Affine-Invariant Symmetry Sets

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by

Paul Andrew Holtom

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Synopsis

Affine-invariant *Symmetry Sets* of planar curves were first introduced and studied in a series of articles by Giblin and Sapiro (see [GS96], [GS98]). The original idea was to mimic the numerous constructions of the Euclidean Symmetry Set to produce analogous affine-invariant symmetry sets for affine plane curves. One of the first, and most striking, observations was that, although the different constructions for the Euclidean Symmetry Set led to *identical* sets, the affine-invariant analogues of these constructions resulted in genuinely *different* sets. Thus there is no single affine-invariant symmetry set, but instead a number of affine-invariant sets which individually capture some aspects of local affine symmetry.

In Chapter 1 we present some preliminary results and discussion concerning affine transformations, planar affine differential geometry, conic sections, and envelopes having high contact with their constituent curves.

Chapter 2 introduces the *Affine Envelope Symmetry Set* (AESS), an affine analogue of the Euclidean Symmetry Set (SS). We begin by formulating a geometrical interpretation of the AESS analogous to the interpretation of the SS: in this way we can justifiably claim that the AESS captures some aspect of local affine (reflexional) symmetry. The local structure of the AESS was classified in [GS96], [GS98] for ovals, and in this thesis we extend the classification to include non-oval and non-simple curves. We introduce the *Mid-Parallel Tangents Locus* (MPTL), another affine symmetry set, in order to understand the structure of the AESS, and this in turn leads to the *Affine Area Symmetry Set*. The interesting links between these affine symmetry sets are discussed.

Chapter 3 introduces the *Affine Distance Symmetry Set* (ADSS), the other affine analogue of the SS considered in [GS96], [GS98]. The local structure of the ADSS was classified in these articles for ovals, and in this thesis we extend this classification to include non-ovals and non-simple curves. The main part of this chapter concerns the study of transitions on the ADSS of 1-parameter families of curves, following the analogous procedure given in [BG86] for the SS. The conclusion uncovers an unexpected distinction between the transitions which can occur on families of *ovals* and the transitions that can occur on families of *generic plane curves*.

Chapter 4 and Chapter 5 are concerned with an interesting problem for the AESS and ADSS respectively, the so-called *Reconstruction Problem*, considered (implicitly) in [BG86]. The Reconstruction Problem asks, *Given a smooth curve segment* S*, how can we reconstruct a smooth plane curve having* S *as its symmetry set?* In effect, it asks, *To what extent does the symmetry set define a curve?* In each of the affine symmetry set cases, we begin by posing and solving a simpler problem, and then applying these ideas to the Reconstruction Problem itself.

Chapter 6 begins the study of the ADSS and AESS for *piecewise-conic* curves, that is, curves comprising segments which are parts of conics. This consideration mirrors the analogous study of the Euclidean Symmetry Set for piecewise-circular curves [BanG94]. In a first small step towards this study, we consider the structure of the ADSS and AESS only for curves comprising two complete ellipses. We consider how segments of the symmetry sets can be created or destroyed, can join or split apart, and suggest some avenues of further research.

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Contents

Chapter 1

Introduction

1.1 Euclidean transformations

Definition 1.1.1 ([BGG85]). *The* **Euclidean Symmetry Set** *(SS) of a smooth, simple, closed plane curve is (the closure of) the locus of centres of circles bitangent to the curve.*

Bitangent circles are circles tangent to a curve at two *or more* distinct points. The Euclidean Symmetry Set is a set of points connected to a plane curve which is invariant under the *Euclidean group* of transformations of $\mathbb{R}^2$, that is, under transformations of the form $\mathbf{x} \mapsto \tilde{\mathbf{x}}$ given by

$$\tilde{\mathbf{x}} = A\mathbf{x} + \mathbf{b},$$

where A is an orthogonal (2×2) matrix, $\mathbf{b}$ is a (2×1) matrix, and $\mathbf{x}, \tilde{\mathbf{x}}$ are points in the Euclidean plane (written as column vectors). The Euclidean group consists of rigid translations, rotations and reflexions in the plane. These transformations preserve the *degree* of a curve, thus mapping circle to circles, preserve *ratios of distances*, thus mapping centres of circles to centres of circles, and preserve *contact* between curves. It follows that centres of bitangent circles are invariant under these transformations, and thus the Euclidean Symmetry Set is invariant under the action of the Euclidean group. The Euclidean Symmetry Set captures the local Euclidean reflexional symmetry of a plane curve (see §2.1.2).

We will generalise the idea of a symmetry set in order to construct a set of points which is invariant under a wider group of transformations, namely the *affine transformation group*.

1.2 The affine transformation group

This section is based on Chapter 1 of [S83], which in turn is based on [Bla23], both of which are excellent references for a general introduction to affine transformations and affine differential geometry.

Let $\mathbf{x} = (x_1, x_2)^T$ be the coordinates of a point P in a 2-dimensional affine space. Consider a non-singular (2×2) matrix $A = (a_{ij}), i, j \in \{1, 2\}$. Then an affine transformation $\mathbf{x} \mapsto \tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)^T$ is one of the form

$$\tilde{x}_i = \sum_{k=1}^{2} a_{ij} x_j + b_i \text{ for } i = 1, 2.$$

If $\mathbf{b}$ denote the matrix $(b_1, b_2)^T$, then this affine transformation can be written as

$$\tilde{\mathbf{x}} = A\mathbf{x} + \mathbf{b}. \tag{1.1}$$

Definition 1.2.1. *Transformations of type (1.1) form a group (under the usual composition of transformations) called the* **affine transformation group** *(or simply* **affine group***), denoted $\mathcal{A}^2$.*

$\mathcal{A}^2$ contains six parameters. An important subgroup of $\mathcal{A}^2$ is defined by $\det(A) = 1$, since any area will be magnified by $\det(A)$ under a transformation of type (1.1), and therefore transformations with the property that $\det(A) = 1$ are *area-preserving*. We call such transformations *equi-affine*.

The transformations in $\mathcal{A}^2$ preserve: the *degree* of a curve – for example, they map conics to conics, and more specifically they preserve the type of conic; *parallelness* – two parallel lines will remain parallel after an affine transformation; *contact* between curves – for example, curves which share the same tangent line, affine tangent or affine normal (see Definition 1.3.1) will continue to do so after an affine transformation; and *ratios of Euclidean distances* along a straight line – in particular, midpoints of chords are invariant.

The Euclidean group is a subgroup of the affine group. However, in general, transformations from $\mathcal{A}^2$ do not preserve lengths, distances or angles, and nor do they map circles to circles. Thus the Euclidean Symmetry Set is not invariant under the group of affine transformations.

1.2.1 Affine reflexions and affine symmetry

Definition 1.2.2. *An* **affine reflexion** *with axis d is an affine transformation of order* 2 *(i.e. equal to its inverse, but not the identity) which leaves d pointwise fixed.*

We now define the affine analogue to Euclidean reflexional (or bilateral) symmetry. Since this is the only form of global affine symmetry we consider, we omit the word 'reflexional'.

Definition 1.2.3. *A plane curve γ is* **affine symmetric** *about axis d if there exists an affine reflexion with axis d which maps γ to itself.*

1.2.2 Using affine transformations

We are able to utilise affine transformations to simplify specific geometrical situations. For example, we will often consider a coordinate system with two local curve segments in general position, with one passing through the origin and tangent to the x-axis there and the other passing through a point (c, d). We simplify this situation without sacrificing generality by applying an affine transformation to make $c = 0$, which corresponds to the second curve segment being translated to cross the y-axis. Where appropriate in the subsequent analysis we will often assume that such an affine transformation has taken place.

1.3 Review of Affine Differential Geometry in the Plane

We present some basic concepts and definitions of planar affine differential geometry.

1.3.1 The affine-invariant arclength parameter

Let $\gamma(t)\colon [0,1] \to \mathbb{R}^2$ be a smooth planar curve parametrised by t. Restricting our analysis to *equi-affine* transformations, it can be shown that the simplest affine-invariant parametrisation s is given by requiring that the relation

$$[\gamma'(s), \gamma''(s)] = 1, \tag{1.2}$$

holds at every curve point $\gamma(s)$, where $'$ (prime) denotes derivative w.r.t. parameter s, and $[*, *]$ denotes the standard vector product, that is, the determinant of the (2×2) matrix defined by the $\mathbb{R}^2$ vectors. *Throughout this thesis we will denote by $'$ (prime) any derivative with respect to the affine-invariant parameter, except where explicitly stated.* This affine-invariant parameterisation is the central idea behind affine differential geometry in the plane.

Definition 1.3.1. *The vectors $\gamma'(s)$ and $\gamma''(s)$ are respectively the* **affine tangent** *and the* **affine normal** *to γ at $\gamma(s)$.*

Geometrically, the affine normal at a point of a curve γ is the locus of centres of conics having (at least) 4-point contact with γ at that point. At an inflexion, we take the affine normal to be in the same direction as the (non-oriented) affine tangent and of infinite length. Since (1.2) cannot hold at inflexion points of γ, affine differential geometry is not defined at inflexion points. However, since inflexions are affine-invariant, we circumvent this problem in practice by segmenting the curve into convex portions.

We now discuss the relationship between Euclidean and affine arclength parametrisations. Let $\dot{}$ (dot) denote derivative w.r.t. t.

Convention: *For brevity, we will often denote $\dot\gamma(t)$ by $\dot\gamma$, $\gamma'(s)$ by γ', and so on. Furthermore, when considering points such as $\gamma(s_1)$ or $\gamma(s_2)$, we will often use γ_1, γ_2 to denote $\gamma(s_1)$, $\gamma(s_2)$, γ_1' to denote $\gamma'(s_1)$, and so on.*

From (1.2) it follows that for arbitrary parametrisation t,

$$ds = [\dot\gamma, \ddot\gamma]^{1/3} dt. \tag{1.3}$$

If we write $k \equiv [\dot{\gamma}, \ddot{\gamma}]$, then it follows from (1.3) that

$$\gamma' = k^{-1/3}\dot{\gamma}. \tag{1.4}$$

This identity expresses the affine tangent vector for an arbitrary regular parametrisation of γ. For example, if t is the Euclidean arclength parameter (so $\langle \dot{\gamma}, \ddot{\gamma} \rangle = 1$), then

$$ds = [\dot{\gamma}, \ddot{\gamma}]^{1/3} dt = [T, \kappa N]^{1/3} dt = \kappa^{1/3} dt,$$

where T, N and κ represent the Euclidean unit tangent, unit normal and curvature respectively. (This relation was introduced in [ST93],[ST94].) It follows that the relationship between the affine tangent γ' and the Euclidean tangent T is given by

$$\gamma' = \kappa^{-1/3} T.$$

Thus two curves share the same *affine tangent* at a point if and only if they have the same *Euclidean tangent* and *curvature* at that point, and this implies the following.

Lemma 1.3.2. *Two curves share the same affine tangent at a point if and only if neither has an inflexion and they have (at least) 3-point contact there.*

We define an affine analogue of Euclidean curvature in the following way. Differentiating (1.2) w.r.t. s we obtain

$$[\gamma'(s), \gamma'''(s)] = 0,$$

for all s, and therefore

$$\gamma'''(s) + \mu\gamma'(s) = 0, \tag{1.5}$$

for some real function $\mu(s)$. We call μ the *affine-invariant curvature* (or just *affine curvature*) of γ. It is the simplest non-trivial affine differential invariant, and defines a curve uniquely up to (equi-) affine transformation (see [Bla23]), just as the Euclidean curvature defines a curve up to Euclidean transformation. Bracketing both sides of (1.5) with $\gamma'(s)$ gives us

$$\mu(s) = [\gamma''(s), \gamma'''(s)]. \tag{1.6}$$

Curves have constant affine curvature if and only if they are *conics*.

Lemma 1.3.3. *The affine curvature of a non-degenerate conic $\mathcal{C}$ is constant and positive if and only if $\mathcal{C}$ is an ellipse, zero if and only if $\mathcal{C}$ is a parabola, and negative if and only if $\mathcal{C}$ is an hyperbola.*

Proof. Consider the ellipse parameterised as

$$\gamma(s) = \left(a\cos\left(\frac{s}{(ab)^{1/3}}\right), b\sin\left(\frac{s}{(ab)^{1/3}}\right)\right), \tag{1.7}$$

for $a, b > 0$. This parametrisation is chosen because it is affine-invariant, that is,

$$\begin{aligned}\left[\frac{d}{ds}\gamma(s), \frac{d^2}{ds^2}\gamma(s)\right] &= \begin{vmatrix} -\frac{a}{(ab)^{1/3}}\sin\left(\frac{s}{(ab)^{1/3}}\right) & -\frac{a}{(ab)^{2/3}}\cos\left(\frac{s}{(ab)^{1/3}}\right) \\ \frac{b}{(ab)^{1/3}}\cos\left(\frac{s}{(ab)^{1/3}}\right) & -\frac{b}{(ab)^{2/3}}\sin\left(\frac{s}{(ab)^{1/3}}\right) \end{vmatrix} \\ &= \frac{ab}{(ab)^{1/3}(ab)^{2/3}}\left(\sin^2\left(\frac{s}{(ab)^{1/3}}\right) + \cos^2\left(\frac{s}{(ab)^{1/3}}\right)\right) \\ &\equiv 1.\end{aligned}$$

We also calculate:

$$\gamma'''(s) = \left(\frac{1}{b}\sin\left(\frac{s}{(ab)^{1/3}}\right), -\frac{1}{a}\cos\left(\frac{s}{(ab)^{1/3}}\right)\right),$$

and so

$$\begin{aligned}\mu(s) \equiv [\gamma''(s), \gamma'''(s)] &= \begin{vmatrix} -\frac{a}{(ab)^{2/3}}\cos\left(\frac{s}{(ab)^{1/3}}\right) & \frac{1}{b}\sin\left(\frac{s}{(ab)^{1/3}}\right) \\ -\frac{b}{(ab)^{2/3}}\sin\left(\frac{s}{(ab)^{1/3}}\right) & -\frac{1}{a}\cos\left(\frac{s}{(ab)^{1/3}}\right) \end{vmatrix} \\ &= (ab)^{-2/3},\end{aligned}$$

and thus $\mu(s) > 0$ for an ellipse. Similarly, consider a branch of an hyperbola parametrised by

$$\gamma(s) = \left(a\sinh\left(\frac{s}{(ab)^{1/3}}\right), b\cosh\left(\frac{s}{(ab)^{1/3}}\right)\right), \tag{1.8}$$

where the parametrisation is affine-invariant. We find that

$$\begin{aligned}\mu(s) \equiv [\gamma''(s), \gamma'''(s)] &= \begin{vmatrix} \frac{a}{(ab)^{2/3}} \sinh\left(\frac{s}{(ab)^{1/3}}\right) & \frac{1}{b} \cosh\left(\frac{s}{(ab)^{1/3}}\right) \\ \\ \frac{b}{(ab)^{2/3}} \cosh\left(\frac{s}{(ab)^{1/3}}\right) & \frac{1}{a} \sinh\left(\frac{s}{(ab)^{1/3}}\right) \end{vmatrix} \\ &= -(ab)^{-2/3},\end{aligned}$$

and thus $\mu(s) < 0$ for an ellipse. The intermediate case when $\mu = 0$ corresponds to a parabola.

Conversely, from (1.5), we know that

$$\gamma'''(s) = -\mu\gamma'(s),$$

where μ is constant, and therefore

$$\gamma''(s) = -\mu\gamma(s) + \boldsymbol{c}, \text{ for some vector } \boldsymbol{c} = (a, b).$$

If we separate $\gamma(s)$ into coordinates $(x(s), y(s))$, then this says that

$$\begin{aligned}x''(s) &= -\mu x(s) + a, \\ y''(s) &= -\mu y(s) + b.\end{aligned}$$

Setting $X(s) = -\mu x(s) + a, Y(s) = -\mu y(s) + b$ we have

$$\begin{aligned}X''(s) &= -\mu X(s), \\ Y''(s) &= -\mu Y(s),\end{aligned}$$

which has well-known solutions. For example, if $\mu > 0$ then

$$(X(s), Y(s)) = \left(A\cos(\sqrt{\mu}s) + B\sin(\sqrt{\mu}s), C\cos(\sqrt{\mu}s) + D\sin(\sqrt{\mu}s)\right),$$

for arbitrary $A, B, C, D \in \mathbb{R}$, which gives us

$$(x(s), y(s)) = \frac{1}{\mu}\left(a - A\cos(\sqrt{\mu}s) - B\sin(\sqrt{\mu}s), b - C\cos(\sqrt{\mu}s) - D\sin(\sqrt{\mu}s)\right),$$

which is an ellipse (or a circle). A similar consideration applies for the case $\mu < 0$, and in the case $\mu = 0$, $x(s)$ and $y(s)$ are both quadratic. □

A point where the affine curvature of γ vanishes is an *affine inflexion*, also known as a *parabolic point* of γ, so called since it is a point at which exists an unique parabola having 5-point contact with γ. If a conic has (at least) 5-point contact with a curve, then it shares the same affine curvature with the curve at the point of contact, and is called an *osculating conic*. The *centre of affine curvature* at $\gamma(s)$ is the centre of the osculating conic at that point, that is, the point $\gamma(s) + (1/\mu(s))\gamma''(s)$, and the locus of these points is the *affine evolute* of γ, the affine-invariant analogue of the Euclidean evolute. Furthermore, with analogy to the Euclidean situation, the affine evolute is the envelope of the affine normal lines to the curve. A point for which $\mu'(s) = 0$ is called an *affine vertex* of a curve, or a *sextactic point*: at such a point there exists a conic having 6-point contact with the curve. The centre of a sextactic conic lies at a cusp of the evolute. There are at least six points on a closed curve for which $\mu'(s) = 0$ (see [Bla23] for a proof of this; see also [F84] for a short exposition on the existence of sextactic points).

1.3.2 The affine distance function

We introduce the concept of *affine distance*, which is based on area and is invariant under equi-affine transformations.

Definition 1.3.4. *Let* $\mathbf{x}$ *be a point in the plane, and* $\gamma(s)$ *a planar curve parametrised by affine-arclength* s. *The* **affine distance** *between* $\mathbf{x}$ *and a non-inflexional point* $\gamma(s)$ *on the curve is given by*

$$d(\mathbf{x}, s) \equiv [\mathbf{x} - \gamma(s), \gamma'(s)]. \tag{1.9}$$

Note that the affine distance is defined between a point $\mathbf{x}$ of $\mathbb{R}^2$ and a curve point $\gamma(s)$. Since the basic geometric (equi-) affine invariant is area, we require three points, or a point and a vector, to define affine-invariant distance. It follows that there is no affine distance between two points in space. For an oval parametrised by affine-arclength, the distance from a point inside the oval to a point of the oval is negative. (By *oval* we will always mean a simple, closed curve having no zeros of Euclidean curvature, that is, a *strictly convex* plane curve.)

In [IS95], this family of functions is used to describe the *affine evolute* in terms of singularity theory, and in [GS96], [GS98], this family of functions is used to study an affine-invariant analogue of the Euclidean Symmetry Set, the ADSS, the subject of Chapters 3, 5 and 6 of this thesis.

Using Arnold's standard A_k notation for singularities of functions of one variable, we have:

Proposition 1.3.5 ([IS95]). *Away from affine inflexion points of γ, the affine distance function d defined on γ exhibits the following singularities:*

$\boldsymbol{A_{\geq 1}}$ $\Longleftrightarrow$ $\mathbf{x} - \gamma(s)$ *is parallel to* $\gamma''(s)$*, and* $\mathbf{x}$ *is then on the affine normal line to* γ *at* $\gamma(s)$*.*

$\boldsymbol{A_{\geq 2}}$ $\Longleftrightarrow$ $\mu(s) \neq 0$ *and* $\mathbf{x} = \gamma(s) + \frac{1}{\mu(s)}\gamma''(s)$*, and* $\mathbf{x}$ *is then at the centre of affine curvature of* γ *at* $\gamma(s)$*, that is, on the affine evolute of* γ*.*

$\boldsymbol{A_{\geq 3}}$ $\Longleftrightarrow$ $\mu(s) \neq 0$, $\mathbf{x} = \gamma(s) + \frac{1}{\mu(s)}\gamma''(s)$ *and* $\mu'(s) = 0$*, and* $\mathbf{x}$ *is then on the affine evolute of* γ *at an affine vertex.*

We now consider the limit of the affine distance function at an inflexion.

Proposition 1.3.6. *The limiting value of the affine distance of a point* $\mathbf{x}$ *from an ordinary inflexion* I *on a plane curve* γ *is*

$$\begin{cases} 0 \text{ if } \mathbf{x} \text{ is along the tangent line to } \gamma \text{ at } I, \text{ and} \\ \infty \text{ otherwise.} \end{cases}$$

Proof. Consider a smooth plane curve segment γ, parametrised by t, and having an inflexion for $t = 0$. Set up a coordinate system with the inflexion at the origin and the inflexional tangent along the x-axis. Then we can write

$$\gamma(t) = (X(t), Y(t)) = (t, at^3 + bt^4 + ct^5 + \ldots),$$

for some $a, b, c, \ldots \in \mathbb{R}$, with $a \neq 0$. Let $\dot{}$ (dot) denote derivative w.r.t. t.

Then

$$\begin{aligned} k(t) &= [\dot{\gamma}(t), \ddot{\gamma}(t)] \\ &= \begin{vmatrix} 1 & 0 \\ 3at^2 + 4bt^3 + \dots & 6at + 12bt^2 + \dots \end{vmatrix} \\ &= 6at + 12bt^2 + \dots \end{aligned}$$

and we calculate that

$$k(t)^{-1/3} = (6a)^{-1/3} \left(t^{-1/3} - \frac{2b}{3a} t^{2/3} + \dots \right).$$

The affine tangent vector to γ is given by (1.3), and thus the affine distance from $\mathbf{x} = (x, y)$ to the curve γ through $\gamma(t)$ is

$$\begin{aligned} d(\mathbf{x}, \gamma(t)) &= [\mathbf{x} - \gamma(t), \gamma'(t)] \\ &= [\mathbf{x} - \gamma(t), k^{-1/3}\dot{\gamma}] \\ &= \begin{vmatrix} x - X(t) & (6a)^{-1/3}(t^{-1/3} - 2bt^{2/3}/3a + \dots) \\ y - Y(t) & (6a)^{-1/3}(t^{-1/3} - 2bt^{2/3}/3a + \dots)(3at^2 + \dots) \end{vmatrix} \\ &= x \left(3a(6a)^{-1/3} t^{5/3} + \dots \right) - y \left((6a)^{-1/3} t^{-1/3} - \frac{2b}{3a} t^{2/3} + \dots \right), \end{aligned}$$

and thus as $t \to 0$, $d(\mathbf{x}, \gamma(t)) \to 0$ if $y = 0$ and $\to \infty$ if $y \neq 0$. $\square$

1.3.3 From an arbitrary regular parametrisation to the affine-invariant parametrisation

Here we present some identities to convert from an arbitrary regular parameterisation to the affine-arclength parametrisation.

Suppose $\gamma(t)$ is an arbitrary regular parametrisation of a plane curve γ. Using $\dot{}$ (dot) for d/dt and $k(t) = [\dot{\gamma}, \ddot{\gamma}]$, by (1.4) we have $\gamma' = k^{-1/3}\dot{\gamma}$.

Differentiating this expression w.r.t. affine-arclength we get

$$\gamma'' = k^{-2/3}\ddot{\gamma} - \frac{1}{3}\dot{k}k^{-5/3}\dot{\gamma}. \tag{1.10}$$

This identity expresses the affine normal vector for any arbitrary regular parametrisation of γ (note that $[\gamma', \gamma''] \equiv 1$). For a smooth curve $\gamma(x) = (x, f(x))$ we have

$$k(x) = \ddot{f}(x),\ \dot{k}(x) = \dddot{f}(x),\ \dot{\gamma}(x) = (1, \dot{f}(x)),\ \text{and}\ \ddot{\gamma}(x) = (0, \ddot{f}(x)),$$

and thus the affine normal to γ at $\gamma(x)$ is

$$\begin{aligned} \gamma''(x) &= \ddot{f}^{-2/3}(0, \ddot{f}) - \frac{1}{3}\dddot{f}\ddot{f}^{-5/3}(1, \dot{f}), \\ &= \left(-\frac{1}{3}\ddot{f}^{-5/3}\dddot{f}, \ddot{f}^{1/3} - \frac{1}{3}\dot{f}\ddot{f}^{-5/3}\dddot{f}\right). \end{aligned}$$

Finally, from (1.5), we have

$$\gamma''' = -\mu\gamma', \tag{1.11}$$

where μ is the affine curvature of γ.

1.3.4 Affine-invariant parametrisation and induced orientation

Affine-invariant parametrisation induces an orientation on curve segments in the affine plane. Fixing $[\gamma', \gamma''] = 1$ means that any affine tangent vector to a curve and the corresponding affine normal vector make a positive frame. The direction of the affine normal to a curve γ at a non-inflexional point γ_1 is the limiting tangent line to the locus of midpoints of chords joining points of γ and parallel to the tangent at γ_1. Using the expression from §1.3.3 for the affine normal vector γ_1'' , we know that it points in the direction as shown in Figure 1.1(a). The affine normal vector always lies to the convex side of the curve, irrespective of any parameterisation of γ. Thus the identity $[\gamma', \gamma''] = 1$ fixes the affine tangent γ' at each point of γ, and therefore (at least locally) induces an orientation on a plane curve.

For example, consider an oval γ of Figure 1.1(b). At each point the affine normal vector points towards the interior of γ, and thus the affine tangent vector must be oriented as shown, which in turn induces the *'positive'* orientation on the oval. In the non-oval case of Figure 1.1(c), a curve is segmented at its (Euclidean) inflexions, and the affine-invariant parametrisation induces a positive orientation on each segment as shown.

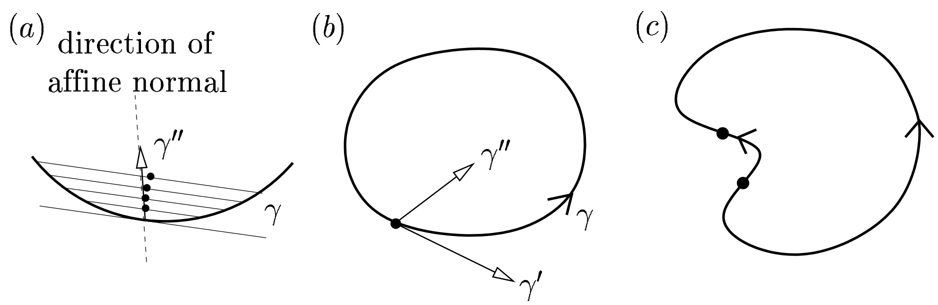


Figure 1.1: *See* §1.3.4.

