singularities of the CSS are related to boundary singularities of functions; see [4, Th.4.3,4.4]. The part Δ_F of the CSS can have cuspidal edge and isolated swallowtail singularities.

For generic surfaces M and N and any bitangent chord at least one of the forms $f^{(2)}, g^{(2)}$ is nondegenerate and does not vanish on the bitangent chord direction. Let this be $f^{(2)}$; then we can make an affine change of coordinates so that

$$f^{(2)} = u^2 \pm v^2, \; g^{(2)} = \alpha u^2 + 2\beta uv + \gamma v^2. \tag{9}$$

If the quadratic form Q_2 is nondegenerate then the caustic Σ_F near the origin is empty. The condition for Q_2 to be degenerate is

$$\lambda_0^2(\beta^2 - \alpha\gamma) - \lambda_0\mu_0(\gamma \pm \alpha) \mp \mu_0^2 = 0. \tag{10}$$

---

[4]However these conditions can be derived by direct calculation or by the methods of [4, p.147].

There is another key quadratic form which governs these singularities:

$$Q_*(\epsilon, u, v) = \lambda_0 \left((\epsilon + \mu_0 u)^2 \pm \mu_0^2 v^2 \right) + \mu_0 \left(\alpha(\epsilon - \lambda_0 u)^2 - 2\beta(\epsilon - \lambda_0 u)\lambda_0 v + \gamma \lambda_0^2 v^2 \right),$$

which is the second order Taylor polynomial at the origin of the function $F|_{x=y=z=0}(\epsilon, u, v)$. The condition for Q_* to be degenerate is

$$\pm \alpha \mu_0 + (\alpha\gamma - \beta^2)\lambda_0 = 0. \tag{11}$$

Until Case 3 below we take $\lambda_0 \neq 0, 1$. We can then choose $x_0 = -\frac{1}{2}$, $x_* = \frac{1}{2}$ so that $\lambda_0 = \mu_0 = \frac{1}{2}$, and (10) and (11) then become

$$
\begin{align}
Q_2 : \quad \beta^2 &= (\alpha + 1)(\gamma \pm 1) \text{ (solutions for } \lambda_0 \text{ are then } \tfrac{1}{2} \text{ and } 1/(\gamma - \alpha)) \tag{12}\\
Q_* : \quad \beta^2 &= \alpha\gamma \pm \alpha. \tag{13}
\end{align}
$$

Case 1: Q_2 degenerate, Q_* nondegenerate. In this case the origin belongs to Σ_F and automatically to Δ_F which is the ruled surface of bitangent lines.

Let

$$\mathbf{w} = (-\lambda_0\beta, \mu_0 + \lambda_0\alpha) = \frac{1}{2}(-\beta, 1 + \alpha) \tag{14}$$

be the kernel direction of the form Q_2.

As above we take $\lambda_0 = \frac{1}{2}$. Similarly let $\overline{\mathbf{w}} = (\mu_0 + \lambda_0\alpha, \lambda_0\beta)) = \frac{1}{2}(1 + \alpha, \beta)$. Let

$$
\begin{align}
Q_3(u, v) &= \mu_0^2 f^{(3)}(u, v) - \lambda_0^2 g^{(3)}(u, v) = \tfrac{1}{4}(f^{(3)}(u, v) - g^{(3)}(u, v)),\\
Q_4(u, v) &= \mu_0^3 f^{(4)}(u, v) + \lambda_0^3 g^{(4)}(u, v) = \tfrac{1}{8}(f^{(4)}(u, v) + g^{(4)}(u, v)).
\end{align}
$$

We find

Case 1(a): If $Q_3(\mathbf{w})$ is nonzero then Σ_F and Δ_F are both nonsingular at the origin and have second-order tangency (inflexional contact) along a curve which is in general transverse to the base chord joining a_0 and b_0. This is type C_3 and is illustrated in Figure 4.