1.4 Conics

A conic is a curve of degree two in two variables, given in general form as

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \tag{1.12}$$

where a, b, c, f, g, h are real homogeneous parameters. Thus a conic has five degrees of freedom, and five points define a conic uniquely. If the expression

$$\nabla = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

is zero, then the conic is degenerate, that is, either an intersecting line-pair, a parallel line-pair, a repeated line-pair, a single point, or the empty set. Otherwise, the conic is non-degenerate and its type is distinguished by the sign of the discriminant $\Delta \equiv ab - h^2$: if $\Delta > 0$ then the conic is an *ellipse*, if $\Delta = 0$ then the conic is a *parabola*, and if $\Delta < 0$ then the conic is an *hyperbola*.

Definition 1.4.1. *A* **centre** *of a conic of the form (1.12) is a solution* (x, y) *to*

$$\begin{aligned} ax + hy + g &= 0, \\ hx + by + f &= 0. \end{aligned}$$

If $\Delta \neq 0$, then this centre is unique and finite. In the case of an hyperbola, the centre is at the intersection of the asymptotes; for a parabola, the centre is the point at infinity along the axis of the parabola; for an intersecting line-pair, the centre is at the intersection point; for a repeated line-pair, the centre is an arbitrary point on the repeated line; for a parallel line-pair, the centre is at infinity. A *central conic* is any non-degenerate conic with a finite centre.

The affine distance from the centre of a central conic to a point of the conic is constant; conversely, if the affine distance from a point to a closed curve is constant then the curve is a conic with the point as its centre.

Definition 1.4.2. *The* **affine radius** σ *of a non-degenerate conic is minus the affine distance from its centre to each of its points.*

Lemma 1.4.3. *The affine radius* σ *of a non-degenerate conic is equal to* $1/\mu$*, where* μ *is the (constant) affine curvature of the conic.*

Proof. Consider an ellipse $\gamma(t) = (a\cos t, b\sin t)$. The affine curvature of γ is defined to be $\mu = [\gamma'', \gamma''']$, and in the proof of Lemma 1.3.3 we calculate that

$$\mu = (ab)^{-2/3}.$$

Now the affine radius of an ellipse γ (centred at $(0, 0)$) is defined to be

$$\begin{aligned} \sigma &= -[(0,0) - \gamma(s), \gamma'(s)] \\ &= (ab)^{-1/3} \begin{vmatrix} a\cos t & -a\sin t \\ b\sin t & b\cos t \end{vmatrix} \\ &= (ab)^{2/3}. \end{aligned}$$

Hence $\sigma = 1/\mu$ in the case of an ellipse.

A similar calculation confirms the result for an hyperbola, using the affine-invariant parametrisation of Lemma 1.3.3. For a parabola, we note that its affine radius is infinite, since its centre is at infinity, and this corresponds with the fact that the affine curvature of a parabola is zero. $\square$

Corollary 1.4.4. *The affine radius of a conic $\mathcal{C}$ is* positive *if and only if $\mathcal{C}$ is an ellipse,* zero *if and only if $\mathcal{C}$ is a parabola, and* negative *if and only if $\mathcal{C}$ is an hyperbola.*

Proof. By Lemma 1.3.3 and Lemma 1.4.3. $\square$

We now present an identity involving the affine radius, discriminant and centre of a conic. Consider a central conic $\mathcal{C}(x, y)$ given by equation (1.12), with centre (p, q), discriminant $\Delta \equiv ab - h^2$ and affine radius σ.

Proposition 1.4.5.

$$\sigma^3 \Delta = \mathcal{C}(p, q)^2,$$

where $\mathcal{C}(p, q)$ denotes the expression for the conic $\mathcal{C}(x, y)$ after the substitution $\{x = p, y = q\}$.

Proof. Suppose $\mathcal{C}(x, y) = 0$ is an ellipse. Substitute $x = X + p, y = Y + p$ in to equation (1.12) for $\mathcal{C}(x, y) = 0$ to get

$$aX^2 + 2hXY + bY^2 + \mathcal{C}(p, q) = 0,$$

and then apply the linear transformation $X \longmapsto X - \frac{h}{a}Y, Y \longmapsto Y$ of determinant 1 to get

$$aX^2 + \frac{\Delta}{a}Y^2 + \mathcal{C}(p, q) = 0.$$

The affine radius of this conic has been preserved. This conic is then mapped to a circle by $X \longmapsto \lambda X, Y \longmapsto \frac{1}{\lambda}Y$, where $\lambda^2 = \sqrt{\Delta}/a$. This circle is parametrised as

$$(X(t), Y(t)) = \left(\sqrt{\frac{-\mathcal{C}(p, q)}{\sqrt{\Delta}}}\cos t, \sqrt{\frac{-\mathcal{C}(p, q)}{\sqrt{\Delta}}}\sin t\right).$$

Note that $\Delta > 0$ for an ellipse, and the value of $\mathcal{C}(p,q)$ for an ellipse is negative. The affine radius σ of this circle is then

$$\begin{aligned}
\sigma &= [(X(t),Y(t)),(X'(t),Y'(t))], \\
&= \begin{vmatrix} \sqrt{\frac{-\mathcal{C}(p,q)}{\sqrt{\Delta}}}\cos t & \sqrt{\frac{-\mathcal{C}(p,q)}{\sqrt{\Delta}}}\sin t \\ -\sqrt{\frac{-\mathcal{C}(p,q)}{\sqrt{\Delta}}}\sin t & \sqrt{\frac{-\mathcal{C}(p,q)}{\sqrt{\Delta}}}\cos t \end{vmatrix} \left(\frac{1}{\frac{-\mathcal{C}(p,q)}{\sqrt{\Delta}}}\right)^{1/3}, \\
&= \left(\frac{C(p,q)^2}{\Delta}\right)^{1/3}.
\end{aligned}$$

A similar procedure can be used for the case when the conic is an hyperbola. □

This confirms that the affine radius σ will always have the same sign as Δ. Compare this with the calculation carried out in [COT96], where the same identity is derived by a different method.

Definition 1.4.6. *A* **diameter** *of a conic is any line through the centre of the conic.*

In the case of an hyperbola, a diameter is any line through the intersection of the asymptotes. For a parabola, a diameter is any line parallel to the axis. We will link the idea of a diameter of a conic with the existence of an affine symmetry of the conic, showing that, with *one exception*, a non-degenerate conic is affine symmetric in any of its diameters.

Lemma 1.4.7. *A central conic $\mathcal{C}$ is affine symmetric about any diameter d, except when $\mathcal{C}$ is an hyperbola with d as an asymptote.*

Proof. Consider $\mathcal{C}$ with centre at the origin and diameter d, which we take to be the x-axis. $\mathcal{C}$ is then given by

$$ax^2 + 2hxy + by^2 = 1, \tag{1.13}$$

for homogeneous coefficients $a, b, h \in \mathbb{R}$. The matrix of a general affine reflexion in the x-axis (that is, which leaves the x-axis pointwise fixed) is

$$\begin{pmatrix} 1 & \lambda \\ 0 & -1 \end{pmatrix},$$

for $\lambda \in \mathbb{R}$, corresponding to the transformation $x \mapsto x + \lambda y$, $y \mapsto -y$. Substituting this into the equation for $\mathcal{C}$ above, we get

$$ax^2 + (2a\lambda - 2h)xy + (a\lambda^2 - 2h\lambda + b)y^2 = 1. \tag{1.14}$$

For the transformation above to be an affine reflexion in the x-axis *fixing* $\mathcal{C}$, we require that the expressions (1.13), (1.14) are identical, i.e. that

$$h = a\lambda - h, \text{ and } b = a\lambda^2 - 2h\lambda + b,$$

where we note that the second condition follows automatically from the first, namely that $\lambda = 2h/a$, assuming that $a \neq 0$. Thus the transformation given by the matrix

$$\begin{pmatrix} 1 & 2h/a \\ 0 & -1 \end{pmatrix},$$

with $a \neq 0$ is the required affine transformation in the x-axis fixing $\mathcal{C}$.

In the case $a = 0$, $\mathcal{C}$ is an hyperbola with the x-axis as one of its asymptotes. The transformation thus defined is not strictly a reflexion in the x-axis, but a limit of such reflexions. It fixes the x-axis, but not pointwise, instead mapping a point $(x, 0) \mapsto (-x, 0)$. Thus an hyperbola is not affine symmetric about an asymptote. □

Lemma 1.4.8. *A parabola $\mathcal{P}$ is affine symmetric about any diameter.*

Proof. Consider parabola $\mathcal{P}$ with diameter any line parallel to the axis of the parabola. Let us take parabola $\mathcal{P}$ to be given by $(y - a)^2 = bx$. Then the axis of $\mathcal{P}$ is parallel to the x-axis, which we will take to be our diameter d. Following a similar approach to the proofs of Lemma 1.4.7, we are able to deduce that the matrix

$$F = \begin{pmatrix} 1 & \frac{4a}{b} \\ 0 & -1 \end{pmatrix},$$

is the required affine reflexion in d fixing $\mathcal{P}$. □

Thus we have shown:

Proposition 1.4.9. *For a non-degenerate conic $\mathcal{K}$ and any diameter d, there exists an (unique) affine reflexion R in d which fixes $\mathcal{K}$,* except *when $\mathcal{K}$ is an hyperbola and d is an asymptote.*

1.4.1 Locus of centres of a pencil of conics

For an oval γ and two fixed points $\gamma(t_1), \gamma(t_2)$ on it, consider the 1-parameter family of all conics $\mathcal{C}$ which are tangent to γ at these two points. If $\mathcal{T}_1 = 0$ and $\mathcal{T}_2 = 0$ are equations of the tangent lines to γ at $\gamma(t_1)$ and $\gamma(t_2)$, and $\mathcal{L} = 0$ is an equation of the line joining these two points, then the general conic of the pencil is

$$\mathcal{T}_1\mathcal{T}_2 + \lambda\mathcal{L}^2 = 0,$$

for some real parameter λ (extended by $\lambda = \infty$). Let m denote the midpoint of the chord joining $\gamma(t_1), \gamma(t_2)$, p denote the intersection of $\mathcal{T}_1 = 0$ and $\mathcal{T}_2 = 0$, and $\mathcal{M} = 0$ denote the equation of the line through p and m.

Proposition 1.4.10. *The locus of centres of the conics in the pencil $\mathcal{T}_1\mathcal{T}_2 + \lambda\mathcal{L}^2 = 0$ (extended by $\lambda = 0$) is the line pair $\mathcal{L}\mathcal{M} = 0$.*

Remark 1.4.11. *The component $\mathcal{L} = 0$ of this locus of centres is degenerate, corresponding to the repeated line-pair $\mathcal{L}^2 = 0$ whose centre is indeterminate along $\mathcal{L} = 0$.*

For the proof, we require the following: suppose $A + \lambda B = 0$ is a pencil of conics in the (x, y)-plane; then differentiating with respect to coordinates x and y in turn we have

$$\frac{\partial A}{\partial x} + \lambda\frac{\partial B}{\partial x} = 0,$$
$$\frac{\partial A}{\partial y} + \lambda\frac{\partial B}{\partial y} = 0.$$

For varying λ, this system defines a pencil of line-pairs, intersecting at the centres of the original pencil of conics, using the definition of the centre of a conic to be the pole of the line at infinity. Thus

$$\frac{\partial A}{\partial x}\cdot\frac{\partial B}{\partial y}=\frac{\partial A}{\partial y}\cdot\frac{\partial B}{\partial x},$$

gives us the locus of centres of the pencil of conics. We may now prove the proposition.

Proof. (of Proposition 1.4.10) The derivatives of $\mathcal{T}_1\mathcal{T}_2+\lambda\mathcal{L}^2=0$ with respect to coordinates x, y respectively are

$$\frac{\partial\mathcal{T}_1}{\partial x}\mathcal{T}_2+\mathcal{T}_1\frac{\partial\mathcal{T}_2}{\partial x}+2\lambda\mathcal{L}\frac{\partial\mathcal{L}}{\partial x}=0,$$

$$\frac{\partial\mathcal{T}_1}{\partial y}\mathcal{T}_2+\mathcal{T}_1\frac{\partial\mathcal{T}_2}{\partial y}+2\lambda\mathcal{L}\frac{\partial\mathcal{L}}{\partial y}=0,$$

and, eliminating λ, the locus of centres of the conics in this pencil is

$$\mathcal{L}\left(\frac{\partial\mathcal{T}_1}{\partial x}\mathcal{T}_2\frac{\partial\mathcal{L}}{\partial y}+\mathcal{T}_1\frac{\partial\mathcal{T}_2}{\partial x}\frac{\partial\mathcal{L}}{\partial y}-\frac{\partial\mathcal{T}_1}{\partial y}\mathcal{T}_2\frac{\partial\mathcal{L}}{\partial x}-\mathcal{T}_1\frac{\partial\mathcal{T}_2}{\partial y}\frac{\partial\mathcal{L}}{\partial x}\right)=0.$$

Thus $\mathcal{L}=0$ is a component of the locus of centres. Denote the other line component by $\mathcal{M}=0$. The intersection p of $\mathcal{T}_1=0$ and $\mathcal{T}_2=0$ lies on $\mathcal{M}=0$. Without loss of generality, suppose $\mathcal{T}_1=0$ and $\mathcal{T}_2=0$ intersect at the origin, and let $\mathcal{L}=0$ be given by $x-k=0$ for some $k\neq 0$. Suppose

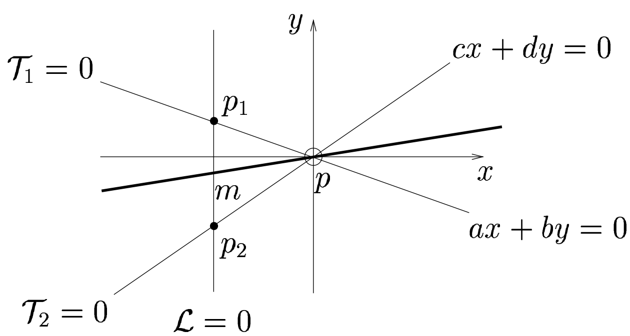


Figure 1.2: *See* §1.4.1.

$\mathcal{T}_1=0$ is $ax+by=0$, and $\mathcal{T}_2=0$ is $cx+dy=0$, for some $a,b,c,d\in\mathbb{R}$. Let p_1, p_2 denote the points $\gamma(t_1), \gamma(t_2)$, situated at $\mathcal{T}_1\cap\mathcal{L}$ and $\mathcal{T}_2\cap\mathcal{L}$ respectively.

Then we see that

$$p_1 = (k, -ak/b)\,,$$
$$p_2 = (k, -ck/d)\,.$$

A short calculation shows that line $\mathcal{M} = 0$ is

$$(ad + bc)x + 2bdy = 0,$$

and hence the midpoint m of the chord p_1p_2, the point $(k, -k(ad+bc)/2bd)$, lies on $\mathcal{M} = 0$. Thus we have shown that the component $\mathcal{M} = 0$ of the locus of centres of the pencil is indeed the line through $p = \mathcal{T}_1 \cap \mathcal{T}_2$ and midpoint m, as required. □

1.5 Envelopes with high contact

Consider a 1-parameter family of smooth curves $F(x, y, u) = 0$, where family parameter u is taken to be in a neighbourhood of U. Suppose that this family has a *smooth* envelope. The constituent curves of the envelope are $F(x, y, u_0) = 0$, and the corresponding envelope point is denoted (x_0, y_0).

Definition 1.5.1. *A smooth envelope having n-point contact with each of the constituent curves of the family is called an* **n-point contact envelope**.

This section contains some general results concerning contact between members of 1-parameter families of curves and the corresponding envelopes. These results will be applied in Chapters 4 and 5, when we consider the problem of creating envelopes having high contact with their constituent curves.

1.5.1 Tangency between implicit curves

Derivatives will be denoted by subscripts.

Lemma 1.5.2. *Two smooth curves $F(x, y) = 0$ and $G(x, y) = 0$ are tangent at (x_0, y_0) if and only if*

$$F(x_0, y_0) = G(x_0, y_0) = 0, \ \ \textit{and} \ \begin{vmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{vmatrix} = 0.$$

Proof. The vectors $(F_x(x_0, y_0), F_y(x_0, y_0))$ and $(G_x(x_0, y_0), G_y(x_0, y_0))$ are parallel to the normal vectors to $F = 0$ and $G = 0$ at (x_0, y_0) respectively. Thus the two curves are tangent if they pass through the point (x_0, y_0) and their normal vectors are parallel there, the necessary and sufficient condition for which is expressed as the determinant condition above. □

1.5.2 Envelopes

For any parameter value $u = u_0$ we have:

Proposition 1.5.3. *The curve $F(x, y, u_0) = 0$ has ≥ 3-point contact with the envelope curve at (x_0, y_0) if and only if the curve $F_u(x, y, u_0) = 0$ has ≥ 2-point contact with the envelope at (x_0, y_0).*

Proof. Consider the surface in (x, y, u)-space defined by $F(x, y, u) = 0$. Locally it can be expressed in the form $y = G(x, u)$ for suitable function G. The curve $F(x, y, u_0) = 0$ is re-expressed as $y = G(x, u_0)$, the 'slice' of the surface $y = G(x, u)$ at level $u = u_0$. The curve $F_u(x, y, u_0) = 0$ is similarly re-expressed as $G_u(x, u_0) = 0$. The envelope is then defined to be the projection of the critical set of this surface to the (x, y)-plane, and is given by solutions to the system

$$\begin{cases} y = G(x, u), \\ 0 = G_u(x, u). \end{cases}$$

Suppose that $G_u(x, u) = 0$ has a non-singular solution $x = X(u)$ for some u in a neighbourhood of u_0: the envelope is then $(X(u), G(X(u), u))$. We may assume that $X'(u_0) \neq 0$, since we require the envelope to be smooth. This gives us the identities

$$y = G(X(u), u), \tag{1.15}$$

$$0 = G_u(X(u), u). \tag{1.16}$$

To measure the contact between the curve $y = G(x, u_0)$ and the envelope at (x_0, y_0) we consider the vanishing at $u = u_0$ of the derivatives w.r.t. u of the expression

$$h(u, u_0) = G(X(u), u) - G(X(u), u_0). \tag{1.17}$$

The curve $y = G(x, u_0)$ and the envelope have $\geq n$-point contact at (x_0, y_0) if and only if the first $(n-1)$ derivatives of h w.r.t. u vanish at $u = u_0$.

To measure the contact between the curve $G_u(x, u_0) = 0$ and the envelope $(x, y) = (X(u), G(X(u), u))$ we consider the vanishing at $u = u_0$ of the derivatives w.r.t. u of the expression

$$f(u, u_0) = G_u(X(u), u_0). \tag{1.18}$$

The curve $G_u(x, u_0) = 0$ and the envelope $(x, y) = (X(u), G(X(u), u))$ have $\geq n$-point contact at (x_0, y_0) if and only if the first $(n-1)$ derivatives of f w.r.t u vanish at $u = u_0$.

The result follows by calculating the required derivatives of functions h and f, evaluated at $u = u_0$, using the identities (1.15) and (1.16). □

Calculating further derivatives of the functions h and f at $u = u_0$ leads to the following extension of Proposition 1.5.3.

Proposition 1.5.4. *The curve $F(x, y, u_0) = 0$ has ≥ 4-point contact with the envelope for all u_0 in a neighbourhood of U if and only if the curve $F_u(x, y, u_0) = 0$ has ≥ 3-point contact with the envelope for all u_0 in a neighbourhood of U.*

Remark 1.5.5. *It is crucial to note that Proposition 1.5.4 links 4-point contact between $F(x, y, u_0) = 0$ for a range of u_0 to 3-point contact between $F_u(x, y, u_0) = 0$ for a range of u_0.*

We use Proposition 1.5.4 to improve Proposition 1.5.3.

Corollary 1.5.6. *A curve $F(x, y, u_0) = 0$ has exactly 3-point contact with the envelope curve at (x_0, y_0) if and only if the curve $F_u(x, y, u_0) = 0$ has exactly 2-point contact with the envelope at (x_0, y_0).*

We now use these results to link the contact between an envelope and a constituent curve $F(x, y, u_0) = 0$ to contact between this curve and the curve $F_u(x, y, u_0) = 0$.

Corollary 1.5.7. *The curve $F(x, y, u_0) = 0$ has $\geq$ 3-point contact with the envelope at (x_0, y_0) if and only if the curve $F(x, y, u_0) = 0$ has $\geq$ 2-point contact with the curve $F_u(x, y, u_0) = 0$ at (x_0, y_0).*

Proof.

($\Rightarrow$) Suppose $F(x, y, u_0) = 0$ has $\geq$ 3-point contact with the envelope at (x_0, y_0). Then $F(x, y, u_0) = 0$ and the envelope share the same 3-jet (in suitable coordinates at (x_0, y_0)). Then, by Proposition 1.5.3, $F_u(x, y, u_0) = 0$ has $\geq$ 2-point contact with the envelope at (x_0, y_0), and thus $F_u(x, y, u_0) = 0$ and the envelope share the same 2-jet. Thus $F(x, y, u_0) = 0$ and $F_u(x, y, u_0) = 0$ share the same 2-jet, and hence have $\geq$ 2-point contact.

($\Leftarrow$) Suppose $F_u(x, y, u_0) = 0$ has $\geq$ 2-point contact with $F(x, y, u_0) = 0$. Then $F(x, y, u_0) = 0$ and $F_u(x, y, u_0) = 0$ share the same 2-jet. We also know by construction that $F(x, y, u_0) = 0$ and the envelope must share the same 2-jet, and thus $F_u(x, y, u_0) = 0$ must share the same 2-jet, that is, $F_u(x, y, u_0) = 0$ has $\geq$ 2-point contact with the envelope. Hence, by Proposition 1.5.3, $F(x, y, u_0) = 0$ has $\geq$ 3-point contact with the envelope.

□

Corollary 1.5.8. *The curve $F(x, y, u_0) = 0$ has $\geq$ 4-point contact with the envelope at (x_0, y_0) for all u_0 in a neighbourhood of U if and only if $F(x, y, u_0) = 0$ has $\geq$ 3-point contact with the curve $F_u(x, y, u_0) = 0$ at (x_0, y_0) for all u_0 in a neighbourhood of U.*

Proof. By similar arguments to above. □

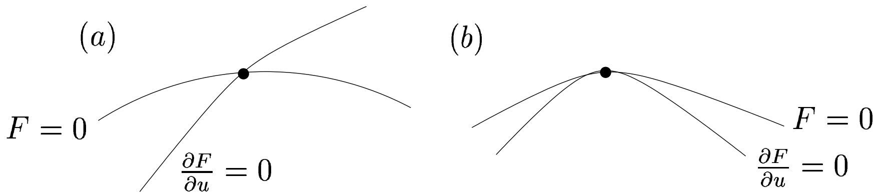


Figure 1.3: *Diagrammatic illustration of the zero-levels of* F *and* $\partial F/\partial u$.

1.5.3 Geometrical interpretation of Proposition 1.5.3 and Proposition 1.5.4

Consider the envelope of a smooth 1-parameter family of plane curves

$$F(x, y, u) = 0.$$

Geometrically, an envelope point occurs at the intersections of the two *'consecutive'* curves $F(x, y, u_0) = 0$ and $F_u(x, y, u_0) = 0$. Normal (2-point) contact envelopes are formed where consecutive curves in the family intersect transversally. Proposition 1.5.3, or rather Corollary 1.5.6, tells that 3-point contact envelope points occur when the zero-levels of $F(x, y, u_0)$ and $F_u(x, y, u_0)$ are tangent. Furthermore, Proposition 1.5.4 tells us that $\geq$ 4-point envelopes occur when the zero-levels of $F(x, y, u_0)$ and $F_u(x, y, u_0)$ have $\geq$ 3-point contact. This interpretation is intuitively attractive, and suggests a general extension to these propositions.

Conjecture 1.5.9. *For integer* $n \geq 2$ *the curve* $F(x, y, u_0) = 0$ *has* n*-point contact with the envelope curve at* (x_0, y_0) *for all* u_0 *if and only if* $F(x, y, u_0) = 0$ *has* $(n-1)$*-point contact with the curve* $F_u(x, y, u_0) = 0$ *at* (x_0, y_0) *for all* u_0.

Remark 1.5.10. *We are assuming that, for a specific family of curves, there are enough degrees of freedom for these higher contact envelopes to occur. In the application to conics in Chapters 4 and 5, we require only Proposition 1.5.3 and Proposition 1.5.4 respectively, since we are considering only 3- and 4-point contact envelopes.*

Chapter 2

Affine Envelope Symmetry Sets

In this chapter we consider the affine-invariant symmetry set whose definition mirrors that of the Euclidean Symmetry Set given in Definition 1.1.1.

The affine-invariant analogue of a circle is a conic section, and since affine transformations preserve conics and contact between curves, we will base the analogous affine-invariant symmetry set on contact between a curve and *conics*. This will include *degenerate* conics, such as intersecting, parallel and repeated line-pairs, and it is often the case that the geometry of the affine symmetry set (as defined in Definition 2.1.1) becomes most interesting when these degenerate conics appear.

Outline of Chapter 2

§**2.1:** We define the *Affine Envelope Symmetry Set* (AESS), the analogue of the Euclidean Symmetry Set as defined in Definition 1.1.1, and introduce the concept of a *'3+3' conic*.

§**2.2:** We derive the *'AESS Condition'*, which defines the *pre-AESS* for ovals. This condition is then considered for non-ovals, and slightly modified. In §2.2.2, we define the *Centre Map*, which maps the pre-AESS to the AESS, and in §2.2.3 we consider the effect of *horizontal* and *vertical tangents* to the pre-AESS. In §2.2.4, we consider *Morse singularities* on the pre-AESS.

§**2.3:** We set up a coordinate system and derive explicit conditions for there to exist 3+3 and 4+3 conics.

§**2.4:** We begin the study of the local structure of the AESS, restricting the study to ovals only. We will see that the structure of the AESS makes more sense when considered in union with another affine-invariant set, the MPTL.

§**2.5:** We extend our classification of the local structure of the AESS to include *non-oval* curves, deducing the structure of the AESS (and MPTL) in situations involving inflexions and double tangents. We also consider the conditions for a cusp on the AESS and the structure of the AESS and MPTL at points where they meet: we show that the AESS $\cup$ MPTL exhibits a *beaks singularity* in this case. Finally, we consider the condition for the AESS to exhibit an inflexion and relate this to the ADSS, the subject of Chapter 3.

§**2.6:** We consider the local structure of the AESS for *non-simple* curves, deducing that the AESS passes smoothly through self-intersections of a curve.

§**2.7:** We reconsider the MPTL, showing that it can be defined as a *bifurcation set* of a family of area functions defined on a curve. This leads to the introduction of another affine-invariant symmetry set, the AASS.

§**2.8:** We illustrate the results of §§2.4-2.7 with some [LSMP] plots.

2.1 Introducing the Affine Envelope Symmetry Set

Definition 2.1.1. *The* **Affine Envelope Symmetry Set** *(AESS) of a simple, closed, smooth plane curve γ is the closure of the locus of centres of conics with (at least) 3-point contact with the curve in two or more distinct points.*

Notation: *We will call a conic having (at least) 3-point contact with a specific curve in two distinct points a '3+3 conic'; if, at one of these points,* *the curve and the conic have (at least) 4-point contact, we will call the conic a '4+3 conic'; if it is essential for us to specify that the conic should have precisely 3-point contact at one point and precisely 4-point contact at the other point, we will say 'exactly 4+3 conic', and so on.*

2.1.1 Discussion of *allowable* 3+3 conics

It is necessary to decide precisely what we will mean by a *3+3 conic*, and make a specific list of which 3+3 conics will contribute their centres to the AESS. The simplest cases are those where the 3+3 conic is either an *ellipse*, an *hyperbola*, or an *intersecting line-pair*, each of which has a finite centre which contributes to the AESS.

Consider the line-pair consisting of two tangents to the curve, where one of these lines is a tangent at an inflexion and cuts the curve again at the point of contact of the other tangent (see Figure 2.1 for an illustration). This line-pair is a legitimate 3+3 conic, and its contribution to the AESS is its centre, c, which sits on the curve at the intersection of the two lines. This phenomenon will occur for a generic non-oval plane curve, since whenever we have an inflexion where the curve crosses its tangent line the inflexional tangent will cut the curve at some other point, resulting in the degenerate 3+3 conic as prescribed. We will study the interesting geometry of this situation in §2.5.4, showing that the AESS at this point will generically exhibit an ordinary cusp.

Another allowable 3+3 conic is the *parabola* which has its centre at infinity, and thus contributes this point to the AESS. A *parallel line-pair* has centre at an arbitrary point of the parallel line midway between the line-pair, but this conic corresponds to the tangent lines at parallel inflexions on the original curve, the appearance of which is a non-generic phenomenon, and thus this case is not considered. It remains for us to consider the possibility that a repeated line pair, which has indeterminate centre at an arbitrary point of the repeated line, could contribute to the AESS. The two situations where a repeated line can have double 3-point contact with a curve are:

- *The case of a repeated tangent at an inflexion of our original curve.* To consider this as a legitimate 3+3 conic would be to include the

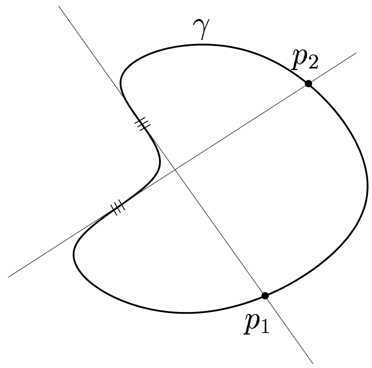


Figure 2.1: *Non-oval curve γ has two inflexional tangents, which cut the curve again at p_1, p_2. The degenerate 3+3 conics centred at the points p_1 and p_2 consist of the inflexional tangents together with the corresponding tangent to γ at p_1 or p_2.*

entire inflexional tangent line as part of the AESS. However, in this case the repeated tangent makes more sense when considered as a *6-point* contact conic, and geometrically the limit of a series of nearby 3+3 conics, with centres tending towards the inflexion point. We then conclude that this repeated inflexional tangent should *not* be classed as a 3+3 conic. The fact that the AESS is defined as the *closure* of the locus of centres of 3+3 conics means that the point of inflexion is included in the AESS, as an *endpoint*.

- *The case of a repeated line-pair tangent to the curve at two distinct points*, that is, the case of a repeated double tangent. This repeated line-pair in fact has *4*-point contact with the curve at each of these two points, and again the corresponding centre is an arbitrary point along this repeated line. However, we will not exclude this conic from our list of legitimate 3+3 conics, for reasons which will become clear in §2.2. Regarding the problem of the contribution of this 3+3 conic to the AESS of the curve, we see that if we view this repeated line as a limit of a series of nearby 3+3 conics, then its contribution is the midpoint of the chord joining points of contact of the double tangent. This situation is considered in detail in §2.5.2.

In summary, our set of *allowable 3+3 conics* will consist of the *ellipse*, the *hy-*

perbola and the *intersecting line-pair* (corresponding to *finite* AESS points), the *parabola* (corresponding to AESS points *at infinity*), and the *repeated line-pair* at a double tangent. The set of allowable 3+3 conics will *not* include the repeated line-pair comprising the tangent at an inflexion counted twice.

2.1.2 Geometrical interpretation of Definition 2.1.1

We have now defined, for any simple, closed, smooth plane curve γ, a set of points which remains invariant under the group of affine transformations. We now ask: *In what respect may we think of this set of points as capturing some aspect of the local affine symmetry of γ?*

The Euclidean Symmetry Set is a means of quantifying the *local Euclidean reflexional* (or *bilateral*) *symmetry* of a plane curve. Consider Figure 2.2,

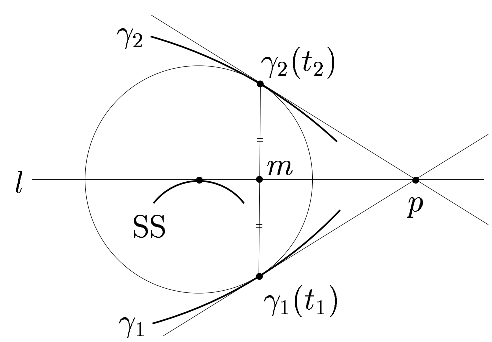


Figure 2.2: *l is an* infinitesimal axis of symmetry *for* $\gamma_1 \cup \gamma_2$.

where γ_1 and γ_2 are two smooth curve segments. Suppose the two points $\gamma_1(t_1), \gamma_2(t_2)$ are such that there exists a circle tangent to γ_1, γ_2 at these two points. The line l, through points m and p, respectively the midpoint of the chord joining $\gamma_1(t_1)$ and $\gamma_2(t_2)$ and the intersection of the tangents at these two points, is tangent to the Symmetry Set at the corresponding point (see Remark 4.1.1 for a geometrical justification of this statement). Then there exists a reflexion in l taking $\gamma_1(t_1)$ *and its tangent line* to $\gamma_2(t_2)$ and *its* tangent line. We call l an *'infinitesimal axis of (reflexional) symmetry'*.

We will now show that an analogous geometrical interpretation can be given for the AESS. Given two smooth curve segments γ_1, γ_2, suppose there exists a non-degenerate conic $\mathcal{C}$ having 3-point contact with γ_1 and γ_2 at $\gamma_1(s_1)$ and $\gamma_2(s_2)$ respectively. We will assume that the tangent lines to γ_1 and γ_2 at these points are not parallel.

Convention: *We will denote the midpoint of the chord joining such points $\gamma_1(s_1)$ and $\gamma_2(s_2)$ by m, and the intersection of the tangent lines at $\gamma_1(s_1)$ and $\gamma_2(s_2)$ by p.*

Proposition 2.1.2. *There exists a (non-degenerate) conic having 3-point contact with curve segments γ_1 and γ_2 at $\gamma_1(s_1)$ and $\gamma_2(s_2)$ if and only if there exists an affine reflexion taking $\gamma_1(s_1)$, and its affine tangent vector, to $\gamma_2(s_2)$ and its affine tangent vector. (The affine tangent vector is introduced in* §1.3.1*, and the definition of an affine reflexion is given in Definition 1.2.2.)*

Proposition 2.1.5 and Proposition 2.1.6 prove Proposition 2.1.2, and this section concerns the proofs of these two propositions. We require an intermediate result.

Lemma 2.1.3. *The tangent to the AESS at the centre of conic $\mathcal{C}$ passes through m and p.*

Proof. See §2.4.3 (Proposition 2.4.3(iv)). ☐

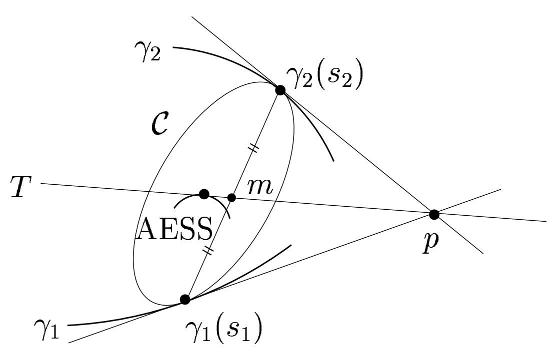


Figure 2.3: *Illustration of Lemma 2.1.3. Line T is tangent to the AESS at the centre of conic $\mathcal{C}$.*

Denote the tangent to the AESS at the centre of conic $\mathcal{C}$ by T. Then T is a diameter of $\mathcal{C}$. In Proposition 1.4.9 we have a result connecting the diameter of a non-degenerate conic to an affine reflexion with the diameter as an axis and fixing the conic, the exception being when the conic is an hyperbola with an asymptote as the axis.

We will now consider the exceptional case of Proposition 1.4.9 in relation to 3+3 conics. Suppose we have an hyperbola $\mathcal{K}$ having 3+3 contact with a curve at finite points p_1, p_2. By our convention, the tangent lines to the curve intersect at point p, and the midpoint of the chord joining p_1, p_2 is m. The intersection of the asymptotes is the centre, c, of hyperbola $\mathcal{K}$, and this is the corresponding AESS point. By Lemma 2.1.3, the tangent T to the AESS at this point passes through m and p, and is a diameter of $\mathcal{K}$. However, T cannot be along an asymptote of $\mathcal{K}$, regardless of whether or not p_1, p_2 are on the same branch of $\mathcal{K}$: for this to happen we would require m and p to lie on an asymptote, and it is clear from Figure 2.4 that in case (i) m cannot lie on an asymptote, and in case (ii) p cannot lie on an asymptote. So if T is a diameter of conic $\mathcal{K}$ having double 3-point contact with a curve

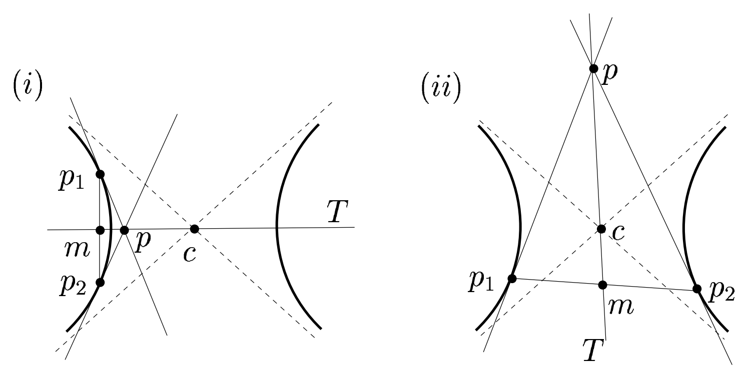


Figure 2.4: *The asymptotes are shown dashed.*

in two distinct, finite points, then T is not an asymptote of $\mathcal{K}$, and hence there exists an affine reflexion in T fixing $\mathcal{K}$ by Proposition 1.4.9, that is, $\mathcal{K}$ is affine symmetric about T. Thus, when we insist that the conic $\mathcal{K}$ in Proposition 1.4.9 is a 3+3 conic, and the diameter is taken to be the tangent T to the AESS at the centre of $\mathcal{K}$, we deduce that the exceptional case cannot occur.

Proposition 2.1.4. *Any non-degenerate conic $\mathcal{K}$ having 3+3 contact with a curve (at finite points) is affine symmetric about the tangent to the AESS through the centre of $\mathcal{K}$.*

We use this result to obtain a geometrical interpretation of the AESS. Consider two smooth curve segments γ_1 and γ_2, parametrised by affine-arclength. Let $\gamma_1(s_1)$ and $\gamma_2(s_2)$ be two fixed points on γ_1 and γ_2, and let $\gamma_1'(s_1)$ and $\gamma_2'(s_2)$ denote the respective affine tangent vectors to the curve segments at these two points. Let T be the unique line through midpoint m and intersection point p.

Proposition 2.1.5. *If there exists a non-degenerate conic $\mathcal{C}$ having 3+3 contact with curve segments γ_1, γ_2 at $\gamma_1(s_1), \gamma_2(s_2)$, then there exists an (unique) affine reflexion R in T (which is a diameter of $\mathcal{C}$) taking $\gamma_1(s_1)$ to $\gamma_2(s_2)$, and $\gamma_1'(s_1)$ to $\gamma_2'(s_2)$.*

Proof. In the case where $\mathcal{C}$ is an hyperbola, T cannot be an asymptote of $\mathcal{C}$, as deduced above. Then by Proposition 1.4.9, there exists an (unique) affine reflexion R in T fixing $\mathcal{C}$. Let T_i denote the tangent direction to γ_i at $\gamma_i(s_i)$, which is also the tangent direction to $\mathcal{C}$ since $\mathcal{C}$ and γ_i have 3-point contact at $\gamma_i(s_i)$ and thus share a tangent line. Clearly R takes $\gamma_1(s_1)$ to $\gamma_2(s_2)$, and vice-versa. Since R leaves T pointwise fixed, and fixes $\mathcal{C}$, it maps the tangent line T_1 to $\mathcal{C}$ at $\gamma_1(s_1)$ to the unique *other* tangent line to $\mathcal{C}$ through p, namely T_2, since the tangents T_1 and T_2 to the points of contact of 3+3 conics, and the tangent T to the AESS are concurrent (see Lemma 2.1.3). Thus R takes $\gamma_1(s_1)$, *and* its tangent line, to $\gamma_2(s_2)$ and *its* tangent line.

Furthermore, since R fixes the conic $\mathcal{C}$, it must map the *affine* tangent to $\mathcal{C}$ at $\gamma_1(s_1)$ to the *affine* tangent to $\mathcal{C}$ at $\gamma_2(s_2)$, and vice-versa. Since $\mathcal{C}$ has 3-point contact with each of γ_1, γ_2 at $\gamma_1(s_1), \gamma_2(s_2)$, Lemma 1.3.2 tells us that $\mathcal{C}$ shares the same affine tangent with each of the curve segments at these points. Also R preserves contact between $\mathcal{C}$ and γ_1, γ_2 at $\gamma_1(s_1), \gamma_2(s_2)$, and thus R maps the affine tangent to γ_1 at $\gamma_1(s_1)$ to the affine tangent to γ_2 at $\gamma_2(s_2)$, and vice-versa. $\square$

By similar arguments we also have the converse:

Proposition 2.1.6. *If there exists an affine reflexion R in line T taking $\gamma_1(s_1)$ to $\gamma_2(s_2)$, and $\gamma_1'(s_1)$ to $\gamma_2'(s_2)$, then there exists an (unique) conic $\mathcal{C}$ (with T as a diameter) having 3+3 contact with γ_1, γ_2 at $\gamma_1(s_1), \gamma_2(s_2)$.*

Proposition 2.1.5 and Proposition 2.1.6 together imply that there exists a non-degenerate conic having 3+3 contact with γ_1, γ_2 at $\gamma_1(s_1), \gamma_2(s_2)$ if and only if there exists an affine reflexion taking $\gamma_1(s_1)$, and the corresponding affine tangent to γ_1, to $\gamma_2(s_2)$ and the corresponding affine tangent to γ_2, and thus prove Proposition 2.1.2. This affine reflexion will be in the line T, through the centre of the 3+3 conic and tangent to the AESS there. Hence we can see that the existence of a 3+3 conic at $\gamma_1(s_1), \gamma_2(s_2)$ is equivalent to the two curve segments γ_1, γ_2 being *locally affine symmetric about* T. The tangents to the AESS can be thought of as *'infinitesimal axes of affine (reflexional) symmetry'*, and in this way we think of the AESS as capturing the local affine reflexional symmetry of a plane curve. This explains why the AESS is named as it is, since the AESS is the *envelope of the infinitesimal axes of affine reflexional symmetry of a curve.*

Note that this interpretation involves the set of tangents to the AESS, the 'dual-AESS', rather than the AESS itself. In §2.4.1 we will study the dual-AESS (augmented by the dual of the *affine evolute*) in order to derive the local structure of the AESS (and the affine evolute) for generic plane curves.

2.2 The pre-AESS

The analysis of the local structure of the AESS begins by locating pairs of points on a given curve γ contributing to the AESS. The pairs of parameter values for such points is called the *'pre-AESS'*, the set of points in the space of pairs of parameters (s_1, s_2) for which there exists a conic having double 3-point contact with γ at $\gamma(s_1), \gamma(s_2)$. Our definition of the AESS requires that γ and a particular conic have two points and their affine tangents in common, and we will use the following lemma to deduce the appropriate condition.

Lemma 2.2.1 ([GS96]). *Let δ be a non-degenerate conic parametrised by affine-arclength s. Then, for any two points $\delta(s_1)$, $\delta(s_2)$ we have*

$$[\delta(s_1) - \delta(s_2), \delta'(s_1) + \delta'(s_2)] = 0.$$

From this lemma we may deduce the following AESS Condition.

Proposition 2.2.2 ([GS96]: The AESS Condition). *Let γ be parametrised by affine-arclength. There is a non-degenerate conic $\mathcal{C}$ having (at least) 3-point contact with γ at two distinct points $\gamma(s_1)$ and $\gamma(s_2)$, neither of which is an inflexion, if and only if*

$$[\gamma(s_1) - \gamma(s_2), \gamma'(s_1) + \gamma'(s_2)] = 0. \tag{2.1}$$

Remark 2.2.3. *If $\mathcal{C}$ has (at least) 3-point contact with γ at an inflexion, then $\mathcal{C}$ has zero Euclidean curvature at this point, and is therefore degenerate, that is, a line-pair. Conversely, if there are no inflexions on γ, then any conic $\mathcal{C}$ having (at least) 3-point contact with γ must be non-degenerate. Thus, if we restrict γ to be an oval, then all 3+3 conics must be non-degenerate.*

2.2.1 The *AESS Condition* for non-ovals

The AESS Condition of Proposition 2.2.2 expresses the condition for a pair of parameter values (s_1, s_2) to contribute to the AESS, allowing us to pick out points of the *pre-AESS*, the set of parameter pairs (s_1, s_2) for which there exists a 3+3 conic. In effect, Proposition 2.2.2 defines the pre-AESS, where we exclude the 'diagonal' in parameter space by insisting that the points $\gamma(s_1)$ and $\gamma(s_2)$ are distinct. This definition of the pre-AESS is suitable if we restrict our curve γ to an oval, since then the affine tangent $\gamma'(s)$ is finite everywhere, and the condition that neither point is an inflexion is redundant. Problems occur when we try to use the AESS Condition when γ is a non-oval, at points of γ which are inflexions, since then the affine tangents will have infinite length. We avoid this problem as follows. Using expression (1.4) from §1.3.1 we can write $\gamma'(s_i) = k_i^{-1/3}\dot{\gamma}(s_i)$ for each of $i = 1, 2$, where $\dot{\gamma}$ is used to denote the derivative of γ with respect to an arbitrary parameter along γ, and $k_i \equiv [\dot{\gamma}(s_i), \ddot{\gamma}(s_i)]$. Then we can rewrite (2.1) as

$$\begin{aligned} & [\gamma(s_1) - \gamma(s_2), k_1^{-1/3}\dot{\gamma}(s_1) + k_2^{-1/3}\dot{\gamma}(s_2)] = 0, \\ \Rightarrow \quad & k_1^{-1/3}k_2^{-1/3}[\gamma(s_1) - \gamma(s_2), k_2^{1/3}\dot{\gamma}(s_1) + k_1^{1/3}\dot{\gamma}(s_2)] = 0. \end{aligned}$$

Using this alternative expression, we have:

Proposition 2.2.4 (Alternative AESS Condition). *Let γ be parametrised by affine-arclength. There is a conic $\mathcal{C}$ having (at least) 3-point contact with γ at two distinct points $\gamma(s_1)$ and $\gamma(s_2)$ if and only if*

$$[\gamma(s_1) - \gamma(s_2), k_2^{1/3}\dot{\gamma}(s_1) + k_1^{1/3}\dot{\gamma}(s_2)] = 0. \tag{2.2}$$

We now have a suitable definition for the pre-AESS for non-ovals. With reference to §2.1.1, it is easy to check that this expression does indeed identify all *allowable* 3+3 conics and thus, for the non-oval case, we are justified in using the Alternative AESS Condition.

2.2.2 The Centre Map

Whichever expression we use to define the pre-AESS, it is this set in (s_1, s_2)-space which is mapped to the plane to produce the AESS. Such a mapping will take the pairs of parameter values of points on the curve having 3+3 point contact conics (found using (2.1) or (2.2)), and map them to the *centre* of the 3+3 conic, the AESS point. We will consider only central conics picked out by these expressions, which requires the following lemma.

Lemma 2.2.5. *Let δ be a central conic, parametrised by affine-arclength s. For any two points $\delta(s_1)$, $\delta(s_2)$ on δ, the vector*

$$\delta(s_2) - \delta(s_1) - [\delta'(s_1), \delta'(s_2)]\delta'(s_1)$$

is along the line joining $\delta(s_1)$ to the centre of the conic.

Proof. The statement is invariant under affine transformations of arbitrary determinant, and thus the condition may be verified directly for the unit circle and the hyperbola $xy = 1$ (similar to proof of Lemma 2.2.1). □

The statement above holds when parameter values s_1 and s_2 are interchanged. The centre of a non-degenerate conic having 3-point contact with γ at $\gamma(s_1)$ and $\gamma(s_2)$ will be at the intersection of the two prescribed lines. Hence Lemma 2.2.5 can be applied to obtain an explicit formula for the centre of a central conic having 3-point contact with a curve γ at points with parameter values s_1 and s_2, which is the required map from the pre-AESS to the AESS.

Proposition 2.2.6. *The centre of a central 3+3 conic is at*

$$\frac{1}{2}\left(\gamma_1 + \gamma_2 + \frac{[\gamma_1 - \gamma_2, \gamma_1'][\gamma_1', \gamma_2']}{2[\gamma_1 - \gamma_2, \gamma_1'] - [\gamma_1', \gamma_2']^2}(\gamma_2' - \gamma_1')\right).$$

Proof. Applying the above lemma twice, with the roles of s_1 and s_2 reversed, we get two equivalent expressions for the centre, c, of a conic having 3-point contact at both $\gamma(s_1), \gamma(s_2)$:

$$\left.\begin{array}{l} c = \gamma_1 + \lambda_1(\gamma_2 - \gamma_1 - [\gamma_1', \gamma_2']\gamma_1') \\ c = \gamma_2 + \lambda_2(\gamma_1 - \gamma_2 - [\gamma_2', \gamma_1']\gamma_2') \end{array}\right\} \tag{2.3}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Summing these two expressions tells us that

$$c = \frac{1}{2}\left(\gamma_1 + \gamma_2 - (\lambda_2 - \lambda_1)(\gamma_2 - \gamma_1) + [\gamma_1', \gamma_2'](\lambda_2\gamma_2' - \lambda_1\gamma_1')\right).$$

Taking the difference of the same two expressions tells us that

$$\gamma_1 - \gamma_2 + (\lambda_1 + \lambda_2)(\gamma_2 - \gamma_1) - (\lambda_1\gamma_1' + \lambda_2\gamma_2')[\gamma_1', \gamma_2'] = 0. \tag{2.4}$$

Bracketing (2.4) with the vector $\gamma_1' + \gamma_2'$ gives us

$$(\lambda_1 + \lambda_2 - 1)[\gamma_2 - \gamma_1, \gamma_1' + \gamma_2'] - [\lambda_1\gamma_1' + \lambda_2\gamma_2', \gamma_1' + \gamma_2'][\gamma_1', \gamma_2'] = 0,$$

and since we know that the 3+3 Conic Condition $[\gamma_1 - \gamma_2, \gamma_1' + \gamma_2'] = 0$ holds, we deduce that

$$[\lambda_1\gamma_1' + \lambda_2\gamma_2', \gamma_1' + \gamma_2'][\gamma_1', \gamma_2'] = 0.$$

If $[\gamma_1', \gamma_2'] = 0$, then the tangent lines at γ_1 and γ_2 are parallel, and the 3+3 conic is a parallel line-pair and thus not central. So, assuming $[\gamma_1', \gamma_2'] \neq 0$, the above holds if and only if

$$[\lambda_1\gamma_1' + \lambda_2\gamma_2', \gamma_1' + \gamma_2'] = 0 \iff \lambda_1 = \lambda_2 (\equiv \lambda \text{ say}).$$

Thus

$$c = \frac{1}{2}\left(\gamma_1 + \gamma_2 + \lambda[\gamma_1', \gamma_2'](\gamma_2' - \gamma_1')\right).$$

Bracketing the equations (2.3) with γ_1', we see that

$$\lambda = \frac{[\gamma_1 - \gamma_2, \gamma_1']}{2[\gamma_1 - \gamma_2, \gamma_1'] - [\gamma_1', \gamma_2']^2},$$

which gives the required expression for the centre. □

Definition 2.2.7. *For (s_1, s_2) on the pre-AESS of a curve γ corresponding to a central 3+3 conic, the* **Centre Map** *from the pre-AESS to the AESS is given by*

$$(s_1, s_2) \longmapsto \frac{1}{2}\left(\gamma_1 + \gamma_2 + \frac{[\gamma_1 - \gamma_2, \gamma_1'][\gamma_1', \gamma_2']}{2[\gamma_1 - \gamma_2, \gamma_1'] - [\gamma_1', \gamma_2']^2}(\gamma_2' - \gamma_1')\right).$$

For finite points γ_1 and γ_2 of an oval, and corresponding finite tangents γ_1' and γ_2', the centre c goes to infinity when

$$2[\gamma_1 - \gamma_2, \gamma_1'] = [\gamma_1', \gamma_2']^2.$$

By calculating each side of this expression for the unit circle and rectangular hyperbola, we have:

Corollary 2.2.8. *Suppose there exists a conic $\mathcal{C}$ having 3-point contact with an oval at points γ_1, γ_2 with corresponding affine tangents γ_1', γ_2'. Then $\mathcal{C}$ is*

- *an* ellipse *if* $2[\gamma_1 - \gamma_2, \gamma_1'] > [\gamma_1', \gamma_2']^2$*;*
- *a* parabola *if* $2[\gamma_1 - \gamma_2, \gamma_1'] = [\gamma_1', \gamma_2']^2$*;*

- *an* hyperbola *if* $2[\gamma_1 - \gamma_2, \gamma_1'] < [\gamma_1', \gamma_2']^2$.

We consider now the cases where the Alternative AESS Condition of (2.2) has identified degenerate conics. For the intersecting line-pair comprising the tangent line at an inflexion and the tangent to the curve where the inflexional tangent intersects the curve, we know that the centre lies at the intersection of these lines, namely at γ_2. In the case of a double tangent, we have $[\gamma_1', \gamma_2'] = 0$ and $[\gamma_1 - \gamma_2, \gamma_1'] = 0$, and hence the above expression for the centre may not be used in its present state. In §2.5.2, we see that the limit of the centres of 3+3 conics tending towards the double tangent is the midpoint of the chord joining the points of contact of the double tangent, and we take this to be the *centre* of this 3+3 conic.

The expression for the centre of a 3+3 conic as given in Proposition 2.2.6 is essential to be able to plot the AESS for explicit curves using a graphics package such as [LSMP]. Examples of such plots are contained in §2.8.