Case 1(b): If $Q_3(\mathbf{w}) = 0$ then generically the caustic Σ_F has a folded Whitney unbrella (also known as a folded crosscap[5]) while the criminant Δ_F is smooth. The singularity of the CSS is then of type C_4. The generic additional conditions which need to be satisfied for exactly a C_4 are as follows. We require that $Q_4(\mathbf{w}) \neq \tfrac{1}{4}(\mu_0 + \lambda_0\alpha)S^2$ where

$$S = \left(\frac{\partial Q_3}{\partial u}(\mathbf{w}), \frac{\partial Q_3}{\partial v}(\mathbf{w}) \right) \cdot \overline{\mathbf{w}}.$$

There is an additional condition for the C_4 singularity to be versally unfolded. This is most simply stated in terms of the cubic form $\overline{Q}_3 = \mu_0 f^{(3)} + \lambda_0 g^{(3)}$: we require $\beta(\partial \overline{Q}_3/\partial u) + (1 - \alpha)\partial \overline{Q}_3/\partial v \neq 0$ when evaluated on the vector $\mathbf{w}$.

To find examples, as in Figure 5, we take $f(u, v) = u^2 \pm v^2$, without higher terms, so that $f^{(3)} = 0$, use (12) to find γ from α, β and then use $Q_3(\mathbf{w}) = 0$ to find the coefficient
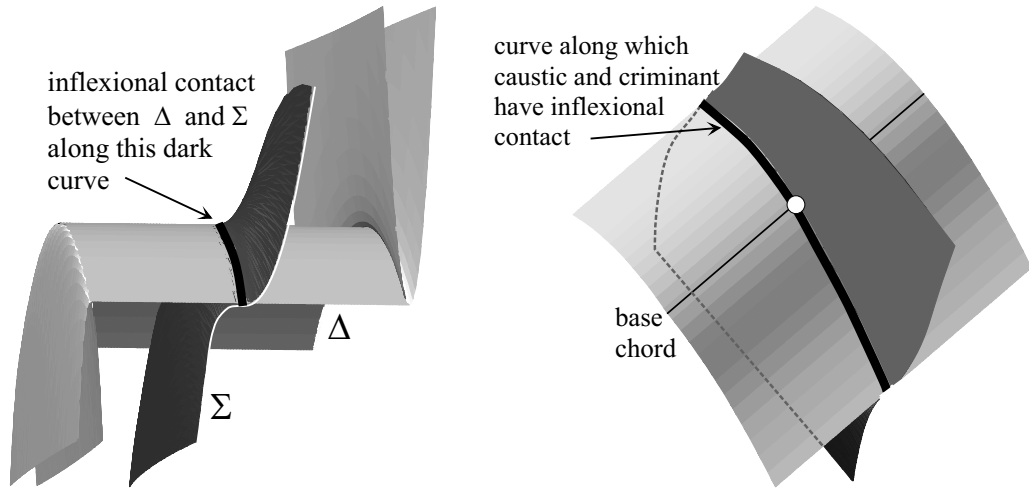
Figure 4: The C_3 case (Case 1(a) in the text), illustrated by $f(u,v) = u^2 - v^2$, $g(u,v) = -2u^2 + 4uv - 3v^2 - u^3 + v^3$. Left: the two surface pieces, left (g) and right (f), together with the ruled surface Δ of bitangent chords on which the base chord is drawn with the origin marked, and also Σ which is tangent to Δ and crosses it. The vertical scale is greatly exaggerated in this figure. Right: just the two components Δ and Σ are shown, with their tangency of order 2 along a curve of C_3 points, again with the vertical scale exaggerated.
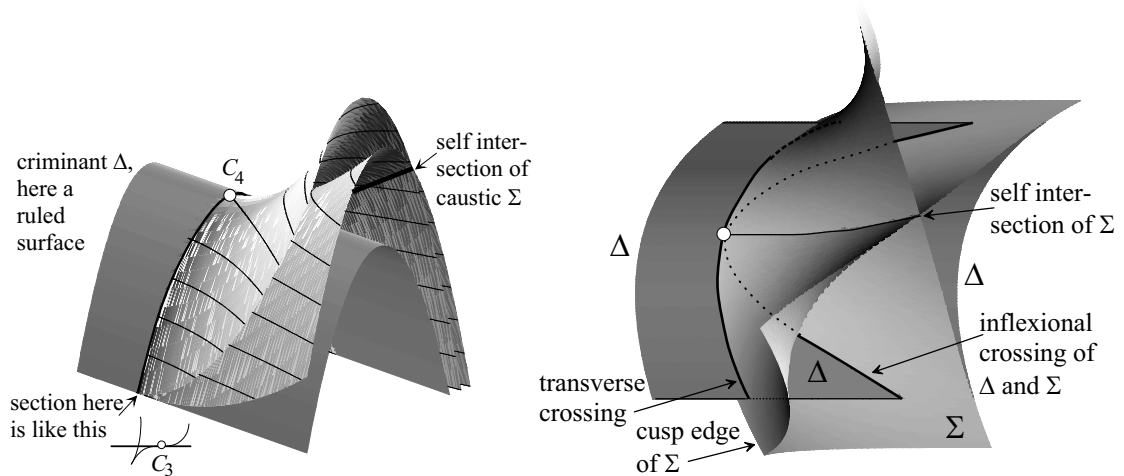