2.2.3 Horizontal and vertical tangents to the pre-AESS

We are able to use Lemma 2.2.5 to relate cusps on the AESS to the structure of the pre-AESS. In §2.4.3 (Proposition 2.4.7), and §2.5.6 (Proposition 2.5.8), we show that a cusp appears on the AESS when the 3+3 conic becomes a 4+3 conic, that is, when the conic has (at least) 4-point contact with the curve at one of the two points of contact $\gamma(s_1), \gamma(s_2)$. Suppose the 4+3 conic has 4-point contact at $\gamma(s_1)$. Then the conic and γ share the same affine normal at $\gamma(s_1)$ (since the affine normal at $\gamma(s_1)$ is the locus of conics having $\geq$ 4-point contact with γ at $\gamma(s_1)$), and it follows that the line joining $\gamma(s_1)$ to the AESS point (at the centre of the 4+3 conic) is the affine normal, $\gamma''(s_1)$. Using Lemma 2.2.5, this tells us that the vector shown is parallel to $\gamma''(s_1)$. We have:

Proposition 2.2.9. *The 3+3 conic has in fact 4-point contact at γ_1 if and only if*

$$[\gamma_2 - \gamma_1, \gamma_1''] = [\gamma_1', \gamma_2'].$$

Proof. From Lemma 2.2.5 we see that $\gamma''(s_1)$ is parallel to the given vector (with $\gamma(s_1), \gamma(s_2)$ for $\delta(s_1), \delta(s_2)$), i.e.

$$\begin{aligned} & \gamma_1 - \gamma_2 - [\gamma_1', \gamma_2']\gamma_1' \text{ is parallel to } \gamma_1'', \\ \iff & [\gamma_2 - \gamma_1 - [\gamma_1', \gamma_2']\gamma_1', \gamma_1''] = 0, \\ \iff & [\gamma_2 - \gamma_1, \gamma_1''] = [\gamma_1', \gamma_2'], \end{aligned}$$

since $[\gamma_1', \gamma_1''] \equiv 1$. □

The condition in Proposition 2.2.9 is precisely the condition for the tangent to the pre-AESS to be parallel to the s_1-axis, which follows directly from differentiating (2.2) with respect to s_1. Similarly, tangents to the pre-AESS parallel to the s_2-axis correspond to a conic having 4-point contact at $\gamma(s_2)$. We will show that one way for the AESS to exhibit a cusp singularity is for there to exist a 4+3 conic. Thus we will be able to predict the existence of this type of cusp on the AESS by picking out *horizontal or vertical* tangents to the AESS. The reader is referred to §2.4.3 and §2.5.6 for details.

Remark 2.2.10. *Not all cusps on the AESS correspond to 4+3 conics, and hence not all cusps can be recognised by observing horizontal or vertical tangents to the pre-AESS.*

2.2.4 Morse singularities on the pre-AESS

We now consider the occurrence of Morse singularities on the pre-AESS of a smooth curve segment γ. Consider the function

$$F(t_1, t_2) \equiv [\gamma(t_1) - \gamma(t_2), \gamma'(t_1) + \gamma'(t_2)]. \tag{2.5}$$

Then $F(t_1, t_2) = 0$ if and only if (t_1, t_2) lies on the pre-AESS. So $F(t_1, t_2) = 0$ is an equation for the pre-AESS, and is the condition for there to exist a conic having 3-point contact with γ at $\gamma(t_1)$ and $\gamma(t_2)$. Away from the diagonal $t_1 = t_2$, we require a condition which distinguishes between *isolated points* and *crossings* on the pre-AESS. Now the pre-AESS exhibits a Morse singularity at a point (t_1, t_2) if

$$F(t_1, t_2) = F_{t_1}(t_1, t_2) = F_{t_2}(t_1, t_2) = 0,$$

and we calculate that

$$F_{t_1}(t_1, t_2) \equiv [\gamma_1', \gamma_2'] + [\gamma_1 - \gamma_2, \gamma_1''], \tag{2.6}$$

$$F_{t_2}(t_1, t_2) \equiv [\gamma_1', \gamma_2'] + [\gamma_1 - \gamma_2, \gamma_2''], \tag{2.7}$$

So we consider only (t_1, t_2) for which

$$F(t_1, t_2) = F_{t_1}(t_1, t_2) = F_{t_2}(t_1, t_2) = 0. \tag{2.8}$$

Taking (2.6) and (2.7) together we have:

$$F_{t_1}(t_1, t_2) - F_{t_2}(t_1, t_2) \equiv 0 \Leftrightarrow [\gamma_1 - \gamma_2, \gamma_1'' - \gamma_2''] = 0, \tag{2.9}$$

which is precisely the condition for (t_1, t_2) to lie on the *pre-ADSS* (see Chapter 3 for details).

Remark 2.2.11. *Thus if a point (t_1, t_2) is at a Morse singularity of the pre-AESS, then it also lies on the* pre-ADSS. *See* Remark 2.5.12 *for a similar link between the AESS and the structure of the pre-ADSS.*

Thus (2.8) says that the conic having 3-point contact with γ at $\gamma(t_1)$ and $\gamma(t_2)$ is in fact a *4+4* conic, in which case the corresponding AESS point $\mathbf{x}_0$ lies at the intersection of the affine normals to γ at γ_1 and γ_2, and at the same affine distance from these points. Expression (2.9) and $F(t_1, t_2) = 0$ together give us

$$[\gamma_1' + \gamma_2', \gamma_1'' - \gamma_2''] = 0. \tag{2.10}$$

We will require the following expressions:

$$\begin{aligned} F_{t_1 t_1}(t_1, t_2) &= 1 + [\gamma_1'', \gamma_2'] - \mu_1[\gamma_1 - \gamma_2, \gamma_1'], \\ F_{t_2 t_2}(t_1, t_2) &= -1 + [\gamma_1', \gamma_2''] - \mu_2[\gamma_1 - \gamma_2, \gamma_2'], \\ \text{and } F_{t_1 t_2}(t_1, t_2) &= [\gamma_1' + \gamma_2', \gamma_2'' - \gamma_1''] = 0 \text{ by (2.10)}, \end{aligned}$$

using the fact that $\gamma_i''' \equiv -\mu_i \gamma_i'$ (see §1.3.3). Let d_0 denote the common affine distance from γ_i to the AESS point, at the centre of the 4+4 conic. Then

$$d_0 \;\; = \;\; -\frac{1}{\mu},$$

where μ is the affine curvature of the 4+4 conic, and we can write

$$\gamma_1 - \gamma_2 = \frac{1}{\mu}(\gamma_2'' - \gamma_1''). \tag{2.11}$$

This follows since the centre $\mathbf{x}_0$ of the 4+4 conic can be written as

$$\mathbf{x}_0 \equiv \gamma_1 - d_0 \gamma_1'' \equiv \gamma_2 - d_0 \gamma_2''.$$

From this, we deduce:

$$[\gamma_1 - \gamma_2, \gamma_1'] \;\; = \;\; \frac{1}{\mu}\left([\gamma_2'', \gamma_1'] + 1\right),$$

$$[\gamma_1 - \gamma_2, \gamma_2'] \;\; = \;\; -\frac{1}{\mu}\left([\gamma_1'', \gamma_2'] + 1\right).$$

The pre-AESS has an isolated point or a crossing at (t_1, t_2) depending upon whether the expression

$$F_{t_1 t_1}(t_1, t_2) F_{t_2 t_2}(t_1, t_2) - F_{t_1 t_2}(t_1, t_2)^2,$$

is positive or negative respectively. We have $F_{t_1 t_2}(t_1, t_2) \equiv 0$, and also

$$\begin{aligned} F_{t_1 t_1}(t_1, t_2) \;\; &= \;\; 1 + [\gamma_1'', \gamma_2'] - \frac{\mu_1}{\mu}\left([\gamma_2'', \gamma_1'] + 1\right), \\ &= \;\; 1 + [\gamma_1'', \gamma_2'] - \frac{\mu_1}{\mu}\left([\gamma_1'', \gamma_2'] + 1\right), \text{ using the fact that } F_{t_1 t_2} = 0, \\ &= \;\; \left(1 - \frac{\mu_1}{\mu}\right)\left(1 + [\gamma_1'', \gamma_2']\right), \end{aligned}$$

and similarly

$$F_{t_2t_2}(t_1,t_2) = -\left(1-\frac{\mu_2}{\mu}\right)(1+[\gamma_1'',\gamma_2']) \, .$$

Thus we have an isolated point on the pre-AESS if and only if

$$F_{t_1t_1}(t_1,t_2)F_{t_2t_2}(t_1,t_2) > 0$$

$$\iff \left(1-\frac{\mu_1}{\mu}\right)\cdot\left(1-\frac{\mu_2}{\mu}\right) < 0,$$

$$\iff \mu \text{ lies between } \mu_1 \text{ and } \mu_2.$$

Thus we have a criterion for distinguishing between the Morse singularities on the pre-AESS. Unfortunately, a lack of intuition for affine geometry hinders any attempt to find a simple geometric interpretation for this criterion. However, we *can* make sense of this condition in the following situation.

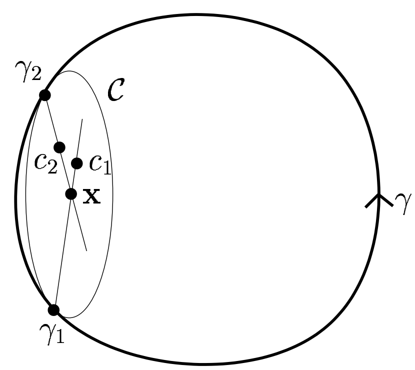


Figure 2.5: *Oval* γ *and 4+4 conic* $\mathcal{C}$*.*

Consider an oval γ, and suppose there exists an *ellipse* $\mathcal{C}$ having 4-point contact with γ in two points γ_1 and γ_2 (see Figure 2.5). Denote the affine curvature of γ at γ_i by μ_i. Then $\mu_i > 0$, and without loss of generality we will assume that $\mu_1 > \mu_2$. Denote the affine curvature of $\mathcal{C}$ by μ (so $\mu > 0$). Thus we have a Morse singularity on the pre-AESS of γ, and the intersection point $\mathbf{x}$ of the affine normals to γ at γ_1 and γ_2 is the corresponding AESS point (at the centre of $\mathcal{C}$). Let $\mathcal{C}_i$ denote the osculating conic to γ at γ_i, that is, the unique conic with 5-point contact with γ at γ_i. Then $\mathcal{C}_i$ has affine curvature μ_i. Each $\mathcal{C}_i$ has 4-point contact with γ at γ_i, and the centres of $\mathcal{C}$ and $\mathcal{C}_i$ lie along the corresponding affine normal. Then the situation $\mu_1 < \mu < \mu_2$ means geometrically that the centre of $\mathcal{C}$ is closer (along the affine normal) to γ_1 than the centre of $\mathcal{C}_1$, the centre of affine curvature of γ at γ_1, and further away (along the affine normal) from γ_2 than the centre of $\mathcal{C}_2$, the centre of affine curvature of γ at γ_2. Thus μ lying between μ_1 and μ_2 in this specific situation means that the 4+4 conic $\mathcal{C}$ lies *inside* the osculating conic at one of the points but *outside* the other. Alternatively, we could say that the points of the evolute of γ corresponding to γ_1 and γ_2 lie *'on the same side'* of the AESS point $\mathbf{x}$.

2.3 Studying the AESS

In §2.4 we begin to study the local structure of the AESS for oval curves, and in §2.5 we extend this analysis to non-ovals, all under a general framework. During this analysis we will occasionally be required to revert to coordinate-wise calculations. In this section we introduce a suitable coordinate system comprising two smooth curve segments γ_1 and γ_2 in general position, and derive an explicit condition (in these coordinates) for there to exists a conic having double 3-point contact with γ_1 and γ_2: §2.3.1 contains these calculations, and the interpretation of this *'3+3 Conic Condition'* of Proposition 2.3.1 in terms of the Euclidean curvatures of the two curve segments. Furthermore, in §2.3.2 we derive an explicit condition (in the same coordinates) for this 3+3 conic to be a 4+3 conic: Proposition 2.3.4 contains this *'4+3 Conic Condition'*.

2.3.1 Double 3-point contact between curve segments and a non-degenerate conic

Consider two smooth curve segments γ_1 and γ_2 given by

$$\gamma_1(t_1) = (t_1, a_2t_1^2 + a_3t_1^3 + \ldots),$$
$$\gamma_2(t_2) = (c + t_2, d + b_1t_2 + b_2t_2^2 + \ldots),$$

where t_1, t_2 are parameters along curve segments γ_1, γ_2 respectively. Consider also a conic $\mathcal{C}$, tangent to the x-axis at the origin, given by

$$ax^2 + by^2 + 2hxy + 2fy = 0. \tag{2.12}$$

where $a, b, h, f \in \mathbb{R}$ are homogeneous coefficients. We first of all require 3-point contact between $\mathcal{C}$ and γ_1 at the origin. Substituting $(x, y) = (t_1, a_2t_1^2 + a_3t_1^3 + \ldots)$ into (2.12) and collecting terms in t_1 we get

$$(a + 2fa_2)t_1^2 + 2(ha_2 + fa_3)t_1^3 + \ldots = 0. \tag{2.13}$$

So we have 3-point contact between $\mathcal{C}$ and γ_1 at the origin if and only if

$$a + 2fa_2 = 0.$$

We will assume that $a_2 \neq 0$ (which is equivalent to assuming that the curvature of γ_1 at the origin in non-zero), and we choose $a = 1$ (note that this assumes that $\mathcal{C}$ is non-degenerate, which follows from the fact that $a_2 \neq 0$). Similarly, we measure contact between $\mathcal{C}$ and γ_2 at (c, d) by substituting $(x, y) = (c + t_2, d + b_1t_2 + b_2t_2^2 + \ldots)$ into (2.12). The table below summarises the conditions on the coefficients for the required degree of contact between $\mathcal{C}$ and γ_2.

Contact between $\mathcal{C}$ and γ_2 at (c, d)	Condition for contact
$(\geq)$1-point	$ac^2 + bd^2 + 2d(hc + f) = 0$
$(\geq)$2-point	$2(ac + bdb_1 + hcb_1 + hd + fb_1) = 0$
$(\geq)$3-point	$a + b(2db_2 + b_1^2)$ $+2(hcb_2 + b_1h + fb_2) = 0$

Substituting $a = 1$, $f = -1/2a_2$ into the 1- and 2-point conditions from the table (which are both equations in unknowns b, h), we can rewrite them as

$$\begin{pmatrix} a_2d^2 & 2cda_2 \\ 2db_1a_2 & 2cb_1a_2 + 2da_2 \end{pmatrix} \begin{pmatrix} b \\ h \end{pmatrix} = \begin{pmatrix} -a_2c^2 + d \\ -2a_2c + b_1 \end{pmatrix}. \tag{2.14}$$

The determinant of the matrix is equal to zero if and only if $d = 0$ or $b_1 = d/c$, since we are assuming $a_2 \neq 0$ throughout. Thus we cannot solve this system for b and h in the following cases (see Figure 2.6):

(a) $d = 0$, where the tangent to curve segment γ_1 (namely the x-axis) cuts the curve segment γ_2, or

(b) $b_1 = d/c$, where the tangent to γ_2 passes through the curve segment γ_1 at the origin.

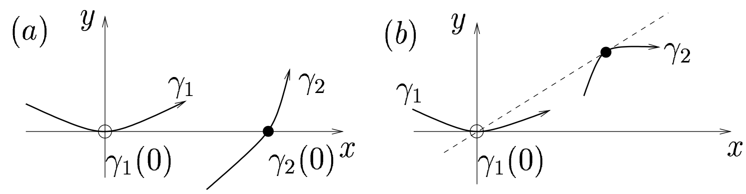


Figure 2.6: *The two situations in which we cannot solve (2.14).*

The two cases present an identical problem. In case (a), the line-pair consisting of the tangent to γ_1 at the origin and the tangent to γ_2 at $\gamma_2(0)$ is the (degenerate) conic having 3-point contact with γ_2 at $\gamma_2(0)$ and 2-point contact with γ_1 at $\gamma_1(0)$. For this line-pair to be a 3+3 conic, we must have an inflexion on γ_1 at $\gamma_1(0)$ (i.e. $a_2 = 0$). In case (b), we see that the line pair consisting of the tangent to γ_1 at the origin (namely the x-axis) and the tangent to γ_2 passing through the origin is the conic having 3-point contact with γ_1 and 2-point contact with γ_2. For this line pair to be a 3+3 conic, we must have an inflexion on γ_2 at $\gamma_2(0)$. Thus both cases relate to the degenerate (but generically occuring, for non-ovals) phenomenon of an inflexional tangent cutting the curve again, in which case the 3+3 conic consists of the inflexional tangent together with the tangent at the point where the inflexional tangent cuts the curve again (Figure 2.1 illustrates this situation). We would not expect to solve the system in these cases since we had already assumed that the conic $\mathcal{C}$ was non-degenerate by taking coefficient $a = 1$. We consider this situation in detail in §2.5.3.

Assuming from now on that $d \neq 0$, and $b_1 \neq d/c$, we solve (2.14) for b and h to get

$$b = \frac{c^2}{d^2} + \frac{1}{a_2(d - cb_1)}, \quad h = -\frac{c}{d} - \frac{b_1}{2a_2(d - cb_1)}.$$

From the table, we use the condition for 3-point contact between $\mathcal{C}$ and γ_2 at (c, d), which reduces to

$$a_2(d - cb_1)^3 + d^3 b_2 = 0.$$

Thus we have the following:

Proposition 2.3.1 (3+3 Conic Condition). *Given two curve segments*

$$\gamma_1(t_1) = (t_1, a_2 t_1^2 + a_3 t_1^3 + \ldots)$$
$$\gamma_2(t_2) = (c + t_2, d + b_1 t_2 + b_2 t_2^2 + \ldots)$$

where $d \neq 0, b_1 \neq d/c$, and $a_2 \neq 0$, there exists a non-degenerate conic having 3-point contact with both γ_1 and γ_2 at $t_1 = t_2 = 0$ if and only if

$$a_2(d - cb_1)^3 + d^3 b_2 = 0.$$

Corollary 2.3.2. *For the curves in Proposition 2.3.1, and $c = 0$, there exists a non-degenerate conic having 3-point contact with γ_1 at $t_1 = 0$ and γ_2 at $t_2 = 0$ if and only if*

$$a_2 + b_2 = 0. \tag{2.15}$$

This can be re-interpreted in terms of the Euclidean curvature of the two curve segments. If $\kappa_i(t_i)$ denote the Euclidean curvature of the curve segment γ_i at $\gamma_i(t_i)$, then it is not hard to see that $\kappa_1(0) = 2a_2$, $\kappa_2(0) = 2b_2$, and (2.15) is equivalent to $\kappa_1(0) + \kappa_2(0) = 0$.

When $a_2 = 0$, then the 3+3 Conic Condition holds if and only if $b_2 = 0$ also, in which case the 3+3 conic is a line-pair comprising the two inflexional tangents at $\gamma_1(0)$ and $\gamma_2(0)$.

Remark 2.3.3. *It may at first seem unusual that this affine condition for the existence of 3+3 conics has an Euclidean interpretation. However, although the individual Euclidean curvatures of the two curve segments are not preserved by affine transformations, the fact that they are equal and opposite* is *preserved, since the affine transformation will change them both by the same factor, namely by multiplying them both by the determinant of the corresponding affine transformation. Thus the above condition is reasonable.*

2.3.2 4+3 conics

We now derive the condition for the 3+3 conic to become a 4+3 conic, where we will assume that the conic achieves 4-point contact with the γ_1 curve segment. Since the calculations for $c = 0$ are so much simpler, *we will from now on assume (by an affine transformation) that* $c = 0$. Now for 4-point contact between $\mathcal{C}$ and γ_1 at the origin, we require the t_1^3 coefficient from (2.13) to vanish, i.e.

$$ha_2 + fa_3 = 0,$$

where $f = -1/2a_2$, $h = -b_1/2a_2d$, which becomes

$$a_2b_1 + a_3d = 0.$$

Proposition 2.3.4 (4+3 Conic Condition). *For the curves in Proposition 2.3.1, with* $c = 0$, $b_1d \neq 0$ *and* $a_2 + b_2 = 0$, *there exists a non-degenerate conic having 4-point contact with* γ_1 *at* $t_1 = 0$ *and 3-point contact with* γ_2 *at* $t_2 = 0$ *if and only if*

$$a_2b_1 + a_3d = 0.$$

A simple calculation shows that it is equivalent to the analogous condition for a 4+3 conic as stated in Proposition 2.2.9. Hence this is a condition, in this coordinate system, for the AESS to exhibit a cusp. Note the following special cases (remember that we have effectively ruled out the case $d = 0$ by assuming that $c = 0$):

- $a_2 = a_3 = 0$: there is an higher inflexion on γ_1 at the origin, and also, since $b_2 = 0$, and inflexion on γ_2 at $\gamma_2(0)$; the 4+3 conic is the line pair consisting of the two inflexional tangents;

- $a_2 = b_1 = 0$: there is an inflexion on γ_1 at the origin, an inflexion on γ_2 at $\gamma_2(0)$, and the inflexional tangent to γ_2 at $\gamma_2(0)$ is parallel to the inflexional tangent to γ_1.

We would not expect either of these situations to occur for a generic plane curve. Thus we have found, in this coordinate system, explicit conditions for

3+3 and 4+3 conics. We will utilise the 3+3 and 4+3 Conic Conditions of Propositions 2.3.1 and 2.3.4 numerous times during the analysis of the local structure of the AESS in §§2.4-2.5, and will also find them indispensible when constructing computer programs to plot the AESS (see Figure 2.7 and §2.8 for examples of such plots on [LSMP]).

2.4 The local structure of the AESS of an oval

We now leave the coordinate-wise calculations of §2.3 and begin the task of classifying the possible events which may occur on the AESS of a generic oval using a general framework. The method involves studying the set of tangents to the AESS, considering this set as a curve in dual-space, and relating the possible local structures of this dual-AESS to the corresponding events on the AESS. Our approach will be analogous to the study of the *Perpendicular Bisector Map*, which gave rise to the dual of the Euclidean Symmetry Set (see [T90], [GT95] for details). It is important to make the following distinction. In the Euclidean case, it is well-known that the SS can be obtained via a *bifurcation set of Euclidean distance functions*, and the local structure of the SS of the curve is found by analysing this bifurcation set. In contrast, the alternative analysis of the perpendicular bisector map leads to the classification of the local structure of the dual-SS. However, in the affine case, the AESS is *not* a bifurcation set: the analogous construction of the *bifurcation set of the affine distance function* leads to an entirely different set, the ADSS, which is the subject of Chapters 3, 5 and 6. Thus in the case of the AESS, our main tool in the classification of the local structure will be to mimic the study of the perpendicular bisector map to probe the structure of the dual-AESS, and then relate our results back to the AESS itself.

2.4.1 Introducing the *'Midline Map'*

We define an affine-invariant replacement for the *Perpendicular Bisector Map* as follows. For any two distinct points $\gamma(t_1), \gamma(t_2)$ on an oval γ, we will call the line joining m and p (by adopted convention, respectively the midpoint of the chord joining $\gamma(t_1)$ and $\gamma(t_2)$ and the point of intersection of the tangents to γ at these points) the *'midline'*. Furthermore, for a repeated pair of points, $\gamma(t)$, we will define the midline to be the affine normal to γ at $\gamma(t)$. The geometric justification for this will be explained presently.

Definition 2.4.1. *For a smooth, simple closed curve γ, the* **Midline Map**

$$M\colon S^1 \times S^1 \to \mathcal{L},$$

where S^1 is a circle parametrising γ, and $\mathcal{L}$ is the dual-plane (whose points are the lines of the ordinary affine plane), is defined by

- *$t_1 \neq t_2$: $M(t_1, t_2)$ is the* midline *of $\gamma(t_1)$ and $\gamma(t_2)$, that is the line joining the point of intersection of the tangents to γ at $\gamma(t_1), \gamma(t_2)$ to the midpoint of the chord between $\gamma(t_1), \gamma(t_2)$;*
- *$t_1 = t_2$: $M(t_1, t_2)$ is the* affine normal line *to γ at $\gamma(t_1)$.*

Remark 2.4.2.

(i) When the tangents at $\gamma(t_1), \gamma(t_2)$ are parallel, then we take the midline to be the unique line through the midpoint of the chord joining $\gamma(t_1)$ and $\gamma(t_2)$, and parallel to the tangent lines.

(ii) This definition of the Midline Map seems suitable for any *generic simple, smooth plane curve, remembering that, at an inflexion of γ, we define the affine normal to be in the same direction as the affine tangent, and of infinite length (see §1.3.1). However, for simplicity we will continue to assume that the curve γ is an oval. The study of the Midline Map for non-ovals is undertaken in §2.5, where it becomes clear that a slightly different approach is required.*

For a fixed and distinct pair t_1, t_2, the line $M(t_1, t_2)$ is the locus of centres of conics having (at least) 2-point contact with γ at $\gamma(t_1), \gamma(t_2)$ (the reader is referred to Proposition 1.4.10 for a justification of this statement). When $t_2 \to t_1$, we shall obtain the locus of centres of conics having (at least) 4-point contact with γ at $\gamma(t_1)$, which is precisely the affine normal at this point. Thus the Midline Map M is *continuous* at points (t, t).

2.4.2 The dual-AESS as (a subset of) the critical locus of the Midline Map

The original exposition of some of this work can be found in [GS96] and [GS98]. Here we present a summary of the methods and conclusions of these articles, and also extend the results where noted.

For an oval γ, and two fixed points $\gamma(t_1), \gamma(t_2)$ on it, consider the pencil of conics $\mathcal{C}$ which are tangent to γ at these two points. From §1.4.1, we know that the general conic of the pencil is $\mathcal{T}_1\mathcal{T}_2 + \lambda\mathcal{L}^2 = 0$, for some real parameter λ, extended as usual by $\lambda = \infty$, where $\mathcal{T}_1 = 0$ and $\mathcal{T}_2 = 0$ are equations of the tangent lines to γ at $\gamma(t_1)$ and $\gamma(t_2)$, and $\mathcal{L} = 0$ is the line joining these two points. Let $\mathcal{M} = 0$ denote the equation of the line passing through p and m. Then by Proposition 1.4.10, the locus of centres of the conics in the pencil $\mathcal{T}_1\mathcal{T}_2 + \lambda\mathcal{L}^2 = 0$ is the line pair $\mathcal{LM} = 0$, consisting of two components, namely the line $\mathcal{M} = 0$, passing through the intersection of $\mathcal{T}_1$ and $\mathcal{T}_2$ and the midpoint of the chord from $\gamma(t_1)$ to $\gamma(t_2)$, and a degenerate component, namely the line $\mathcal{L} = 0$, taken as a repeated line $\mathcal{L}^2 = 0$ of the pencil, whose centre is indeterminate on $\mathcal{L} = 0$. Thus for each pair (t_1, t_2), the locus of centres of the pencil of conics thus prescribed defines the midline $\mathcal{M}(t_1, t_2) = 0$, and it is this locus of points which we will study in more detail.