Figure 5: The CSS in the C_4 case (Case 1(b) in the text). Left: a computer rendering of the case $f(u,v) = u^2 - v^2$, $g(u,v) = -2u^2 + 4uv - 3v^2 - u^3 + 2u^2v + uv^2 - 2v^3 + 2u^4$, with the shape of the caustic indicated by contour lines. Right: a standard model of the caustic Σ and criminant Δ of a C_4, showing the two curves of intersection between them, one crossing being transverse except at the C_4 point and the other being inflexional at C_3 points. This model is based on the normal form for C_4 given in §1, distorted by a global diffeomorphism to make Δ more closely resemble that for the CSS on the left.

14

of v^3 in g from the other cubic coefficients. In the example of Figure 5 we have also added a quartic term to g to obtain a better picture.

Case 2: Q_2 nondegenerate, Q_* degenerate. The caustic Σ_F is empty close to the origin and the whole CSS is the ruled surface of bitangents. Since the rank of Q_2 is maximal the variables u, v can be eliminated by a stabilization, as in [4], and the generating family can be written in the form

$$F(\epsilon, x, y, z) = -z + c_3\epsilon^3 + c_4\epsilon^4 + c_{2,1}x\epsilon^2 + y(c_{1,2}\epsilon + c_{2,2}\epsilon^2) + \phi(\epsilon, x, y),$$

where ϕ contains terms divisible by one of $\epsilon^5, x^2, xy, y^2, x\epsilon^3, y\epsilon^3$ and

$$
\begin{aligned}
bc_{1,2} &= 2(\pm\alpha - \alpha\gamma + \beta^2) \neq 0, \\
c_{2,1} &= \mp\frac{2\alpha^2}{\beta^2\lambda_0^2\mu_0} + \frac{3\alpha^2}{\beta^3\lambda_0^2}f_{3,0} + \frac{3}{\beta^3\mu_0^2}g^{(3)}(\beta, -\alpha), \\
c_3 &= \mp\frac{\alpha^2}{\beta^2\lambda_0^2\mu_0} + \frac{\alpha^3}{\beta^3\lambda_0^2}f_{3,0} + \frac{1}{\beta^3\mu_0^2}g^{(3)}(\beta, -\alpha).
\end{aligned}
$$

The condition for the CSS (that is Δ_F) to have a B_3 singularity is $c_3 \neq 0$. In that case the ruled surface of bitangent lines has a cuspidal edge. If $c_3 = 0$ then generically c_4 can also be expressed in terms of the coefficients in f and g, and when $c_4 \neq 0$ there is a B_4 singularity so that the ruled surface has a swallowtail singularity.