To link the $\mathcal{M} = 0$ component of the locus of centres to the Midline Map M, consider the conic $\mathcal{C}$ of the form

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0, \qquad (2.16)$$

for homogeneous coefficients $a, b, h, g, f, c \in \mathbb{R}$. We write $\gamma(t) = (X(t), Y(t))$, and introduce some abbreviations in order to write down our later results succinctly. By $\mathcal{C}(t_i)$ we will mean the vector

$$\left(X^2, Y^2, 2XY, 2X, 2Y, 1\right)$$

evaluated at $t = t_i$ $(i = 1, 2)$, and by $\mathcal{C}_t(t_i)$ we mean the vector

$$(2XX', 2YY', 2(XY' + X'Y), 2X', 2Y', 0)$$

evaluated at $t = t_i$, where $'$ (prime) denotes the derivative with respect to t. Now $\mathcal{C}$ is tangent to γ at $\gamma(t_1)$, and we can express this with the following two equations:

$$\mathcal{C}(t_1) \cdot (a, b, h, g, f, c)^T = 0,$$
$$\mathcal{C}_t(t_1) \cdot (a, b, h, g, f, c)^T = 0.$$

Similarly, the fact that $\mathcal{C}$ is tangent to γ at $\gamma(t_2)$ can be expressed by the pair of linear equations below:

$$\mathcal{C}(t_2) \cdot (a, b, h, g, f, c)^T = 0,$$
$$\mathcal{C}_t(t_2) \cdot (a, b, h, g, f, c)^T = 0.$$

We also introduce vectors $\mathcal{C}_x, \mathcal{C}_y$ which are given by

$$\mathcal{C}_x(x, y) = (2x, 0, 2y, 2, 0, 0),$$
$$\mathcal{C}_y(x, y) = (0, 2y, 2x, 0, 2, 0),$$

obtained by differentiating (2.16) with respect to x and y and writing the coefficients of a, b, h, g, f, c in vector form. The coordinates of the centre (p, q) of conic $\mathcal{C}$ satisfy the two linear equations obtained by differentiating (2.16) w.r.t. x and y and substituting $x = p, y = q$, and these equations can be expressed as

$$\mathcal{C}_x(p, q) \cdot (a, b, h, g, f, c)^T = 0,$$
$$\mathcal{C}_y(p, q) \cdot (a, b, h, g, f, c)^T = 0.$$

Thus we have six linear equations which must hold for $\mathcal{C}$ to have 2-point contact with γ at $\gamma(t_1)$ and $\gamma(t_2)$ and centre at $(x, y) = (p, q)$, from which we may eliminate a, b, h, g, f, c. The determinant condition that there should exist a conic $\mathcal{C}$ as in (2.16), with these properties (and with not all coefficients zero) is

$$\mathcal{G}(p,q,t_1,t_2) := \begin{vmatrix} \mathcal{C}(t_1) \\ \mathcal{C}_t(t_1) \\ \mathcal{C}(t_2) \\ \mathcal{C}_t(t_2) \\ \mathcal{C}_x(p,q) \\ \mathcal{C}_y(p,q) \end{vmatrix} = 0. \qquad (2.17)$$

Expression (2.17) is the equation, in (p,q)-coordinates, for the locus of centres of conics $\mathcal{C}$ having 2-point contact with γ at $\gamma(t_1)$ and $\gamma(t_2)$. Then Proposition 1.4.10 tells us that

$$\mathcal{G} \equiv \mathcal{M}\mathcal{L}.$$

Suppose that $\mathcal{G}(p,q,t_1,t_2) = 0$, that is suppose that (p,q) is the centre of a conic tangent to γ at $\gamma(t_1), \gamma(t_2)$. So long as (p,q) is not at the midpoint of the chord joining these points on γ (i.e. the intersection of $\mathcal{M} = 0$ and $\mathcal{L} = 0$), we can deduce that

$$\mathcal{L}(p,q,t_1,t_2) \neq 0.$$

Now chord $\mathcal{L} = 0$ is a diameter of the conic, and since tangents to a conic at opposite ends of a diameter are parallel, the only case that causes problems here is where the tangents to γ at $\gamma(t_1)$ and $\gamma(t_2)$ are parallel. Away from this situation, the zeros of $\mathcal{G}$ coincide with those of $\mathcal{M} = 0$, and the same holds for derivatives: for example, $\mathcal{G} = \mathcal{G}_t = 0$ is equivalent, away from $\mathcal{L} = 0$, to $\mathcal{M} = \mathcal{M}_t = 0$.

The function $\mathcal{M}$ defines for us a 2-parameter family of lines in the plane, parametrised by pairs of (distinct) points of γ. We now ask: *Can $\mathcal{M}$ be extended smoothly to* all *pairs of points?* The answer is that it can, by defining $\mathcal{M}(p,q,t,t) = 0$ to be an equation of the *affine normal* to γ at $\gamma(t)$. The geometric reason for the continuity of M at (t,t) was outlined in §2.4.1, and the smoothness of M is proved in §2.5.1. In this way, $\mathcal{M}$ naturally gives rise to the *Midline Map* as defined in §2.4.1 (Definition 2.4.1), where $M(t_1,t_2)$ is the line whose equation in the current coordinates (p,q) is $\mathcal{M}(p,q,t_1,t_2) = 0$.

We can view $\mathcal{M}$ (or M) as defining two envelopes of lines, one by fixing t_1 and the other by fixing t_2. We use the function $\mathcal{G}$ to measure the contact between conic $\mathcal{C}$ and γ, and use the relation $\mathcal{G} = \mathcal{ML}$ to compare this contact with the properties of the two envelopes. The Midline Map M is the affine-invariant analogue of the Perpendicular Bisector Map. Since M is a map from the plane to the plane (at least locally), its critical points are given by the vanishing of a (2×2) Jacobian determinant. The *critical locus* of M is then the image under M of this critical set, and is a set of lines in the plane, the dual-AESS.

2.4.3 The Midline Map for an oval

We now link the *Midline Map M* with the AESS, or more accurately the dual-AESS, the set of tangents to the AESS. In what follows in this section, $\mathcal{C}$ will always denote the conic tangent to γ at $\gamma(t_1)$ and $\gamma(t_2)$, and having centre at (p, q). Using the notation of §2.4.2, we have the following series of results:

Proposition 2.4.3.

(i) $\mathcal{G} = \mathcal{G}_{t_1} = 0$ *if and only if the conic* $\mathcal{C}$ *has (at least) 3-point contact with* γ *at* $\gamma(t_1)$.

(ii) $\mathcal{M} = \mathcal{M}_{t_1} = 0$ *if and only if* $\mathcal{G} = \mathcal{G}_{t_1} = 0$ or *the tangent lines to* γ *at* $\gamma(t_1)$ *and* $\gamma(t_2)$ *are parallel.*

(iii) *Let* $E \equiv \{(t_1, t_2, p, q) : \mathcal{M} = \mathcal{M}_{t_1} = \mathcal{M}_{t_2} = 0\}$. *The projection of* E *onto the* (p, q)*-plane consists of the* AESS *together with the* affine evolute *(corresponding to* $t_1 = t_2$*) and another affine-invariant symmetry set, the* MPTL*, defined to be the* set midpoints of the chords joining points of contact of parallel tangent pairs *(see Definition 2.4.8).*

(iv) *The tangent to the AESS is the line* $\mathcal{M} = 0$*, which passes through the midpoint of the chord joining the points of* γ *where* $\mathcal{C}$ *has 3-point contact, and through the intersection of the tangents to* γ *at these two points.*

(v) *The set of critical values of the Midline Map M consists of the tangents to the AESS and to the affine evolute, together with the lines lying half-way between pairs of parallel tangents of γ, that is, the set of tangents to the MPTL.*

(vi) *Fixing t_2, p, q, the function $\mathcal{G}$ is zero and has a singularity of type A_k at t_1 (i.e. the first k derivatives with respect to t_1 vanish but the $(k+1)$-th derivative is non-zero), if and only if the conic $\mathcal{C}$ has $(k+2)$-point contact with γ at $\gamma(t_1)$.*

Proof. Without loss of generality, we fix $\gamma(t_2)$ at the origin, and the tangent there as the x-axis. Conic $\mathcal{C}$ then has the special form

$$ax^2 + 2hxy + by^2 + 2fy = 0,$$

with four homogeneous coefficients $a, b, f, h \in \mathbb{R}$. The line $\mathcal{M}$ joins the intersection of the tangent at $\gamma(t_1) = (X(t_1), Y(t_1))$ and the x-axis to the midpoint of the segment from the origin to $\gamma(t_1)$. The function $\mathcal{G}$ is given by eliminating a, b, f, h, and is the (4×4) determinant (omitting the columns consisting only of zeros)

$$\mathcal{G}(t_1, p, q) \equiv \begin{vmatrix} \mathcal{C}(t_1) \\ \mathcal{C}_t(t_1) \\ \mathcal{C}_x(p, q) \\ \mathcal{C}_y(p, q) \end{vmatrix}, \tag{2.18}$$

As noted earlier, $\mathcal{G} = 0$ is the equation (in (p, q) coordinates) of the locus of centres of conics tangent to γ at $\gamma(t_1)$ and tangent to the x-axis at the origin.

(i) We need to connect the number of vanishing derivatives of $\mathcal{G}$ to the order of contact of conic $\mathcal{C}$ with γ at $\gamma(t_1)$. By differentiating $\mathcal{G}$ w.r.t.

t we get

$$\mathcal{G}_t(t_1,p,q) \equiv \begin{vmatrix} \mathcal{C}_t(t_1) \\ \mathcal{C}_t(t_1) \\ \mathcal{C}_x(p,q) \\ \mathcal{C}_y(p,q) \end{vmatrix} + \begin{vmatrix} \mathcal{C}(t_1) \\ \mathcal{C}_{tt}(t_1) \\ \mathcal{C}_x(p,q) \\ \mathcal{C}_y(p,q) \end{vmatrix} = \begin{vmatrix} \mathcal{C}(t_1) \\ \mathcal{C}_{tt}(t_1) \\ \mathcal{C}_x(p,q) \\ \mathcal{C}_y(p,q) \end{vmatrix}. \tag{2.19}$$

Suppose that $\mathcal{G} = \mathcal{G}_t = 0$ at (t_1, p, q). Then $\mathcal{G} = 0$ tells us that $\mathcal{C}_t(t_1)$ is a linear combination of linearly independent vectors $\mathcal{C}(t_1)$, $\mathcal{C}_x(p,q)$ and $\mathcal{C}_y(p,q)$ (from (2.18)). Similarly, $\mathcal{G}_t = 0$ tells us that $\mathcal{C}_{tt}(t_1)$ is a linear combination of $\mathcal{C}(t_1)$, $\mathcal{C}_x(p,q)$ and $\mathcal{C}_y(p,q)$ (from (2.19)). Thus the matrix

$$\begin{pmatrix} \mathcal{C}(t_1) \\ \mathcal{C}_t(t_1) \\ \mathcal{C}_{tt}(t_1) \\ \mathcal{C}_x(p,q) \\ \mathcal{C}_y(p,q) \end{pmatrix} \tag{2.20}$$

has rank ≤ 3. However, the rank of matrix (2.20) being ≤ 3 is precisely the condition for conic $\mathcal{C}$ to have 3-point contact with γ at $\gamma(t_1)$, that is, that the same values of a, b, f, h should allow $\mathcal{C} = \mathcal{C}_t = \mathcal{C}_{tt} = 0$. The result then follows.

(ii) By Proposition 1.4.10, we can write $\mathcal{G}(t_1, p, q)$ from (2.18) as

$$\mathcal{G} = \mathcal{M}(Xq - Yp),$$

for some suitable equation $\mathcal{M} = 0$ of the line $\mathcal{M}$. Suppose the second factor is non-zero, that is, (p,q) is not the midpoint of the chord, which corresponds to saying that the tangent to γ at $\gamma(t_1)$ is not parallel to the x-axis. Then $\mathcal{G}$ and its first k derivatives w.r.t. t vanish at (t_1, p, q) if and only if the same applies to $\mathcal{M}$, that is, the functions $\mathcal{G}$ and $\mathcal{M}$ have the same singularity type so long as $\mathcal{M} = 0$ and (p,q) is not on the line $\mathcal{L}$.

Naturally, both (i) and (ii) hold when t_1 is replaced everywhere by t_2.

(iii) This follows from (i) and (ii), except for the case where the tangents to

γ at $\gamma(t_1)$ and $\gamma(t_2)$ are parallel. In this case, we may, without loss of generality, perform an initial affine transformation so that the points of contact of the parallel tangents are at $(0,0)$ and $(0, f(0))$, where $y = f(x)$ is the equation of γ near $\gamma(t_1)$, the parallel tangents are respectively the x-axis and the line $y = f(0)$, and $f'(0) = 0$. Fixing t_2 to give the origin, the equation of the line $\mathcal{M}(t_1, t_2, p, q) = 0$ joining the intersection of the x-axis and the tangent to γ at $\gamma(t_1)$ to the midpoint of the chord joining the origin to $\gamma(t_1)$, is given by

$$f(t_1)f'(t_1)p + (t_1 f'(t_1) - 2f(t_1))q - t_1 f(t_1)f'(t_1) + f(t_1)^2 = 0.$$

It then follows that

$$\mathcal{M}(0,0,p,q) = \mathcal{M}_{t_1}(0,0,p,q) = 0 \Longleftrightarrow (p,q) = \left(0, \frac{1}{2}f(0)\right).$$

By symmetry, the same holds when the roles of t_1 and t_2 are reversed.

(iv) This follows immediately from (i) and (ii), except the case where the tangents to γ at $\gamma(t_1)$ and $\gamma(t_2)$ are parallel, which follows from (iii): here, the limiting tangent to the AESS at the midpoint $(0, f(0)/2)$ will be the line $\mathcal{M}$, by continuity.

(v) This follows from the observation that the set of critical values of M is the line $\mathcal{M}$ for which there exists t_1, t_2, p, q such $\mathcal{M} = \mathcal{M}_{t_1} = \mathcal{M}_{t_2} = 0$.

(vi) Follows by repeating the argument of (i) with more derivatives of $\mathcal{G}$.

□

Remark 2.4.4. *Regarding Proposition 2.4.3(iii), the appearance of the MPTL in the projection of the set E is due to the fact that, although the three conditions defining E are usually enough to guarantee that there is a 3+3 conic at $\gamma(t_1)$ and $\gamma(t_2)$, we find that when the tangents are parallel then the three conditions are automatically satisfied by the midpoint of the chord of contact. Of course it could be happen that there is a 3+3 conic at parallel tangent points, and in this case it is indeed true that the midpoint of the chord is the centre of that conic.*

Proposition 2.4.3(iv) asserts the following:

Proposition 2.4.5 (Concurrent Tangents Condition). *The tangents to the curve at points of contact of a 3+3 conic are concurrent with the tangent to the AESS at the centre of this conic.*

Part (v) of Proposition 2.4.3 is crucial, since it asserts that the *critical locus of the Midline Map* is the *dual of the union of the AESS, the MPTL and the affine evolute.* It allows us to redefine the AESS.

Definition 2.4.6. *The AESS $\cup$ MPTL $\cup$ affine evolute is the* dual of the critical locus of the Midline Map M.

We are now in a position to prove the following:

Proposition 2.4.7. *The AESS of a generic smooth plane curve γ through points $\gamma(t_1)$ and $\gamma(t_2)$, where neither point is an inflexion, exhibits a cusp singularity (that is, the dual-AESS exhibits an inflexion) when either of the following happen:*

- **(a)** *The conic $\mathcal{C}$ has 4-point contact with γ at either $\gamma(t_1)$ or $\gamma(t_2)$.*
- **(b)** *The tangents to γ at $\gamma(t_1)$ and $\gamma(t_2)$ are parallel.*

(See §2.5 *for the excluded cases where one or more point is an inflexion.)*

Proof. (outline)

- **(a)** Let us suppose we do not have parallel tangents, and consider the set E in Proposition 2.4.3(iii). We are looking for points where the tangent line to E projects to a point under the projection from (t_1, t_2, p, q) to (p, q). Writing down the Jacobian matrix of the equations defining E we find that that this happens precisely when $\mathcal{M}_{t_1t_1} = 0$ or $\mathcal{M}_{t_2t_2} = 0$, and the result follows from Proposition 2.4.3(vi).
- **(b)** This is proved in §2.5.7, where we will see that, at parallel tangent points, it is natural to study the AESS augmented by the MPTL, as introduced in Proposition 2.4.3(iii), and discussed again in §2.4.4 and §2.7. We show in Proposition 2.5.9 that the dual-AESS $\cup$ dual-MPTL exhibits a dual beaks singularity at points of contact of 3+3 conics which also have parallel tangents.

□

Earlier work in [GS96] and [GS98] omitted the consideration of the MPTL in Proposition 2.4.3(iii) and (v), and part (b) of the above Proposition 2.4.7. These amendments demonstrate the important and unexpected role that parallel tangents play in this construction of the AESS. The MPTL will prove to be a very important affine-invariant set, and it will often provide a neat geometrical link between numerous apparently unrelated affine-invariant symmetry sets, in particular linking the AESS and the AASS of §2.7.

The pre-AESS revisited

In §2.2.3 it is shown that horizontal and vertical tangents to the pre-AESS correspond with cusps on the AESS, enabling us to spot cusps on the AESS by studying at the pre-AESS. However, we have just shown that cusps of the AESS can appear in another situation, namely when we have parallel tangent lines at the corresponding curve points, and these cusps cannot (presently) be identified by looking at the pre-AESS. We will require also the *pre-MPTL*, defined to be the set of parameter values for pairs of points which share a tangent direction. This type of cusp will then correspond to points which lie on *both* the pre-AESS and the pre-MPTL.

We will see in §2.5.4, when we consider the Midline Map for non-ovals, that there is another situation in which a cusp may appear on the AESS.

2.4.4 The Mid-Parallel-Tangents Locus

Definition 2.4.8. *The* **MPTL** *of a smooth plane curve γ is the locus of midpoints of chords joining points of contact of parallel tangents to γ.*

The MPTL is an interesting set in its own right, but its true value to us is in its relationship with the AESS. We will see that studying the AESS and the MPTL together aids our analysis, since we are often able to deduce or explain facts about the AESS with reference to the MPTL. On a visual level, it is interesting to see how neatly the AESS and the MPTL fit together (see Figure 2.7 for an example), and this strengthens our assertion that these two sets should be studied together. We begin our analysis of the MPTL with a

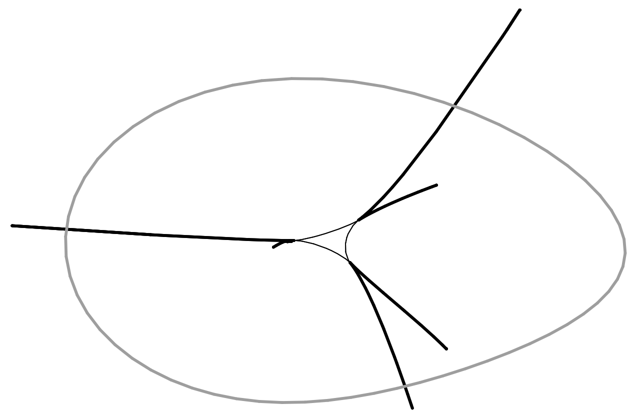


Figure 2.7: *The AESS ∪ MPTL for an oval. The original curve is shown grey, the thinner, 3-cusped closed curve is the MPTL, and the thicker curve the AESS.*

view to probing its links with the AESS.

Proposition 2.4.9. *The MPTL generically has an ordinary cusp at the centre of a 3+3 conic.*

Proof. We must revert to coordinate-wise calculations, using the standard coordinate system as set up in §2.3, and where we take $c = 0$, without loss of generality, to make calculations simpler. If γ_1 and γ_2 have parallel tangents at parameter values $t_1 = t_2 = 0$, then $b_1 = 0$. Taking

$$\begin{aligned} f(t_1) &= a_2 t_1^2 + a_3 t_1^3 + \ldots, \\ g(t_2) &= b_2 t_2^2 + b_3 t_2^3 + \ldots, \end{aligned}$$

the *parallel tangents condition* is $f'(t_1) = g'(t_2)$, where $'$ (prime) denotes differentiation with respect to the corresponding parameter t_1 or t_2. We solve this for t_2 as a function of t_1, say

$$t_2 = u(t_1) \equiv u_1 t_1 + u_2 t_1^2 + u_3 t_1^3 + \ldots.$$

We find that

$$\begin{aligned}
u_1 &= \frac{a_2}{b_2},\\
u_2 &= \frac{3(a_3b_2^2 - b_3a_2^2)}{2b_2^3},\\
u_3 &= \frac{1}{b_2^5}\left(2b_2(a_4b_2^3 - b_4a_2^3) - \frac{9}{2}a_2b_3(a_3b_2^2 - b_3a_2^2)\right).
\end{aligned}$$

So we are now able to parametrise the MPTL by parameter t_1. We will denote the MPTL by $M(t_1)$, where

$$\begin{aligned}
M(t_1) &= \frac{1}{2}\left(t_1 + t_2, d + f(t_1) + g(t_2)\right)\\
&= \frac{1}{2}\left(t_1 + u(t_1), d + f(t_1) + g(u(t_1))\right)\\
&= \frac{1}{2}\left(t_1 + \frac{a_2}{b_2}t_1 + \frac{3}{2b_2^3}(a_3b_2^2 - b_3a_2^2)t_1^2\right.\\
&\quad + \frac{1}{b_2^5}\left(2b_2(a_4b_2^3 - b_4a_2^3) - \frac{9}{2}a_2b_3(a_3b_2^2 - b_3a_2^2)\right)t_1^3 + \ldots,\\
&\quad \left. a_2t_1^2 + a_3t_1^3 + d + b_2\left(\frac{a_2^2}{b_2^2}t_1^2 + \left(\frac{3a_2}{b_2^4}(a_3b_2^2 - b_3a_2^2)\right)t_1^3\right) + b_3\frac{a_2^3}{b_2^3}t_1^3 + \ldots\right)
\end{aligned}$$

and thus

$$\begin{aligned}
M'(0) &= \frac{1}{2}\left(1 + \frac{a_2}{b_2}, 0\right)\\
&= (0,0) \iff a_2 = -b_2.
\end{aligned}$$

So the MPTL is smooth near $t_1 = 0$ unless $a_2 = -b_2$, which is precisely the condition (2.15) of the corollary to the '3+3 Conic Condition' for there to be a 3+3 conic at parameter values $t_1 = t_2 = 0$. Hence *the MPTL is singular at the centre of a 3+3 conic.*

To show that this singularity is an ordinary cusp, we must check that the vectors $M''(0), M'''(0)$ are linearly independent when $a_2 = -b_2$, and a short calculation tells us that this is true so long as $a_3 \neq b_3$, assuming that $a_2 = b_2 \neq 0$. So the MPTL fails to have an ordinary cusp if and only if $a_3 = b_3$, i.e. if and only if $\kappa_1'(0) = \kappa_2'(0)$, which is a non-generic occurence.

Hence the MPTL generically has a cusp at the centre of a 3+3 conic. □

Remark 2.4.10. *It is interesting to note that the condition for the MPTL to have an ordinary cusp can be expressed in terms of Euclidean curvatures, even though the condition itself should be affine. Although Euclidean curvature, and the rate of change of Euclidean curvature, are not affine-invariants,* equality *of Euclidean curvature and its rate of change* are *affine-invariants between two pairs of points on a curve having parallel tangents since, if two points with parallel tangents have* $\kappa_1 = -\kappa_2$*, and* $\kappa_1' = \kappa_2'$*, then this will remain true after an affine transformation. Thus the Euclidean interpretation of this affine condition makes sense.*

The observant reader will have noted the interesting duality of this situation. In §2.4.3 (Proposition 2.4.7), we showed that the AESS has a cusp when we have parallel tangents at the points of contact of a 3+3 conic, and here we have shown that the MPTL has a cusp at the centre of a 3+3 conic. Hence the AESS and the MPTL exhibit cusps at the points where they meet. We will show later, in §2.5.7 that the (dual-) AESS and the (dual-) MPTL taken together here form a (dual-) beaks singularity.

In §2.7, we consider the MPTL again, showing that it can also be defined as the *bifurcation set* of a two parameter family of smooth 'Area' functions defined on the curve, and studying it as part of a full bifurcation set in conjunction with another affine symmetry set, the AASS. Thus we have the interesting fact that the MPTL can be defined as part of the dual of the *critical locus* of a map from the plane to the plane (together with the AESS and the affine evolute), and also as part of a full bifurcation set (along with the AASS). This is reminiscent of the multiple definitions of the Euclidean Symmetry Set. This dual definition of the MPTL allows us to link two affine-invariant symmetry sets, and increases our understanding of both.

2.5 The Local Structure of the AESS and MPTL for Non-Ovals

In this section we will consider how the Midline Map behaves when we move away from studying strictly convex curves, and allow our curve to exhibit non-oval structures, namely *inflexions* and *double tangents*.

We approach the Midline Map from an alternative angle. This conceptual change is necessary to modify the definition of the Midline Map for certain non-oval situations.

Convention:

- l will always denote the line through m and p;
- v will always denote the direction of l; and
- $'$ (prime) will always denote the derivative w.r.t. the corresponding parameter along γ_1, γ_2.

Consider two smooth curve segments γ_1 and γ_2. We have

$$m = \frac{1}{2}\left(\gamma_1 + \gamma_2\right),$$

and

$$\begin{aligned} p &= \gamma_1 + \lambda_1 \gamma_1' && \text{for } \lambda_1 \in \mathbb{R} \\ &= \gamma_2 + \lambda_2 \gamma_2' && \text{for } \lambda_2 \in \mathbb{R} \end{aligned}$$

which gives

$$\gamma_1 - \gamma_2 = \lambda_2 \gamma_2' - \lambda_1 \gamma_1'.$$

Bracketing this expression with γ_1' and γ_2', we have

$$[\gamma_1 - \gamma_2, \gamma_1'] = \lambda_2 [\gamma_2', \gamma_1'], \tag{2.21}$$

$$[\gamma_1 - \gamma_2, \gamma_2'] = -\lambda_1 [\gamma_1', \gamma_2']. \tag{2.22}$$

We see that (2.21) and (2.22) are valid expressions for λ_1 and λ_2 as long as $[\gamma_1', \gamma_2'] \neq 0$, that is, as long as the tangents to the two curve segments are not parallel, and therefore p is a finite point. We may assume generically that this is the case. So

$$p = \gamma_1 - \frac{[\gamma_1 - \gamma_2, \gamma_2']}{[\gamma_1', \gamma_2']} \gamma_1' = \gamma_2 - \frac{[\gamma_1 - \gamma_2, \gamma_1']}{[\gamma_1', \gamma_2']} \gamma_2',$$

and therefore we may write

$$p = \frac{1}{2}(\gamma_1 + \gamma_2) - \frac{1}{2[\gamma_1', \gamma_2']}\left([\gamma_1 - \gamma_2, \gamma_2']\gamma_1' + [\gamma_1 - \gamma_2, \gamma_1']\gamma_2'\right).$$

Thus we have

$$p - m = -\frac{1}{2[\gamma_1', \gamma_2']}\left([\gamma_1 - \gamma_2, \gamma_2']\gamma_1' + [\gamma_1 - \gamma_2, \gamma_1']\gamma_2'\right).$$

Now the line l consists of points x which satisfy

$$[x - m, p - m] = 0. \tag{2.23}$$

Note that this expression is valid for any parametrisation of the curve segments γ_1, γ_2. Let vector v be given by

$$v \equiv [\gamma_1 - \gamma_2, \gamma_2']\gamma_1' + [\gamma_1 - \gamma_2, \gamma_1']\gamma_2'.$$

Then v is parallel to $p - m$, since we are assuming that $[\gamma_1', \gamma_2'] \neq 0$. Then (2.23) can be rewritten as

$$[x - m, v] = 0,$$

which in turn can be re-written as

$$[x, v] = r \in \mathbb{R}. \tag{2.24}$$

So the line l is defined by the pair

$$\begin{aligned} v &= [\gamma_1 - \gamma_2, \gamma_2']\gamma_1' + [\gamma_1 - \gamma_2, \gamma_1']\gamma_2', && (2.25) \\ r &= \frac{1}{2}[\gamma_1 + \gamma_2, [\gamma_1 - \gamma_2, \gamma_2']\gamma_1' + [\gamma_1 - \gamma_2, \gamma_1']\gamma_2'], \\ &= \frac{1}{2}[\gamma_1 + \gamma_2, v], && (2.26) \end{aligned}$$

with equivalence relation $(v, r) \sim (\lambda v, \lambda r)$ for $\lambda \in \mathbb{R}\backslash\{0\}$. We see that $v = (0, 0)$ only in the following two situations:

(i) $\gamma_1 = \gamma_2$, which occurs when we are considering the Midline Map on a single curve segment;

(ii) $\gamma_1 - \gamma_2, \gamma_1'$ and γ_2' are parallel, which is the situation where there exists a double tangent to the curve segment.

These are the two cases in which the Midline Map, as presently defined, is not immediately suitable, namely at a *single curve segment*, and at a *double tangent to the curve segment*. Geometrically, v determines the direction of line l, and in both of the above cases it is not clear what the limit of the direction v should be, and hence the Midline Map as it is presently expressed is unsuitable. We will now consider the limiting value of vector v in each of these two cases, with the aim of finding a suitable expression for the Midline Map in any situation.

2.5.1 Single curve segment

To find a suitable expression for the direction of vector v in the situation where we have a single curve segment, consider γ_1 and γ_2 to be identical curve segments given by

$$\begin{aligned}\gamma_1(t_1) &= (t_1, f(t_1)),\\ \gamma_2(t_2) &= (t_2, f(t_2)),\end{aligned}$$

where $f(0) = f'(0) = 0$. We will consider the limiting value of the vector v at $(t_1, t_2) = (0, 0)$. From (2.25) we have

$$v(t_1,t_2) = \begin{vmatrix} t_1 - t_2 & f(t_1) - f(t_2) \\ 1 & f'(t_2) \end{vmatrix} (1, f'(t_1)) + \begin{vmatrix} t_1 - t_2 & f(t_1) - f(t_2) \\ 1 & f'(t_1) \end{vmatrix} (1, f'(t_2)),$$

and splitting this into coordinates v_1 and v_2 of v gives us

$$\begin{aligned}v_1 &= (t_1 - t_2)f'(t_2) - f(t_1) + f(t_2) + (t_1 - t_2)f'(t_1) - f(t_1) + f(t_2),\\ v_2 &= ((t_1 - t_2)f'(t_2) - f(t_1) + f(t_2))f'(t_1)\\ &\quad +((t_1 - t_2)f'(t_1) - f(t_1) + f(t_2))f'(t_2).\end{aligned}$$

Clearly $v(0,0) = (0,0)$. The idea is to remove factors of $(t_1 - t_2)$ from $v(t_1, t_2)$ until we are left with factors of v_1 and v_2 which are non-zero at $(0, 0)$. By

Hadamard's lemma, we can write

$$f(t_1) - f(t_2) = (t_1 - t_2)h(t_1, t_2), \tag{2.27}$$

where h is a smooth function of t_1 and t_2. Differentiating (2.27) w.r.t. t_1 and t_2 in turn gives us

$$\begin{aligned} f'(t_1) &= h(t_1, t_2) + (t_1 - t_2)h_1(t_1, t_2), & (2.28) \\ f'(t_2) &= h(t_1, t_2) - (t_1 - t_2)h_2(t_1, t_2), & (2.29) \end{aligned}$$

where h_i denotes $\partial h/\partial t_i$. Note that $f'(0) = 0$ implies that $h(0, 0) = 0$, and shows that h is of order ≥ 1 in t_1, t_2. From now on we will omit the variables (t_1, t_2) in the expressions for h and its derivatives. We will also need the sum and product of expressions (2.28) and (2.29), respectively

$$\begin{aligned} f'(t_1) + f'(t_2) &= 2h + (t_1 - t_2)(h_1 - h_2), & (2.30) \\ 2f'(t_1)f'(t_2) &= 2h^2 - 2h(t_1 - t_2)(h_2 - h_1) - 2h_1h_2(t_1 - t_2)^2. & (2.31) \end{aligned}$$

We are now in a position to remove factors of $(t_1 - t_2)$ from v:

$$\begin{aligned} v_1(t_1, t_2) &\equiv (t_1 - t_2)(f'(t_2) + f'(t_1)) - 2(f(t_1) - f(t_2)), \\ &= (t_1 - t_2)(f'(t_2) + f'(t_1)) - 2(t_1 - t_2)h, \text{ by (2.27)}, \\ &= (t_1 - t_2)^2(h_1 - h_2), \text{ by (2.30)}, \end{aligned}$$

and similarly

$$v_2(t_1, t_2) = (t_1 - t_2)^2(h(h_1 - h_2) - 2h_1h_2(t_1 - t_2)).$$

Thus we can remove a factor of $(t_1 - t_2)^2$ from vector v. However, differentiating expressions (2.28) and (2.29) w.r.t. t_1 and t_2 gives us

$$\begin{aligned} f''(t_1) &= 2h_1 + (t_1 - t_2)h_{11}, & (2.32) \\ f''(t_2) &= 2h_2 - (t_1 - t_2)h_{22}, \end{aligned}$$

respectively, and thus

$$h_1(0,0) = h_2(0,0) = \frac{1}{2} f''(0). \tag{2.33}$$

Thus we must reduce v further. Differentiating (2.28) w.r.t. t_2 gives us

$$0 = h_2 - h_1 + (t_1 - t_2) h_{12}, \tag{2.34}$$

and substituting (2.34) into v we can write

$$v = (t_1 - t_2)^3 \left(h_{12}, h h_{12} - 2 h_1 h_2 \right). \tag{2.35}$$

Now differentiating (2.34) w.r.t. t_1 we see that

$$2h_{12} - h_{11} + (t_1 - t_2) h_{112} = 0,$$

and thus

$$h_{12}(0,0) = \frac{1}{2} h_{11}(0,0).$$

Furthermore, differentiating (2.32) w.r.t. t_1 we have

$$f'''(t_1) = 3h_{11} + (t_1 - t_2) h_{111},$$

and thus

$$h_{11}(0,0) = \frac{1}{3} f'''(0),$$

and therefore

$$h_{12}(0,0) = \frac{1}{6} f'''(0).$$

Using this, and (2.33), the limiting value of v as $t_1 \to 0, t_2 \to 0$ is

$$v(0,0) = \left(\frac{1}{6} f'''(0), -\frac{1}{2} f''(0)^2 \right),$$

and of course this is $(0,0)$ if and only if we have $f''(0) = f'''(0) = 0$, that is, there is a higher inflexion on the curve. Furthermore, from §1.3.3, we know that this is in the direction of the affine normal vector to the curve at the origin.

Expression (2.35) shows that v is of the form $(t_1 - t_2)^3(\alpha(t_1, t_2), \beta(t_1, t_2))$, where α and β are smooth functions of t_1, t_2 and generically are not both zero at $t_1 = t_2 = 0$. Thus we have shown:

Proposition 2.5.1. *The limiting midline at a diagonal point (t, t) is in the direction of the affine normal to the curve at t, and the Midline Map is smooth there.*

Note that this is true even when there is an inflexion on the curve, in which case $f''(0) = 0$, and the limiting value of v is along the inflexional tangent line, which of course is also in the direction of the affine normal there. Thus we have:

Proposition 2.5.2. *In the case of a single inflexional curve segment, the limiting value of v at the inflexion is in the direction of the inflexional tangent, and thus the limiting midline is the inflexional tangent line.*

By symmetry, the limiting point of the AESS at an inflexion on the original curve is at the inflexion itself, and we have shown above that the AESS approaches this point tangentially (see Figure 2.8). Thus we have:

Proposition 2.5.3. *The AESS approaches an inflexion tangentially and has an endpoint there.*

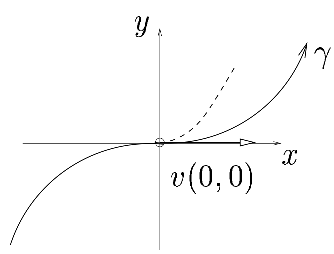


Figure 2.8: *Illustrating the conclusions of* §2.5.1. *The AESS is dashed.*

2.5.2 Double tangent to the original curve segment

It is natural to ask whether we can apply a similar procedure to derive the limiting value of v in the case of a double tangent, reducing v in such a way to find the limiting midline in this case. However, it can be shown that the Midline Map is discontinuous in this situation, and that there is no unique limiting direction for the midline l at a double tangent. Although no unique limiting midline exists, we can see that all possible limiting midlines pass through the midpoint of the chord joining the points of contact of the double tangent. It is thus possible to use the technique of 'blowing up' this point in the plane, thus changing the domain of definition of the Midline Map in such a way as to make it well defined.

However, we are able to deduce structure of AESS at a double tangent using a different method. Let γ_1 and γ_2 be two smooth curve segments parametrised by s_1 and s_2 respectively, given by

$$\gamma_1(s_1) = (X_1(s_1), Y_1(s_1)),$$
$$\gamma_2(s_2) = (X_2(s_2), Y_2(s_2)),$$

where

$$X_1(s_1) = s_1, \qquad Y_1(s_1) = a_2 s_1^2 + a_3 s_1^3 + \dots$$
$$X_2(s_2) = 1 + s_2, \quad Y_2(s_2) = b_2 s_2^2 + b_3 s_2^3 + \dots$$

The double tangent is thus along the x-axis. We will assume that $a_2 b_2 \neq 0$, so that neither point of contact is an inflexion. Then

$$\begin{aligned} k_1(s_1) &= \dot{X}_1(s_1)\ddot{Y}_1(s_1) - \ddot{X}_1(s_1)\dot{Y}_1(s_1), \\ &= 2a_2 + 6a_3 s_1 + 12a_4 s_2^2 + \dots \end{aligned}$$

and similarly

$$k_2(s_2) = 2b_2 + 6b_3 s_2 + 12b_4 s_2^2 + \dots$$

Then the condition (2.2) for s_1, s_2 to be on the pre-AESS can be written as

$$\Big((X_1 - X_2)\dot{Y}_1 - (Y_1 - Y_2)\dot{X}_1\Big)^3 k_2 + \Big((X_1 - X_2)\dot{Y}_2 - (Y_1 - Y_2)\dot{X}_2\Big)^3 k_1 = 0.$$

where we have assumed that $k_1 k_2 \neq 0$, and have have just written X_1, Y_1 for $X_1(s_1), Y_1(s_1)$, etc. for brevity. We would like to solve this equation for s_2

as a function of s_1, say

$$s_2(s_1) = u_1 s_1 + u_2 s_2^2 + \dots \tag{2.36}$$

Now we can write

$$k_1(s_1) = 2a_2 \left(1 + \frac{3a_3}{a_2} s_1 + \frac{6a_4}{a_2} s_1^2 + \dots\right),$$

Then

$$k_1(s_1)^{1/3} \;=\; (2a_2)^{1/3} \left(1 + \frac{a_3}{a_2} s_1 + \left(\frac{2a_2a_4 - a_3^2}{a_2^2}\right) s_1^2 + \dots\right).$$

Similarly,

$$k_2(s_2)^{1/3} = (2b_2)^{1/3} \left(1 + \frac{b_3}{b_2} s_2 + \left(\frac{2b_2b_4 - b_3^2}{b_2^2}\right) s_2^2 + \dots\right).$$

So the equation for the pre-AESS is, up to degree 2,

$$\begin{aligned} &((s_1 - 1 - s_2)\ (2a_2s_1 + 3a_3s_1^2) - a_2s_1^2 + b_2s_2^2)\,(2b_2)^{1/3}\left(1 + \frac{b_3}{b_2}s_2\right) \\ &+ \left((s_1 - 1 - s_2)(2b_2s_2 + 3b_3s_2^2) - a_2s_1^2 + b_2s_2^2\right)(2a_2)^{1/3}\left(1 + \frac{a_3}{a_2}s_2\right) + \dots = 0. \end{aligned}$$

Substituting (2.36) into this expression, and collecting terms, we find that

$$\begin{aligned} u_1 &= -a_2^{2/3}b_2^{-2/3}, \\ u_2 &= \frac{1}{2}\left(a_2^{4/3}b_2^{-4/3} - a_2^{2/3}b_2^{-2/3} - a_2^{4/3}b_2^{-7/3}b_3 - a_2^{-1/3}b_2^{-2/3}a_3\right). \end{aligned}$$

We now have a suitable expression for the pre-AESS, the set of parameter values which give 3+3 conics. We calculate the centres of these 3+3 conics using the expression given in Proposition 2.2.6 which maps two points on a curve to the centre of the conic having 3+3 contact at these two points. Of course, this was originally derived to find the centre of a *central* 3+3 conic, and so we will have to amend it so as to be able to use it in this case. This will then give us the locus of centres of 3+3 conics, that is, the AESS.

We then show that this locus is parametrised by s_1, and therefore smooth. From Proposition 2.2.6, we know that the centre of the conic having 3-point contact at both $\gamma_1(s_1)$ and $\gamma_2(s_2)$ is at

$$\frac{1}{2}(\gamma_1+\gamma_2)+\frac{1}{2}\left(\frac{[\gamma_1-\gamma_2,\gamma_1'][\gamma_1',\gamma_2']}{2[\gamma_1-\gamma_2,\gamma_1']-[\gamma_1',\gamma_2']^2}(\gamma_2'-\gamma_1')\right)$$

where γ_1, γ_2 denote $\gamma_1(s_1), \gamma_2(s_2)$, etc. Using identity $\gamma_i' = k_i^{-1/3}\dot{\gamma}_i$, we see that the centre is given by

$$\frac{1}{2}(\gamma_1+\gamma_2)+\frac{1}{2}\left(\frac{k_1^{-2/3}k_2^{-1/3}[\gamma_1-\gamma_2,\dot{\gamma}_1][\dot{\gamma}_1,\dot{\gamma}_2]}{2k_1^{-1/3}[\gamma_1-\gamma_2,\dot{\gamma}_1]-k_1^{-2/3}k_2^{-2/3}[\dot{\gamma}_1,\dot{\gamma}_2]^2}(k_2^{-1/3}\dot{\gamma}_2-k_1^{-1/3}\dot{\gamma}_1)\right),$$

We can rewrite this as

$$\frac{1}{2}(\gamma_1+\gamma_2)+\frac{1}{2}\left(\frac{k_1^{-1/3}[\gamma_1-\gamma_2,\dot{\gamma}_1][\dot{\gamma}_1,\dot{\gamma}_2]}{2k_1^{1/3}k_2^{2/3}[\gamma_1-\gamma_2,\dot{\gamma}_1]-[\dot{\gamma}_1,\dot{\gamma}_2]^2}(k_1^{1/3}\dot{\gamma}_2-k_2^{1/3}\dot{\gamma}_1)\right) \quad (2.37)$$

Now

$$[\dot{\gamma}_1,\dot{\gamma}_2] = -\left(2b_2\frac{a_2^{2/3}}{b_2^{2/3}}+2a_2\right)s_1+\dots$$

using the the fact that we have s_2 as a function of s_1 from (2.36). Thus $[\dot{\gamma}_1,\dot{\gamma}_2]$ is first order in s_1, assuming $a_2 \neq -b_2$, which is condition that the Euclidean curvatures of the two curve segments are not equal and opposite at the points of contact of the double tangent. We may assume generically that this is true. Furthermore, calculation gives us

$$[\gamma_1-\gamma_2,\dot{\gamma}_1] = -2a_2s_1+\left(a_2-2a_2\left(-\frac{a_2^{2/3}}{b_2^{2/3}}\right)-b_2\frac{a_2^{4/3}}{b_2^{4/3}}\right)s_1^2+\dots,$$

and thus $[\gamma_1-\gamma_2,\dot{\gamma}_1]$ is first order in s_1, since we assume that $a_2 \neq 0$. Thus

$$\frac{k_1^{-1/3}[\gamma_1-\gamma_2,\dot{\gamma}_1][\dot{\gamma}_1,\dot{\gamma}_2]}{2k_1^{1/3}k_2^{2/3}[\gamma_1-\gamma_2,\dot{\gamma}_1]-[\dot{\gamma}_1,\dot{\gamma}_2]^2}$$

is of first order in s_1, and thus expression (2.37) is smooth. Finally, we see that

$$\frac{1}{2}\left(\frac{k_1^{-1/3}[\gamma_1-\gamma_2,\dot{\gamma}_1][\dot{\gamma}_1,\dot{\gamma}_2]}{2k_1^{1/3}k_2^{2/3}[\gamma_1-\gamma_2,\dot{\gamma}_1]-[\dot{\gamma}_1,\dot{\gamma}_2]^2}(k_1^{1/3}\dot{\gamma}_2-k_2^{1/3}\dot{\gamma}_1)\right)\longrightarrow 0 \text{ as } s_1\longrightarrow 0.$$

Hence the centre, given in expression (2.37), tends to

$$\frac{1}{2}(\gamma_1(0)+\gamma_2(0)),$$

as $s_1 \to 0$, which is the midpoint of the chord joining the points of contact of the double tangent with γ_1 and γ_2.

Proposition 2.5.4. *The AESS at a double tangent is a smooth curve passing through the midpoint of the chord joining the points of contact of the double tangent.*

2.5.3 Degenerate 3+3 Conic Situations

We now consider the following degenerate 3+3 conic situations using the methods developed in previous sections to probe the resulting geometry of the AESS:

(i) The *inflexional tangent to the curve cuts the curve again.* The 3+3 conic is the inflexional tangent together with the tangent at the point of intersection, and the centre of the conic is at the intersection point.

(ii) *Two inflexional tangents meet.* The 3+3 conic is the two inflexional tangents, and its centre is at the intersection of these tangents.

Each of these situations occur generically for a non-oval plane curve. In Case (i), any inflexional tangent will cut the curve again, and this crossing will generically be transversal. In Case (ii), since inflexions are created in pairs, any two inflexions will contribute a finite point to the AESS, since generically these inflexional tangents will not be parallel and will intersect at a finite point.

Outline of Method

We will analyse Cases (i) and (ii) in turn. The procedure is as follows:

- We develop a local formula for the Midline Map as redefined in §2.5, given in terms of $v_1(t_1,t_2)$ and $v_2(t_1,t_2)$, the components of the direction of the midline, and denoted by $B(t_1,t_2)$.
- We calculate the partial derivatives of v_1 and v_2 up to third order (these are listed in Appendix A). We require these to define and analyse the structure of the critical set of B, denoted Σ_B, and given by

$$\Sigma_B \equiv \{(t_1,t_2):\ \det(JB)=0\},$$

where JB denotes the (2×2) Jacobian matrix of B.

- We consider the structure of the locus of points in (t_1,t_2)-space defined by an equation for the critical set Σ_B. This locus is the *pre-dual-AESS*.
- The *dual-AESS* is then the image of $B(\Sigma_B)$, and from this we can deduce the local structure of the AESS itself in each of the Cases (i) and (ii) listed above.

We begin by deriving a local form for the Midline Map. Consider the curve segments γ_1,γ_2 to be given by

$$\gamma_1(t_1)=(t_1,f(t_1)),\quad \gamma_2(t_2)=(c+t_2,d+g(t_2)),$$

where

$$f(t_1)=a_2t_1^2+a_3t_1^3+\dots,\quad g(t_2)=b_1t_2+b_2t_2^2+b_3t_2^3+\dots$$

From (2.25) we calculate the direction $v=(v_1,v_2)$ of the 'midline' to be given by

$$v_1(t_1,t_2) = (t_1-c-t_2)(f'(t_1)+g'(t_2))-2(f(t_1)-d-g(t_2)), \quad (2.38)$$

$$v_2(t_1,t_2) = (2(t_1-c-t_2)f'(t_1)g'(t_2) - (f(t_1)-d-g(t_2))(f'(t_1)+g'(t_2)). \quad (2.39)$$

We are interested in v_1, v_2, and their derivatives w.r.t. t_1, t_2, evaluated at $t_1 = t_2 = 0$, so we will use the following convention:

Convention: *The superscript '0' will represent an expression evaluated at* $t_1 = t_2 = 0$. *For example*

$$v_1^0 \equiv v_1(0,0) = -cg'(0) + 2d, \quad v_2^0 \equiv v_2(0,0) = dg'(0).$$

Thus we see that $v^0 \equiv (v_1^0, v_2^0) = (0,0)$ when $(c,d) = (0,0)$ (the case of a single inflexional curve segment), or $d = g'(0) = 0$ (the case of the double tangent). We will assume from now on that we have $v_1^0 \neq 0$ (this holds generically for Cases (i) and (ii)), and we have the *local Midline Map*, which we will denote by B, given locally as

$$(t_1, t_2) \mapsto \left(\frac{v_2}{v_1}, \frac{r}{v_1}\right) = \left(\frac{v_2}{v_1}, (t_1 + c + t_2)\frac{v_2}{v_1} - (f(t_1) + d + g(t_2))\right), \quad (2.40)$$

where v_1, v_2 are functions of t_1, t_2 as given in (2.38) and (2.39), and where the expression for r comes from (2.26), after omitting the factor of 1/2. We will use the shorthand

$$B(t_1, t_2) = (a(t_1, t_2), b(t_1, t_2)), \quad (2.41)$$

to express B in terms of functions a and b given by

$$\begin{aligned} a(t_1, t_2) &= \frac{v_2(t_1, t_2)}{v_1(t_1, t_2)}, \\ b(t_1, t_2) &= (t_1 + c + t_2)\frac{v_2(t_1, t_2)}{v_1(t_1, t_2)} - (f(t_1) + d + g(t_2)). \end{aligned}$$

During the analysis of §§2.5.4-2.5.7, we will require expressions for the partial derivatives of v_1 and v_2 with respect to t_1 and t_2, up to the third order derivative, and then evaluate each of these expressions at $t_1 = t_2 = 0$. This amounts to a list of forty expressions, which is contained in Appendix A for easy reference. We will denote derivatives by subscripts: for example, the second partial derivative of v_1 with respect to t_1 and then t_2 will be denoted

$v_{1_{t_1 t_2}}$. We will omit the parameters t_1, t_2 for brevity.

The critical set Σ_B of the local Midline Map B is given by the vanishing of the determinant of the Jacobian matrix of B. This requires us to calculate the derivatives of the coordinates of B as shown in expression (2.40), which are

$$a_{t_1} = \frac{v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}}}{v_1^2} \tag{2.42}$$

$$a_{t_2} = \frac{v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}}}{v_1^2} \tag{2.43}$$

$$b_{t_1} = \frac{v_2}{v_1} + (t_1 + c + t_2)\left(\frac{v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}}}{v_1^2}\right) - f' \tag{2.44}$$

$$b_{t_2} = \frac{v_2}{v_1} + (t_1 + c + t_2)\left(\frac{v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}}}{v_1^2}\right) - g' \tag{2.45}$$

We can now write down the Jacobian matrix,

$$JB = \begin{pmatrix} a_{t_1} & a_{t_2} \\ b_{t_1} & b_{t_2} \end{pmatrix},$$

which is

$$\begin{pmatrix} \frac{v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}}}{v_1^2} & \frac{v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}}}{v_1^2} \\ \frac{v_2}{v_1} + (t_1 + c + t_2)\left(\frac{v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}}}{v_1^2}\right) - f' & \frac{v_2}{v_1} + (t_1 + c + t_2)\left(\frac{v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}}}{v_1^2}\right) - g' \end{pmatrix}$$

which becomes

$$\begin{pmatrix} v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}} & v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}} \\ v_1 v_2 + (t_1 + c + t_2)(v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}}) - f' v_1^2 & v_1 v_2 + (t_1 + c + t_2)(v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}}) - g' v_1^2 \end{pmatrix}$$

upon removing a factor of $1/v_1^2$ from each entry. We are able to do this since we are taking v_1 to be non-zero at $t_1 = t_2 = 0$, and therefore in a neighbourhood of $(t_1, t_2) = (0, 0)$. We then calculate that

$$v_1^3 \det(JB) = (v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}})(v_2 - g' v_1) - (v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}})(v_2 - f' v_1).$$

Thus we are able to define the *critical set* of the mapping B, which we will denote Σ_B, to be

$$\begin{aligned}\Sigma_B &\equiv \{(t_1,t_2): \quad \det(JB)=0\},\\ &= \{(t_1,t_2): (v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}})(v_2 - g' v_1) = (v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}})(v_2 - f' v_1)\},\end{aligned}$$

If we define

$$F(t_1,t_2) \equiv (v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}})(v_2 - g' v_1) - (v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}})(v_2 - f' v_1), \quad (2.46)$$

then $F(t_1,t_2)=0$ is an equation for the set Σ_B, which defines the *pre-dual-AESS* as a subset of (t_1,t_2)-space. We are interested in the structure of this curve at $t_1=t_2=0$, and thus require expressions for F_{t_1} and F_{t_2}. Calculation shows that

$$\begin{aligned}F_{t_1}(t_1,t_2) &= \left(v_1 v_{2_{t_1 t_1}} - v_2 v_{1_{t_1 t_1}}\right)(v_2 - g' v_1)\\ &\quad + \left(v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}}\right)\left(v_{2_{t_1}} - g' v_{1_{t_1}}\right)\\ &\quad - \left(v_{1_{t_1}} v_{2_{t_2}} + v_1 v_{2_{t_1 t_2}} - v_{2_{t_1}} v_{1_{t_2}} - v_2 v_{1_{t_1 t_2}}\right)(v_2 - f' v_1)\\ &\quad - \left(v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}}\right)\left(v_{2_{t_1}} - f'' v_1 - f' v_{1_{t_1}}\right), \qquad (2.47)\\ F_{t_2}(t_1,t_2) &= \left(v_{1_{t_2}} v_{2_{t_1}} + v_1 v_{2_{t_1 t_2}} - v_{2_{t_2}} v_{1_{t_1}} - v_2 v_{1_{t_1 t_2}}\right)(v_2 - g' v_1)\\ &\quad + \left(v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}}\right)\left(v_{2_{t_2}} - g'' v_1 - g' v_{1_{t_2}}\right)\\ &\quad - \left(v_1 v_{2_{t_2 t_2}} - v_2 v_{1_{t_2 t_2}}\right)(v_2 - f' v_1)\\ &\quad - \left(v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}}\right)\left(v_{2_{t_2}} - f' v_{1_{t_2}}\right). \qquad (2.48)\end{aligned}$$

In the following sections, we continually refer to the expressions for Σ_B, F, F_{t_1} and F_{t_2}. We will consider Case (i) and Case (ii) in turn, in each case deducing the structure of the pre-dual-AESS given by the critical set of the local Midline Map B, which will give us the local structure of the dual-AESS, and in turn leads to the local structure of the AESS itself using the ideas of [R87] (see also [T90]).