Case 3: Q_2, Q_* both degenerate. Then we have (10) and (11) both holding. Here we have to admit the possibility that λ_0 or μ_0 is zero so that one surface point is parabolic. But note that from (10) we cannot have $\lambda_0 = 0$ (since $\lambda_0 + \mu_0 = 1$) so that, from (11), $\alpha = 0$ implies $\beta = 0$. But the base points we are considering are points of contact of bitangent lines and these occur along curves on our surface which meet the parabolic set only at isolated points. Generically then the asymptotic direction at such a parabolic base point will not be along the bitangent line, that is $\alpha = \beta = 0$ can be ruled out on genericity grounds. We can therefore take $\alpha \neq 0$ and $\lambda_0 \neq 0$. Manipulating (10) and (11) we now find the following cases:

Case 3(a) $\beta = 0, \beta^2 \neq \alpha\gamma$. The chord is then a principal direction for both surface pieces. We require in this case $Q_3(\mathbf{w}) \neq 0$, as for C_3 (Case 1(a) above). This generically gives an F_4 singularity, where the criminant (ruled surface of bitangent lines) has a cuspidal edge and the caustic has a Whitney umbrella singularity. There is of course an additional condition to guarantee that the F_4 is versally unfolded; this is complicated but in the special case $\beta = 0$ (see (9)) it reduces to $(f_{2,1} - g_{2,1})(\alpha + 1)(\gamma \mp 1) \neq 0$. See Figure 6 for an example of this case.

Case 3(b) $\beta^2 = \alpha\gamma$, $\mu_0 = 0$. Thus the CSS point is a_0. The generating family in this case can be reduced to $F = -z + \epsilon(\epsilon + uy + x + u^3 + v^2)$. The caustic is smooth and the criminant is a folded Whitney umbrella: the other way round from C_4 (Case 1(b) above). The genericity conditions in this case come to $g^{(3)}(\mathbf{w}) \neq \mathbf{0}$ and $\beta \neq 0$; see (14), (9) above. In the present case $\mathbf{w} = (-\beta, \alpha)$. See Figure 7 for an example.

---

[5]This can be obtained from the standard versal unfolding of a C_4 singularity, $f = w^4 + \epsilon w + q_2 w^2 + q_1\epsilon + q_0$, and is given in (q_0, q_1, q_2)-space by $f = \partial f/\partial w = \partial^2 f/\partial w^2 = 0$.
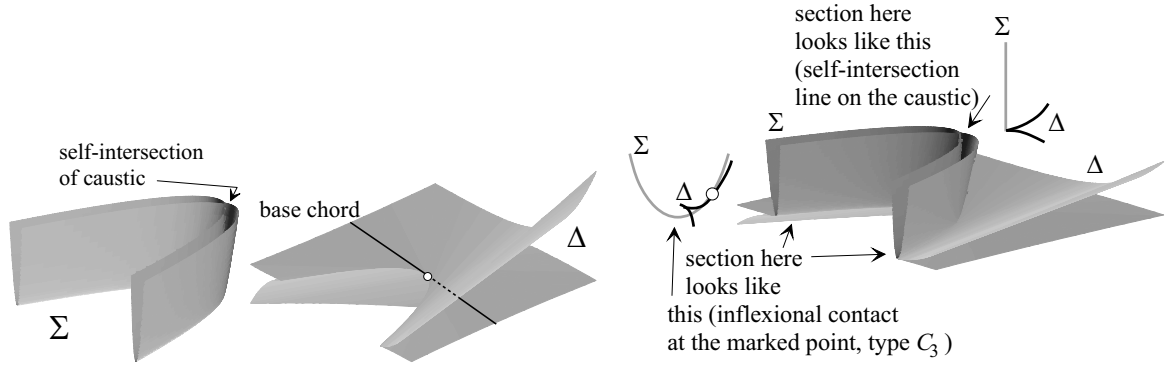
Figure 6: The F_4 case (Case 3(a) in the text) in which the caustic Σ is a Whitney umbrella and the criminant Δ has a cuspidal edge. This illustrates the case $f = u^2 - v^2, g = -2u^2 + v^2 - u^3 + 2\,u^2v + uv^2 + v^3$. Left: the caustic; centre: the criminant, with part of the base chord joining the base points a_0 and b_0. This chord is tangent to the cusp edge. Right: the caustic and criminant: the detailed structure is as indicated, with the inflexional crossings being C_3 points, as in Figure 4.
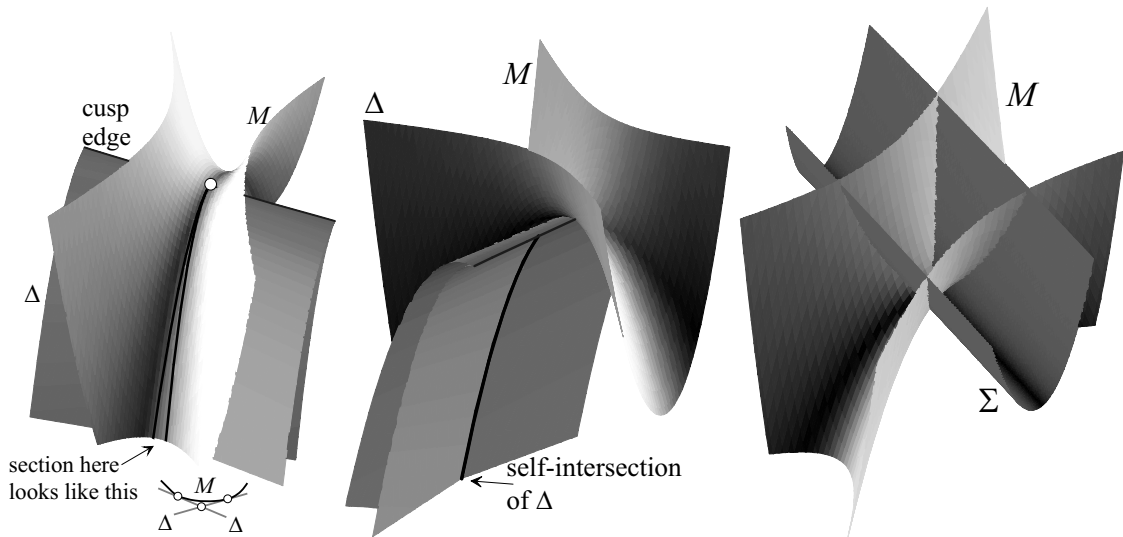