2.5.4 Case (i): Inflexional tangent cuts the curve again

Consider two curve segments γ_1 and γ_2 given by

$$\gamma_1(t_1) = (t_1, f(t_1)), \quad \gamma_2(t_2) = (c + t_2, g(t_2)),$$

where

$$f(t_1) = a_3 t_1^3 + a_4 t_1^4 + \ldots, \quad g(t_2) = b_1 t_2 + b_2 t_2^2 + \ldots.$$

We will assume $a_3 \neq 0$, so γ_1 has an ordinary inflexion at the origin, and that $b_1 b_2 \neq 0$, which means that the tangent to γ_2 at $t_2 = 0$ is not parallel to the tangent to γ_1 at the origin and that γ_2 is non-inflexional at $\gamma_2(0)$. These are all generic assumptions. So we have

$$f(0) = f'(0) = f''(0) = 0, f'''(0) = 6a_3 \neq 0,$$

and

$$g(0) = 0, g'(0) = b_1 \neq 0, g''(0) = 2b_2 \neq 0.$$

To find Σ_B, we require (from the expressions listed in Appendix A):

$$v_1^0 = -cg'(0), v_2^0 = 0, v_{1_{t_1}}^0 = g'(0), v_{2_{t_1}}^0 = 0, v_{1_{t_2}}^0 = g'(0) - cg''(0), v_{2_{t_2}}^0 = g'(0)^2.$$

Substituting these expressions into the expression for the Jacobian of B we get

$$\begin{aligned} JB(0,0) &= \begin{pmatrix} 0 & (-cg'(0))(g'(0)^2) \\ 0 & c(-cg'(0)^3) - g(0)'(-cg'(0))^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -cg'(0)^3 \\ 0 & -2c^2 g'(0)^3 \end{pmatrix}. \end{aligned}$$

As expected, $JB(0,0)$ is a singular matrix, and we can see that $JB(0,0)$ has rank 1 unless $c = 0$ or $g'(0) = 0$, and thus we have a corank 1 singularity here unless $g'(0) = 0$ (recall that we are assuming otherwise). Thus $(t_1, t_2) = (0,0)$ lies on Σ_B.

From (2.47) we have

$$F_{t_1}(0,0) = 2c^3 f'''(0) g'(0)^4.$$

Since we are assuming that there is an ordinary inflexion on γ_1 at the origin, we have $f'''(0) \neq 0$, and $g'(0) \neq 0$ is a running assumption, and thus $F_{t_1}(0,0) \neq 0$. Thus the pre-dual-AESS (i.e. the critical set Σ_B) is smooth in a neighbourhood of $(t_1, t_2) = (0,0)$, and we can parametrise the pre-dual-AESS by t_2 in a neighbourhood of $(t_1, t_2) = (0,0)$, that is, write $t_1 = t_1(t_2)$ in a neighbourhood of $(t_1, t_2) = (0,0)$, with $t_1'(0) = 0$.

Hence we deduce the structure of the dual-AESS. From [LSMP] plots, we suspect that the AESS should exhibit an ordinary cusp in this situation, and we would like to confirm this by showing that the dual-AESS exhibits an inflexion, that is, that $Im(B(\Sigma_B))$ has an inflexion at $B(0,0) \equiv (a(0,0), b(0,0))$. Critical set Σ_B is given by $\overline{F}^{-1}(0)$, where $\overline{F} \equiv a_{t_1} b_{t_2} - a_{t_2} b_{t_1}$, and $a_{t_1}, a_{t_2}, b_{t_1}$ and b_{t_2} are given in (2.42), (2.43), (2.44) and (2.45). We define

$$\delta(t_2) \equiv B(t_1(t_2), t_2) = (a(t_1(t_2), t_2), b(t_1(t_2), t_2)),$$

which is a parametrisation of the image of Σ_B under B, that is, the dual-AESS. Then the condition for this to have an inflexion is that $\delta'(t_2), \delta''(t_2)$ are not independent at $t_2 = 0$. Now

$$\begin{aligned}
\delta'(t_2) &= (a_{t_1} t_1'(t_2) + a_{t_2}, b_{t_1} t_1'(t_2) + b_{t_2}), \\
\delta''(t_2) &= \big(a_{t_1 t_1} t_1'(t_2)^2 + a_{t_1} t_1''(t_2) + 2a_{t_1 t_2} t_1'(t_2) + a_{t_2 t_2}, \\
&\qquad b_{t_1 t_1} t_1'(t_2)^2 + b_{t_1} t_1''(t_2) + 2b_{t_1 t_2} t_1'(t_2) + b_{t_2 t_2}\big).
\end{aligned}$$

The condition for these two vectors $\delta'(0), \delta''(0)$ to be dependent is that

$$[\delta'(0), \delta''(0)] = 0 \equiv a_{t_2}(0) b_{t_2 t_2}(0) - b_{t_2}(0) a_{t_2 t_2}(0) = 0.$$

Using the list of expressions in Appendix A, we calculate that

$$a_{t_2}(0) = -\frac{g'}{c}, \quad a_{t_2 t_2}(0) = -\tfrac{g''(0)}{c} - \tfrac{2g'(0)}{c^2},$$
$$b_{t_2}(0) = -2g'(0), \qquad b_{t_2 t_2}(0) = -\tfrac{4g'}{c} - 2g'',$$

and so

$$a^0_{t_2} b^0_{t_2 t_2} = \frac{4g'(0)^2}{c^2} + \frac{2g'(0)g''(0)}{c},$$

$$b^0_{t_2} a^0_{t_2 t_2} = \frac{2g'(0)g''(0)}{c} + \frac{4g'(0)^2}{c^2},$$

and hence $a^0_{t_2} b^0_{t_2 t_2} - b^0_{t_2} a^0_{t_2 t_2} \equiv 0$ as required. Thus we have shown that the dual-AESS, $Im(B(\Sigma_B))$, has an inflexion at $t_1 = t_2 = 0$.

We deduce ([R87], [T90]) that the AESS exhibits a cusp at $t_1 = t_2 = 0$. The cusp point is at $(c, 0)$ on the γ_2 curve segment and, since $v_1(0,0) \neq 0$ and $v_2(0,0) = 0$, this tells us the direction of the tangent to the AESS at this point is along the inflexional tangent line.

Proposition 2.5.5. *The AESS generically exhibits an ordinary cusp at the intersection of an inflexional tangent with the original curve, and the tangent at the cusp is the inflexional tangent line itself.*

2.5.5 Case (ii): Two inflexional tangents meet

We will follow the same procedure as given in §2.5.3. Consider two curve segments γ_1 and γ_2 given by

$$\gamma_1(t_1) = (t_1, f(t_1)), \quad \gamma_2(t_2) = (c + t_2, d + g(t_2)),$$

where

$$f(t_1) = a_3 t_1^3 + \ldots (a_3 \neq 0), \quad g(t_2) = b_1 t_2 + b_3 t_2^3 + \ldots .$$

We may assume that $b_1 \neq 0$, that is, that the inflexional tangents are not parallel, and also, without loss of generality, that $c = 0$ and $d \neq 0$. We may also assume that the tangent line to γ_2 at $t_2 = 0$ does not pass through $\gamma_1(0)$. We find that, under these assumptions, the (2×2) matrix $JB(0,0)$, which we expect to be singular, has precisely rank 1. Evaluating the expressions (2.47) and (2.48) at $t_1 = t_2 = 0$ we deduce that Σ_B is smooth if and only if at least one of $F_{t_1}(0,0)$ or $F_{t_2}(0,0)$ is non-zero, that is, at least one of the inflexions on curve segments γ_1 and γ_2 is ordinary, and generically we may assume this to be the case.

Consideration of the dual-AESS leads us to deduce that the dual-AESS is smooth if both inflexions are ordinary, and exhibits an inflexion (generically an ordinary inflexion) if one of the inflexions is a higher inflexion. We may deduce ([R87], [T90]) the following.

Proposition 2.5.6. *The AESS is smooth at the intersection of two inflexional tangents if both inflexions are ordinary, and generically has a cusp singularity when one of the inflexions is a higher inflexion.*

We are also able to use the local form of the Midline Map as derived in §2.5.3 to deduce the condition for a cusp on the AESS (as previously found in Proposition 2.4.7), and also to analyse the structure of the AESS $\cup$ MPTL at points where these two sets meet, which corresponds to the situation where there exists a 3+3 conic having contact with a curve in two points which share tangent directions: §2.5.6 and §2.5.7 respectively contain this analysis. Then in §2.5.8 we use the local form of the Midline Map to deduce the condition for an inflexion to appear on the AESS.

2.5.6 Condition for a *cusp* on the AESS

Following the procedure given in §2.5.3, we may show that the existence of a *4+3* conic implies the existence of a *cusp* on the AESS. Consider two curve segments γ_1 and γ_2 given by

$$\gamma_1(t_1) = (t_1, f(t_1)), \quad \gamma_2(t_2) = (c + t_2, d + g(t_2)),$$

where

$$f(t_1) = a_2 t_1^2 + a_3 t_1^3 + \dots \quad g(t_2) = b_1 t_2 + b_2 t_2^2 + b_3 t_2^3 + \dots$$

We will assume that $a_2 \neq 0$ and $b_1 \neq 0$, and that $c = 0$. The *3+3 Conic Condition* of Corollary 2.3.2 tells us that we must take $a_2 = -b_2$ in order for there to exist a conic having 3-point contact with γ_1 and γ_2 at $t_1 = 0$ and $t_2 = 0$ respectively. Furthermore, the *4+3 Conic Condition* of Proposition 2.3.4 tells us that we must take $a_3 = -\frac{a_2 b_1}{d}$ for the conic to have 4-point contact with γ_1 at $t_1 = 0$. This ensures the existence of a conic having 4-point contact with γ_1 at $t_1 = 0$ and 3-point contact with γ_2 at $t_2 = 0$. From (2.48) we are able to deduce that

$$F_{t_2}(0,0) = 0 \Longleftrightarrow f'''(0) + g'''(0) = 0.$$

So the pre-dual-AESS $F(t_1, t_2) = 0$ is smooth at $(t_1, t_2) = (0, 0)$ as long as $g'''(0) \neq -f'''(0)$, which holds generically.

Remark 2.5.7. *The situation $g'''(0) = -f'''(0)$ (along with the 3+3 Conic Condition $f''(0) = -g''(0)$) corresponds to the affine normals to γ_1 and γ_2 at $t_1 = t_2 = 0$ being parallel, the affine normals being in the direction*

$$\left(-\frac{1}{3}f'''(0), f''(0)^2\right) \text{ and } \left(\frac{1}{3}g'''(0), -g''(0)^2\right),$$

respectively (see §1.3.3).

A short calculation shows that the dual-AESS exhibits an *inflexion* in this case. We can thus deduce:

Proposition 2.5.8. *The AESS exhibits a cusp at the centre of a 4+3 conic.*

This confirms part (a) of Proposition 2.4.7. In §2.5.7, we will prove part (b) of this proposition.

2.5.7 Structure of the AESS ∪ MPTL at parallel tangents

We will now show that the AESS ∪ MPTL exhibits a *beaks singularity* when the condition for a 3+3 conic holds at points of contact of parallel tangents, which proves the assertion of Proposition 2.4.7. Consider two curve segments γ_1 and γ_2 given by

$$\gamma_1(t_1) = (t_1, f(t_1)), \quad \gamma_2(t_2) = (t_2, d + g(t_2)),$$

where

$$f(t_1) = a_2 t_1^2 + a_3 t_1^3 + \dots, \quad g(t_2) = b_2 t_2^2 + b_3 t_2^3 + \dots,$$

and we will assume that $a_2 \neq 0$. The *3+3 Conic Condition* of Corollary 2.3.2 tells us that we must take $a_2 = -b_2$ for there to exist a conic having 3-point contact with γ_1 and γ_2 at $t_1 = 0$ and $t_2 = 0$ respectively. Doing so ensures that the curve segments have parallel tangents at non-inflexional points, and that there exists a 3+3 conic there. Under these assumptions, the Jacobian matrix $JB(0,0)$ has rank 1. From expressions (2.47), (2.48) we may deduce that the critical set Σ_B is non-smooth, and thus the $\{$pre-dual-AESS$\} \cup \{$pre-dual-MPTL$\}$ is non-smooth when we have parallel tangents *and* the 3+3 Conic Condition holds.

To determine the structure of the $\{$dual-AESS$\} \cup \{$dual-MPTL$\}$ in this case, we require expressions for $F_{t_1t_1}(t_1,t_2)$, $F_{t_1t_2}(t_1,t_2)$ and $F_{t_2t_2}(t_1,t_2)$. These expressions are given in Appendix A. Evaluating these expressions at $t_1 = t_2 = 0$ gives us

$$\begin{aligned} F_{t_1t_1}(0,0) &= 2d^3 f'''(0)(3f''(0) + g''(0)), \\ F_{t_2t_2}(0,0) &= -2d^3 g'''(0)(f''(0) + 3g''(0)), \\ F_{t_1t_2}(0,0) &= 2d^3(f''(0)g'''(0) - f'''(0)g''(0)). \end{aligned}$$

The Midline Map B has a *beaks* singularity (for details, see [R87]) if

$$F_{t_1t_1}(0,0)F_{t_2t_2}(0,0) < F_{t_1t_2}(0,0)^2.$$

It follows that the $\{$dual-AESS$\} \cup \{$dual-MPTL$\}$ exhibits a cusp singularity if and only if

$$0 < (f'''(0) - g'''(0))^2.$$

This holds generically, since $f'''(0) \neq g'''(0)$ in general. Thus the $\{$dual-AESS$\} \cup \{$dual-MPTL$\}$ generically has a *beaks singularity* when the conditions for a 3+3 conic hold at parallel tangents.

We deduce the following:

Proposition 2.5.9. *The AESS $\cup$ MPTL has a dual-beaks singularity when the condition for a 3+3 conic holds at parallel tangents.*

Remark 2.5.10. *The dual of a beaks singularity is in fact two ordinary cusps with the same cuspidal tangent line. We have already deduced that the MPTL* *has an ordinary cusp in this situation (see Proposition 2.4.9 of* §2.4.4*), and so Proposition 2.5.9 tallies with this fact.*

2.5.8 Condition for an *inflexion* on the AESS

Consider two curve segments γ_1 and γ_2 given by

$$\gamma_1(t_1) = (t_1, f(t_1)), \quad \gamma_2(t_2) = (t_2, d + g(t_2)),$$

where

$$f(t_1) = a_2 t_1^2 + a_3 t_1^3 + \ldots, \quad g(t_2) = b_1 t + b_2 t_2^2 + b_3 t_2^3 + \ldots$$

We will assume that $a_2 b_1 \neq 0$. A short calculation shows that

$$\det(JB(0,0)) = 0 \Longleftrightarrow f''(0) + g''(0) = 0,$$

which is simply the 3+3 Condition $a_2 = -b_2$ of Corollary 2.3.2, as expected. Assume this holds. Then a short calculation shows that

$$\begin{aligned} F_{t_1}(0,0) &= -2d^2 g'(0) \left(3f''(0)g'(0) + df'''(0)\right), \\ F_{t_2}(0,0) &= 2d^2 g'(0) \left(3g'(0)g''(0) + dg'''(0)\right). \end{aligned}$$

Substituting $g''(0) = -f''(0)$, we deduce that

$$\begin{aligned} F_{t_1}(0,0) = 0 \quad &\Longleftrightarrow \quad f'''(0) = -\frac{3f''(0)g'(0)}{d}, \\ F_{t_2}(0,0) = 0 \quad &\Longleftrightarrow \quad g'''(0) = \frac{3f''(0)g'(0)}{d}. \end{aligned}$$

Note that

$$f'''(0) = -\frac{3f''(0)g'(0)}{d} \quad \Longleftrightarrow \quad 6a_3 = 0 \Longleftrightarrow a_2 b_1 + a_3 d = 0,$$

which is precisely the condition for the 3+3 conic to have *4*-point contact with γ_1 at $t_1 = 0$ (see the *4+3 Conic Condition* of Proposition 2.3.4) and we assume generically that this does not happen. We will also assume the same for γ_2, and thus both F_{t_1} and F_{t_2} are generically non-zero at $(t_1, t_2) = (0, 0)$. A short calculation shows that the dual-AESS has a cusp in this situation if and only if $a_3 = -b_3$.

Now consider the expression

$$[\gamma(t_1) - \gamma(t_2), \gamma''(t_1) - \gamma''(t_2)] = 0.$$

This defines the pre-set for the *Affine Distance Symmetry Set* (ADSS), considered in Chapter 3. In this case, it is equivalent to

$$\left| \begin{array}{cc} t_1 - t_2 & -\frac{1}{3}k_1^{-5/3}\dot{k}_1 + \frac{1}{3}k_2^{-5/3}\dot{k}_2 \\ f(t_1) - d - g(t_2) & k_1^{-2/3}f''(t_1) - \frac{1}{3}k_1^{-5/3}\dot{k}_1 f'(t_1) - k_2^{-2/3}g''(t_2) + \frac{1}{3}k_2^{-5/2}\dot{k}g'(t_2) \end{array} \right|_{(0,0)} = 0,$$

where $k_i \equiv [\dot{k}_i, \ddot{k}_i]$. Upon expansion, this reduces to

$$k_1(0)^{-5/3}\dot{k}_1(0) - k_2(0)^{-5/3}\dot{k}_2(0) = 0,$$

and thus this is the condition for $t_1 = t_2 = 0$ to give an ADSS point in this situation. Now $k_1(0) = 2a_2, k_2(0) = 2b_2, \dot{k}_1(0) = 6a_3, \dot{k}_2(0) = 6b_3$ and, along with the 3+3 Conic Condition $a_2 = -b_2$, the condition for $t_1 = t_2 = 0$ to give an ADSS point is

$$(2a_2)^{-5/3}(6a_3 + 6b_3) = 0, \quad \text{i.e. } a_3 + b_3 = 0,$$

which is precisely the condition for the dual-AESS to exhibit a cusp in this situation.

The dual of a cusp is an inflexion (see [R87]), and thus we may deduce:

Proposition 2.5.11. *The AESS of a curve γ has an* inflexion *at the centre of a conic having 3+3 contact with γ at $\gamma(t_1)$ and $\gamma(t_2)$ if and only if*

$$[\gamma(t_1) - \gamma(t_2), \gamma''(t_1) - \gamma''(t_2)] = 0,$$

that is, if and only if the points $\gamma(t_1)$ and $\gamma(t_2)$ also contribute to the ADSS *of γ.*

Remark 2.5.12. *The importance of this result is as follows: we have proved* *that the AESS exhibits an inflexion when the ADSS Condition also holds. Geometrically, we can interpret this result in terms of the pre-sets of both symmetry sets. Referring back to Remark 2.2.11, we saw that Morse singularities on the pre-AESS lie on the pre-ADSS, and in* §2.2.3 *we saw that horizontal and vertical tangents to the pre-AESS correspond to cusps on the AESS. Under perturbation of the curve, an isolated point an the pre-AESS expands to form a smooth closed loop on the pre-AESS which crosses the pre-ADSS twice and has two vertical and two horizontal tangents (see Figure 2.9*(a)*). The corresponding AESS segment is thus a closed curve with two inflexions and four cusps, as illustrated in Figure 2.9*(b).

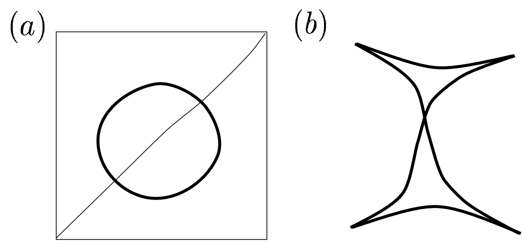


Figure 2.9: *See Remark 2.5.12.*

2.6 The AESS for non-simple curves

Until now, we have had the running assumption that the curves for which we are finding the AESS (and MPTL) are simple, that is, do not self-intersect. However, we will extend this now to study the local structure of the AESS at a crossing on the original curve. Consider two curve segments γ_1 and γ_2 given by

$$\gamma_1(s) = (s, f(s)), \quad \gamma_2(t) = (t, g(t)),$$

where

$$f(s) = a_2 s^2 + a_3 s^3 + \dots, \quad g(t) = b_1 t + b_2 t^2 + b_3 t^3 + \dots,$$

and where we will assume that $a_2 b_1 \neq 0$, that is, γ_1 is non-inflexional at $s = 0$, and γ_1 and γ_2 intersect but are not tangent at the origin. The pre-AESS is defined by solutions (s, t) to

$$[\gamma_1(s) - \gamma_2(t), \gamma_1'(s) + \gamma_2'(t)] = 0, \tag{2.49}$$

(from (2.1)). Expanding (2.49) as a power series in s, t and, setting $K = 2^{1/3}$, $A = a_2^{1/3}$ and $B = b_2^{1/3}$, we can write the equation of the pre-AESS as

$$K\left(Ab_1(s-t) + (B-A)(a_2 s^2 + 2AB(A+B)st + b_2 t^2) + b_1 t\left(\frac{a_3}{a_2}s + \frac{b_3}{b_2}t\right) + \ldots\right) = 0. \tag{2.50}$$

Note that the left-hand side of (2.50) is zero at $s = t = 0$, which confirms that the intersection point of the curves γ_1 and γ_2 contributes to the AESS. Furthermore, the pre-AESS is smooth, unless the first degree terms in (2.50) vanish, which happens if and only if $a_2 b_1 = 0$. Since generically we assume otherwise, we have:

Proposition 2.6.1. *The AESS passes smoothly through the self-intersection points of a non-simple plane curve (see Figure 2.10(a)).*

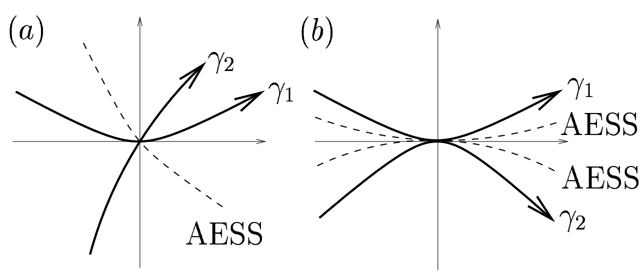


Figure 2.10: *Illustrating the conclusions of* §2.6.

If $b_1 = 0$, then the original curve segments are tangent at the origin, and the pre-AESS exhibits a Morse singularity. Then, assuming that $a_2 \neq b_2$ (that is, that the affine tangents to the curve segments at the origin are not equal), the pre-AESS is given by the equation

$$K(B-A)(A^3 s^2 + 2AB(A+b)st + B^3 t^2) + \ldots = 0,$$

with $A \neq B$, and the discriminant is

$$A^2B^2(A^2 + AB + B^2),$$

which is always positive. Thus we have:

Proposition 2.6.2. *When the branches of the original curve are tangent, the pre-AESS exhibits a crossing and the AESS comprises two tangential branches through the origin (see Figure 2.10(b)).*

2.7 The MPTL (Reprise)

In §2.4.4, we defined the *Mid-Parallel-Tangents Locus* (MPTL), and analysed the local structure of the MPTL, together with the AESS, as part of the *dual of the critical locus* of the Midline Map as defined in Definition 2.4.1. In this way, we saw that it is natural to study the MPTL and the AESS together.

In this section, we will consider an alternative definition of the MPTL for an oval γ, showing that it can be considered as part of the *full bifurcation set* of a 2-parameter family of functions defined on γ, together with another set, the *Affine Area Symmetry Set* (AASS). In §2.7.1, we define the family of *area functions* A on an oval, parametrised by points in the plane.

2.7.1 The family of *Area functions* A on an oval

Consider a positively oriented oval γ, parametrised by t, as illustrated in Figure 2.11(a). (For example, t could be the affine-arclength parameter along γ.) Suppose we fix a point $\mathbf{x}$ inside γ, and take a point $\gamma(t_1)$ on γ. Then the unique line through $\gamma(t_1)$ and $\mathbf{x}$ cuts γ again in an unique point $\gamma(t_2)$. Now, for small δt we have

$$\gamma(t_1 + \delta t) - \gamma(t_1) = \gamma'(t_1)\delta t.$$

So the elemental area shown shaded in Figure 2.11(b) is

$$\frac{1}{2}\left[\gamma(t_1) - \mathbf{x}, \gamma'(t_1)\delta t\right]. \tag{2.51}$$

Thus the area shown shaded in Figure 2.11(a) is

$$\frac{1}{2}\int_{t_1}^{t_2} [\gamma(t) - \mathbf{x}, \gamma'(t)]\, dt. \qquad (2.52)$$

Fixing $\mathbf{x}$, we define the *Area Function* on the curve γ to be

$$A(t_1) = \int_{t_1}^{t_2} [\gamma(t) - \mathbf{x}, \gamma'(t)]\, dt,$$

where we take t_2 to be determined by t_1 and $\mathbf{x}$. (We remove the factor of 1/2 since this does not change things conceptually.) Now, suppose we vary

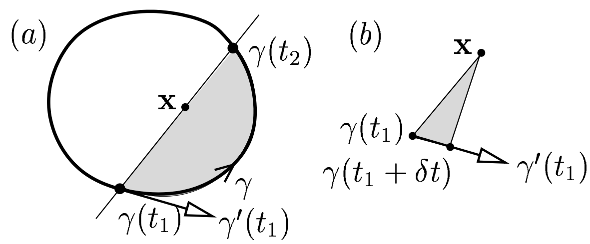


Figure 2.11: *Illustrating the construction of the family of Area functions defined on an oval, outlined in* §2.7.1.

parameter t_1 *and* point $\mathbf{x}$ inside γ. Then the parameter value t_2 of the point where the line through $\mathbf{x}$ and $\gamma(t_1)$ cuts γ again is a function of t_1 (and $\mathbf{x}$). Consider the *2-parameter family of area functions* defined on the oval γ by

$$A(t_1, \mathbf{x}) = \int_{t_1}^{t_2(t_1)} [\gamma(t) - \mathbf{x}, \gamma'(t)] dt,$$

where $t_2(t_1)$ denotes that t_2 is considered to be a function of t_1. In §2.7.2, we will show that the *Critical points* of $A(t_1, \mathbf{x})$ occur when $\mathbf{x}$ is the midpoint of the chord joining $\gamma(t_1)$ and $\gamma(t_2(t_1))$ (see Proposition 2.7.1), a critical point of $A(t_1, \mathbf{x})$ is *degenerate* if the tangents to γ at $\gamma(t_1)$ and $\gamma(t_2(t_1))$ are parallel (see Proposition 2.7.2), and that $A(t_1, \mathbf{x})$ has a higher degenerate critical point if the condition for $\gamma(t_1)$ and $\gamma(t_2(t_1))$ to give an AESS point also holds (see Proposition 2.7.3). Thus the bifurcation set of A is the MPTL, which is *singular* when the AESS Condition holds, and generically has a cusp at the centre of the corresponding 3+3 conic.