Figure 7: The $\widetilde{C}_4$ case (Case 3(b)), using the example $f = u^2 - v^2, g = 4u^2 + 4uv + v^2 - u^3 + 2u^2v + uv^2 + v^3$. Left and middle: two views of the criminant Δ (ruled surface of bitangent lines) and one of the surface pieces M, corresponding to f. The criminant is a folded Whitney umbrella (cuspidal crosscap). The cusped space curve along which the surfaces M and Δ are tangential is marked heavily in the left-hand figure, and the cuspidal edge and self-intersection of Δ are marked heavily in the right-hand figure. The cuspidal edge on the criminant has ordinary (2-point) contact with M. Right: M together with the smooth caustic surface Σ, which has roughly the shape of a parabolic cylinder here. The intersection between caustic and criminant is as depicted in Figure 5 but with the roles of caustic and criminant reversed.

References

[1] V.I. Arnol'd, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of Differentiable Mappings I*, translated by I.R.Porteous, Birkhäuser 1985.

[2] P.J.Giblin and P.A.Holtom, 'The centre symmetry set', *Geometry and Topology of Caustics, Banach Center Publications*, Vol **50**, ed. S.Janeczko and V.M.Zakalyukin, Warsaw, 1999, pp.91–105.

[3] P.J.Giblin and V.M.Zakalyukin, 'Singularities of systems of chords', *Functional Analysis and its Applications*, **36** (2002), 220–224.

[4] P.J.Giblin and V.M.Zakalyukin, 'Singularities of centre symmetry sets', *Proc. London Math. Soc* **90** (2005), 132–166.

[5] V.V.Goryunov,'Singularities of projections of complete intersections',*Journal of Soviet Math.* **27** (1984), 2867-2906.

[6] P.A.Holtom, *Local Central Symmetry for Euclidean Plane Curves*, M.Sc. Dissertation, University of Liverpool, Sept. 1997.

[7] S.Janeczko, 'Bifurcations of the center of symmetry', *Geom. Dedicata* **60** (1996), 9-16.

P.J.Giblin
Department of Mathematical Sciences
The University of Liverpool
Liverpool L69 7ZL, U.K.

V.M.Zakalyukin
The University of Liverpool and
Department of Mechanics and Mathematics
Moscow State University
1, Leninskie Gory
119992 Moscow GSP-2, Russia