2.7.2 Singularities of A

Consider an oval γ and a point $\mathbf{x}$ inside γ, and suppose that γ is parametrised by t_1. The line L through $\mathbf{x}$ and $\gamma_1 \equiv \gamma(t_1)$ is the set of points $\mathbf{p}$ such that

$$[\mathbf{p} - \mathbf{x}, \mathbf{p} - \gamma_1] = 0.$$

So, for a point $\gamma_2 \equiv \gamma(t_2)$ to lie on L we need

$$[\gamma_2 - \mathbf{x}, \gamma_2 - \gamma_1] = 0.$$

Thus, to find parameter t_2 as a function of t_1, we take

$$[\gamma(t_2) - \mathbf{x}, \gamma(t_2) - \gamma(t_1)] = 0, \tag{2.53}$$

to hold identically for all t_1. This defines our point $\gamma(t_2)$ in terms of t_1, for fixed $\mathbf{x}$. We thus have $t_2 \equiv t_2(t_1)$. We will require expressions for the derivatives of t_2 w.r.t. t_1. Denote by $'$ (prime) derivatives w.r.t. t_1, and by $\dot{}$ (dot) derivatives w.r.t. t_2. Differentiating (2.53) w.r.t t_1 we deduce that

$$\begin{aligned} & t_2'[\dot\gamma_2, \gamma_2 - \gamma_1] + [\gamma_2 - \mathbf{x}, t_2'\dot\gamma_2 - \gamma_1'] = 0, \\ \iff \quad & t_2'[\gamma_1 - \mathbf{x}, \dot\gamma_2] - [\gamma_2 - \mathbf{x}, \gamma_1'] = 0. \end{aligned} \tag{2.54}$$

So

$$t_2'(t_1) = \frac{[\gamma_2 - \mathbf{x}, \gamma_1']}{[\gamma_1 - \mathbf{x}, \dot\gamma_2]}. \tag{2.55}$$

Furthermore, since (2.54) holds identically for all t_1, we may deduce that

$$\frac{d}{dt_1}\{(2.54)\} = 0 \iff t_2''[\gamma_1 - \mathbf{x}, \dot\gamma_2] + t_2'^2[\gamma_1 - \mathbf{x}, \ddot\gamma_2] + 2t_2'[\gamma_1', \dot\gamma_2] - [\gamma_2 - \mathbf{x}, \gamma_1''] = 0, \tag{2.56}$$

$$\iff t_2''(t_1) = \frac{t_2'^2[\mathbf{x} - \gamma_1, \ddot\gamma_2] - 2t_2'[\gamma_1', \dot\gamma_2] + [\gamma_2 - \mathbf{x}, \gamma_1'']}{[\gamma_1 - \mathbf{x}, \dot\gamma_2]}. \tag{2.57}$$

Also, since (2.56) holds identically, we can differentiate w.r.t. t_1, and some calculation then shows that

$$\begin{aligned} t_2'''(t_1) &= \frac{3t_2't_2''[\gamma_1 - \mathbf{x}, \ddot{\gamma}_2]}{[\mathbf{x}-\gamma_1, \dot{\gamma}_2]} + \frac{3t_2'^2[\gamma_1', \ddot{\gamma}_2]}{[\mathbf{x}-\gamma_1, \dot{\gamma}_2]} + \frac{3t_2''[\gamma_1', \dot{\gamma}_2]}{[\mathbf{x}-\gamma_1, \dot{\gamma}_2]} \\ &\quad + \frac{3t_2'[\gamma_1'', \dot{\gamma}_2]}{[\mathbf{x}-\gamma_1, \dot{\gamma}_2]} + \frac{t_2'^3[\gamma_1 - \mathbf{x}, \dddot{\gamma}_2]}{[\mathbf{x}-\gamma_1, \dot{\gamma}_2]} - \frac{[\gamma_2 - \mathbf{x}, \gamma_1''']}{[\mathbf{x}-\gamma_1, \dot{\gamma}_2]}. \end{aligned} \tag{2.58}$$

We are now in a position to prove:

Proposition 2.7.1.

$$\frac{dA}{dt_1} = 0 \Longleftrightarrow \mathbf{x} \textit{ is the } \text{midpoint} \textit{ of the chord joining } \gamma(t_1), \gamma(t_2(t_1)).$$

Proof. We have

$$\begin{aligned} \frac{dA}{dt_1} &= t_2'[\gamma_2 - \mathbf{x}, \dot{\gamma}_2] - [\gamma_1 - \mathbf{x}, \gamma_1'], \\ &= \frac{[\gamma_2 - \mathbf{x}, \gamma_1']}{[\gamma_1 - \mathbf{x}, \dot{\gamma}_2]}[\gamma_2 - \mathbf{x}, \dot{\gamma}_2] - [\gamma_1 - \mathbf{x}, \gamma_1'], \end{aligned} \tag{2.59}$$

using the expression for t_2' from (2.55), and thus

$$\begin{aligned} \frac{dA}{dt_1} = 0 \quad &\Longleftrightarrow \quad [\gamma_1 - \mathbf{x}, \gamma_1'][\gamma_1 - \mathbf{x}, \dot{\gamma}_2] - [\gamma_2 - \mathbf{x}, \gamma_1'][\gamma_2 - \mathbf{x}, \dot{\gamma}_2] = 0, \\ &\Longleftrightarrow \quad \mathbf{x} - \gamma_1 = \pm(\mathbf{x} - \gamma_2), \\ &\Longleftrightarrow \quad \gamma_1 \equiv \gamma_2 \text{ or } \mathbf{x} = \frac{1}{2}(\gamma_1 + \gamma_2). \end{aligned}$$

The former is ruled out, since we assume that $\gamma(t_1)$ and $\gamma(t_2(t_1))$ are distinct points (since $\mathbf{x}$ is taken to be inside γ). The latter is the midpoint of the chord joining γ_1 and γ_2, as required. □

Thus $A(t_1)$ has a critical point if and only if $\mathbf{x}$ is the midpoint of the chord joining $\gamma(t_1)$ and $\gamma(t_2(t_1))$. Furthermore, we have:

Proposition 2.7.2.

$$\frac{dA}{dt_1} = \frac{d^2A}{dt_1^2} = 0 \Longleftrightarrow \begin{cases} \bullet\ \mathbf{x} \textit{ is the midpoint of the chord joining } \gamma(t_1), \gamma(t_2(t_1)), \\ \bullet\ \gamma \textit{ has } \text{parallel tangents} \textit{ at these two points.} \end{cases}$$

Proof. By Proposition 2.7.1, we may assume that $\mathbf{x}$ is the midpoint of the chord joining $\gamma(t_1)$ and $\gamma(t_2(t_1))$, that is, $\mathbf{x} = (\gamma_1+\gamma_2)/2$. Differentiating the expression for dA/dt_1 in (2.59) w.r.t. t_1, we get

$$\frac{d^2A}{dt_1^2} = t_2''[\gamma_2 - \mathbf{x}, \dot\gamma_2] + t_2'^2[\gamma_2 - \mathbf{x}, \ddot\gamma_2] - [\gamma_1 - \mathbf{x}, \gamma_1''], \tag{2.60}$$

Substituting $\mathbf{x}$ as the midpoint of the chord in (2.54) we have

$$\begin{aligned} t_2''(t_1) \quad = \quad & \frac{[\gamma_2-\gamma_1,\gamma_1']^2[\gamma_2-\gamma_1,\ddot\gamma_2]}{[\gamma_1-\gamma_2,\dot\gamma_2]^3} \\ & -\frac{4[\gamma_2-\gamma_1,\gamma_1'][\gamma_1-\gamma_2,\dot\gamma_2][\gamma_1',\dot\gamma_2]}{[\gamma_1-\gamma_2,\dot\gamma_2]^3} \\ & +\frac{[\gamma_2-\gamma_1,\gamma_1''][\gamma_1-\gamma_2,\dot\gamma_2]^2}{[\gamma_1-\gamma_2,\dot\gamma_2]^3}, \end{aligned} \tag{2.61}$$

using the fact that

$$t_2'(t_1) = \frac{[\gamma_2-\gamma_1,\gamma_1']}{[\gamma_1-\gamma_2,\dot\gamma_2]}, \tag{2.62}$$

from (2.55) with $\mathbf{x} = (\gamma_1+\gamma_2)/2$. Then calculation shows that

$$\frac{d^2A}{dt_1^2} \quad = \quad 2\frac{[\gamma_2-\gamma_1,\gamma_1']}{[\gamma_1-\gamma_2,\dot\gamma_2]}[\gamma_1',\dot\gamma_2] = 0 \Longleftrightarrow [\gamma_1',\dot\gamma_2] = 0,$$

since $[\gamma_1-\gamma_2,\gamma_1'] \neq 0$ and $[\gamma_1-\gamma_2,\dot\gamma_2] \neq 0$ for an oval. Thus $\frac{dA}{dt_1} = \frac{d^2A}{dt_1^2} = 0$ if an only if $\mathbf{x} = \frac{1}{2}(\gamma_1+\gamma_2)$ *and* we have parallel tangents at γ_1 and γ_2, as required. □

Thus we have shown that the Bifurcation Set of the family of area functions defined on an oval γ and parametrised by points in the plane is identical to the set of midpoints of chords joining points of γ that have parallel tangents, the MPTL of γ.

Calculating further derivatives of A and t_2 we may also deduce:

Proposition 2.7.3.

$$\frac{dA}{dt_1} = \frac{d^2A}{dt_1^2} = \frac{d^3A}{dt_1^3} = 0 \Longleftrightarrow \begin{cases} \bullet\ \mathbf{x} \textit{ is the midpoint of the chord joining } \gamma_1 \textit{ and } \gamma_2, \\ \bullet\ \textit{the tangents to } \gamma \textit{ at } \gamma_1 \textit{ and } \gamma_2 \textit{ are parallel,} \\ \bullet\ \textit{the AESS Condition holds for } \gamma_1 \textit{ and } \gamma_2. \end{cases}$$

Proof. By Proposition 2.7.2, we may assume that

$$\mathbf{x} = \frac{1}{2}(\gamma_1 + \gamma_2), \text{ and } [\gamma_1', \dot{\gamma}_2] = 0.$$

Calculation gives us

$$\frac{d^3A}{dt_1^3} = [\gamma_1', \gamma_1''] - 3\frac{[\gamma_2 - \gamma_1, \gamma_1']^2}{[\gamma_1 - \gamma_2, \dot{\gamma}_2]^2}[\gamma_1', \ddot{\gamma}_2] - 3\frac{[\gamma_2 - \gamma_1, \gamma_1']}{[\gamma_1 - \gamma_2, \dot{\gamma}_2]}[\gamma_1'', \dot{\gamma}_2] - \frac{[\gamma_2 - \gamma_1, \gamma_1']^3}{[\gamma_1 - \gamma_2, \dot{\gamma}_3]^2}[\dot{\gamma}_2, \ddot{\gamma}_2].$$

Now $[\gamma_1', \dot{\gamma}_2]$ is identically zero, so we can differentiate w.r.t. t_1 to get

$$[\gamma_1'', \dot{\gamma}_2] + t_2'[\gamma_1', \ddot{\gamma}_2] = 0,$$

which (using expression (2.62) for t_2' at $\mathbf{x} = (\gamma_1 + \gamma_2)/2$) gives us

$$[\gamma_1'', \dot{\gamma}_2] = -\frac{[\gamma_2 - \gamma_1, \gamma_1']}{[\gamma_1 - \gamma_2, \dot{\gamma}_2]}[\gamma_1', \ddot{\gamma}_2], \tag{2.63}$$

and using (2.63) we can show that

$$\frac{d^3A}{dt_1^3} = [\gamma_1', \gamma_1''] - \frac{[\gamma_2 - \gamma_1, \gamma_1']^3}{[\gamma_1 - \gamma_2, \dot{\gamma}_2]^3}[\dot{\gamma}_2, \ddot{\gamma}_2].$$

Now to link this with the AESS, we must introduce the affine-arclength parametrisation. Without loss of generality, we may suppose t_1 is the affine-arclength parameter along γ. Then $[\gamma_1', \gamma_1''] \equiv 1$, and we have the identity $\frac{d}{dt_1}(\gamma_2) \equiv \gamma_2' = k_2^{-1/3}\dot{\gamma}_2$. Using this, we see that

$$\frac{d^3A}{dt_1^3} = 1 - \frac{[\gamma_2 - \gamma_1, \gamma_1']^3}{k_2[\gamma_1 - \gamma_2, \gamma_2']^3}k_2,$$

and thus

$$\frac{d^3A}{dt_1^3} = 0 \iff [\gamma_1 - \gamma_2, \gamma_1' + \gamma_2'] = 0,$$

which is precisely the 'AESS Condition' of Proposition 2.2.2, the condition for $\gamma(t_1)$ and $\gamma(t_2(t_1))$ to give an AESS point. Thus if $\mathbf{x}$ is the midpoint of the chord joining parallel tangent pairs, then $d^3A/dt_1^3 = 0$ if and only if the AESS Condition holds, as required. □

Remark 2.7.4. *Since*

$$\frac{dA}{dt_1} = \frac{d^2A}{dt_1^2} = \frac{d^3A}{dt_1^3} = 0,$$

is the condition for the bifurcation set of A to have a singularity (which we expect to be an ordinary cusp). Thus we have shown that the MPTL exhibits a cusp, or worse, when the AESS Condition holds, that is, when there exists a conic having 3-point contact with the curve at two points with parallel tangents, and hence we have re-proved Proposition 2.4.9.

2.7.3 The *Affine Area Symmetry Set* (AASS)

Definition 2.7.5. *The* **Affine Area Symmetry Set** (AASS) *of an oval γ is the levels bifurcation set of the family of area functions A defined on γ, that is,*

$$AASS(\gamma) \equiv \{\mathbf{x} \in \mathbb{R}\colon \exists t_i, t_j \ s.t. \ A(t_i) = A(t_j) \ and \ A'(t_i) = A'(t_j) = 0\}.$$

We have shown in Proposition 2.7.1 that A has a singularity for $t = t_i$ if and only if point $\mathbf{x}$ lies at the midpoint of a chord with one end at $\gamma(t_i)$, the other end being at some point $\gamma(g(t_i))$. Thus, for $\mathbf{x}$ to be on the AASS of γ corresponding to parameter values t_i and t_j, then it must lie at the common midpoint of two chords, one based at $\gamma(t_i)$ and the other based at $\gamma(t_j)$, with the added condition that $A(t_i) = A(t_j)$, that is, that these chords 'cut off' equal areas of the oval. See Figure 2.12(a). It is then trivial to note that the vectors

$$\gamma(t_i) - \gamma(t_j) \text{ and } \gamma(g(t_i)) - \gamma(g(t_j))$$

are parallel and of the same length, as are the vectors

$$\gamma(t_i) - \gamma(g(t_i)) \text{ and } \gamma(t_j) - \gamma(g(t_j)).$$

Thus the points $\gamma(t_i)$, $\gamma(t_j)$, $\gamma(g(t_i))$ and $\gamma(g(t_j))$ are vertices of a *parallelogram* (see Figure 2.12(*b*)). This observation provides us with an approximate method of constructing the AASS.

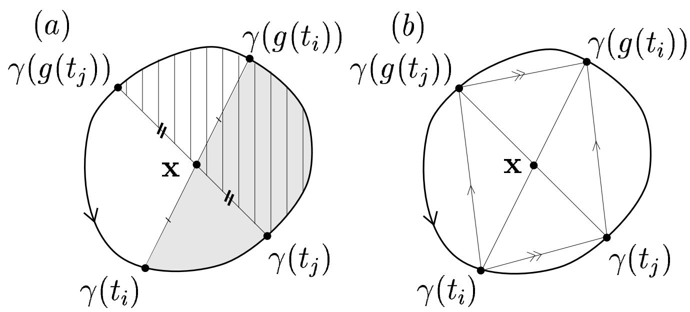


Figure 2.12: *(a) A point* **x** *lies on the AASS of oval γ if it is the midpoint of two chords which 'cut off' equal areas. The two equal areas $A(t_i)$ and $A(t_j)$ are shown shaded and hatched respectively. (b) If $\gamma(t_i)$ and $\gamma(t_j)$ contribute to the AASS, then the four points $\gamma(t_i), \gamma(g(t_i)), \gamma(t_j)$ and $\gamma(g(t_j))$ form a parallelogram.*

Remark 2.7.6. *It is also worth noting that when $\gamma(t_i), \gamma(g(t_i)), \gamma(t_j)$ and $\gamma(g(t_j))$ form a parallelogram on an oval γ, then the areas $A(t_i)$ and $A(t_j)$ are equal if and only if the area bounded by γ and the chord joining $\gamma(t_i)$ and $\gamma(t_j)$ is equal to the area bounded by the chord joining $\gamma(g(t_i))$ and $\gamma(g(t_j))$ (see Figure 2.13).*

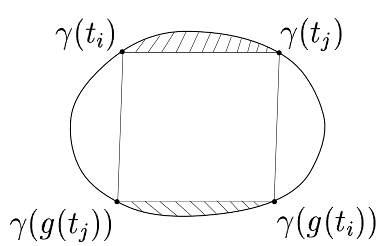


Figure 2.13: *If $\gamma(t_i)$ and $\gamma(t_j)$ contribute to the AASS, then the two hatched areas are equal.*

Constructing the AASS

To construct the AASS, consider a positively oriented parametrised oval γ. For a given point $\gamma(t_1)$ on γ, let $\bar{t}_1$ be the parameter value of the unique point of γ with tangent parallel to the tangent at $\gamma(t_1)$. To construct the AASS of γ we apply the procedure as follows (note that we use s_i in the role of $g(t_i)$ here):

- For each parameter value t_1 do:
    - For each parameter value t_2 between t_1 and $\bar{t}_1$ do:
        - Find parameter values s_1 and s_2 (ordered such that $s_1 < s_2$) such that the chord $\gamma(s_1) - \gamma(s_2)$ is *parallel* to $\gamma(t_1) - \gamma(t_2)$ and has the *same length*.
        - Calculate areas A_1 and A_2.
        - If $A_1 = A_2$ then point
$$\frac{1}{2}(\gamma(s_1) + \gamma(t_1)),$$
is on the AASS of γ.

In practice, we must increment s_2 between s_1 and $\bar{s}_1$, and then find suitable approximate values of t_1 and t_2. Then, for this to give an AASS point, the difference in areas A_1 and A_2 must be less than some chosen small constant. We need only consider s_2 between s_1 and $\bar{s}_1$ due to the symmetry of the construction: considering all s_2 simply results in the AASS being covered twice. This procedure was followed to create a [MAPLE] program used to make the plots contained in §2.7.4.

To calculate the direction of the tangent to the AASS, consider an oval γ, and suppose $\gamma(t_1)$ and $\gamma(t_2)$ contribute a point to the AASS, that is, the chords through $\gamma(t_i)$ and some point $\mathbf{x} = (x, y)$ both have $\mathbf{x}$ as their midpoint and the areas cut off by these chords are equal.

Proposition 2.7.7. *The tangent to the AASS through* $\mathbf{x}$ *is parallel to the chord from* $\gamma(t_1)$ *to* $\gamma(t_2)$.

Proof. Consider the set

$$S \equiv \{(t_1, t_2, \mathbf{x}) : \mathbf{x} \text{ is the midpoint of the chords based at } \gamma(t_1), \gamma(t_2) \text{ with } A(t_1) = A(t_2)\}.$$

Then the AASS is the projection of S to the (x, y)-plane, and the tangent to S in $\mathbb{R}^4$ projects to the tangent to the AASS in $\mathbb{R}^2$. The conditions on $(t_1, t_2, \mathbf{x})$ are

$$\begin{aligned}
&\frac{\partial A}{\partial t}(t_1, \mathbf{x}) = 0, \text{ i.e. } \mathbf{x} = \frac{1}{2}\left(\gamma(t_1) + \gamma(g(t_1))\right), \\
&\frac{\partial A}{\partial t}(t_2, \mathbf{x}) = 0, \text{ i.e. } \mathbf{x} = \frac{1}{2}\left(\gamma(t_2) + \gamma(g(t_2))\right), \\
&A(t_1, \mathbf{x}) = A(t_2, \mathbf{x}).
\end{aligned}$$

We will use the shorthand A_i to denote $A(t_i, \mathbf{x})$, $\frac{\partial A_i}{\partial t}$ to denote $\frac{\partial A}{\partial t}(t_i, \mathbf{x})$, etc. Consider the map F given by

$$(t_1, t_2, \mathbf{x}) \longmapsto \frac{\partial A_1}{\partial t}, \frac{\partial A_2}{\partial t}, A_1 - A_2.$$

The kernel of the Jacobian matrix of F, projected to $\mathbf{x}$-space, is the tangent line to the AASS. We calculate:

$$J(F) = \begin{pmatrix} \frac{\partial^2 A_1}{\partial t^2} & 0 & \frac{\partial^2 A_1}{\partial x \partial t} & \frac{\partial^2 A_1}{\partial y \partial t} \\ 0 & \frac{\partial^2 A_2}{\partial t^2} & \frac{\partial^2 A_2}{\partial x \partial t} & \frac{\partial^2 A_2}{\partial y \partial t} \\ 0 & 0 & \frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial x} & \frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial y} \end{pmatrix},$$

using $\frac{\partial A_i}{\partial t} \equiv 0$. We assume generically that

$$\frac{\partial^2 A_i}{\partial t^2} \neq 0,$$

which is equivalent to assuming that $\mathbf{x}$ is not an endpoint of the AASS. Suppose that $(\zeta_1, \zeta_2, \zeta_x, \zeta_y)^T \in \ker(JF)$. Then the first two rows of JF determine ζ_1, ζ_2 uniquely, and the tangent to the AASS is $(\zeta_x, \zeta_y)^T$ where

$$\left(\frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial x}\right)\zeta_x + \left(\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial y}\right)\zeta_y = 0,$$

and so the tangent vector to the AASS is parallel to the vector

$$\left(-\frac{\partial A_1}{\partial y} + \frac{\partial A_2}{\partial y}, \frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial x}\right).$$

Let us write $\gamma(s) = (X(s), Y(s))$. Then

$$[\gamma(s) - \mathbf{x}, \gamma'(s)] = (X(s) - x)Y'(s) - (Y(s) - y)X'(s),$$

and so

$$\begin{aligned}
\frac{\partial A_1}{\partial x} &= \frac{\partial}{\partial x}\int_{t_1}^{g(t_1)} [\gamma(s) - \mathbf{x}, \gamma'(s)] ds \\
&= \int_{t_1}^{g(t_1)} \frac{\partial}{\partial x}\left((X(s) - x)Y'(s) - (Y(s) - y)X'(s)\right) ds \\
&= \int_{t_1}^{g(t_1)} -Y'(s) ds = -Y(g(t_1)) + Y(t_1).
\end{aligned}$$

Similarly

$$\frac{\partial A_1}{\partial y} = X(g(t_1)) - X(t_1),$$

and thus

$$\begin{aligned}
\left(-\frac{\partial A_1}{\partial y}, \frac{\partial A_1}{\partial x}\right) &= (-X(g(t_1)) + X(t_1), -Y(g(t_1)) + Y(t_1)) \\
&= \gamma(t_1) - \gamma(g(t_1)),
\end{aligned}$$

and hence this vector is along the chord joining $\gamma(t_1)$ to $\gamma(g(t_1))$. Similarly,

$$\left(\frac{\partial A_2}{\partial y}, -\frac{\partial A_2}{\partial x}\right) = -\gamma(t_2) + \gamma(g(t_2)),$$

and thus (ζ_x, ζ_y) is parallel to the vector

$$(\gamma(t_1) - \gamma(t_2)) + (\gamma(g(t_2)) - \gamma(g(t_1))).$$

But the vectors in the brackets of the above expression are parallel and equal. Thus the tangent to the AASS is parallel to the chord $\gamma(t_1) - \gamma(t_2)$

(or equivalently $\gamma(g(t_2)) - \gamma(g(t_1))$). □

2.7.4 MPTL and AASS in terms of *Area Parallels*

The following interpretation of the MPTL and AASS is a simple but interesting application of the general theory of bifurcation sets.

Area-Parallels are the level-sets of the family of Area Functions $A(t, \mathbf{x})$ defined on an oval γ. The area-parallels of an ellipse are themselves ellipses, or exceptionally the point at the centre of the ellipse, which is the area-parallel at level equal to a half of the total area of the ellipse. The *cusps* of the area-parallels to γ lie on the *MPTL* of γ (see Figure 2.14(a)). The *self-intersections* of the area-parallels lie on the *Affine Area Symmetry Set* (see Figure 2.14(b)).

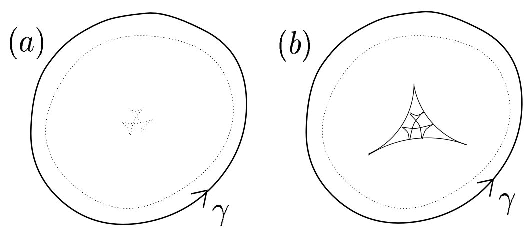


Figure 2.14: *(a) Two area parallels to γ are shown: one is smooth (for small negative distance d_0), the other is singular, with six cusps and three crossings. (b) The cusps of the singular area parallels lie on the MPTL of γ. (c) The self-intersections of the area parallels lie on the AASS of γ (which is not shown).*

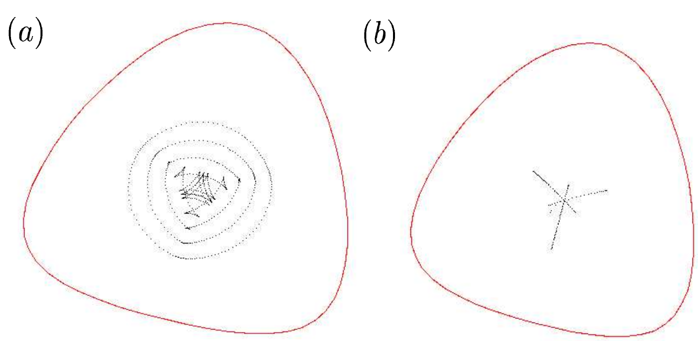


Figure 2.15: *(a) Some area-parallels for an oval. (b) An AASS for an oval.*

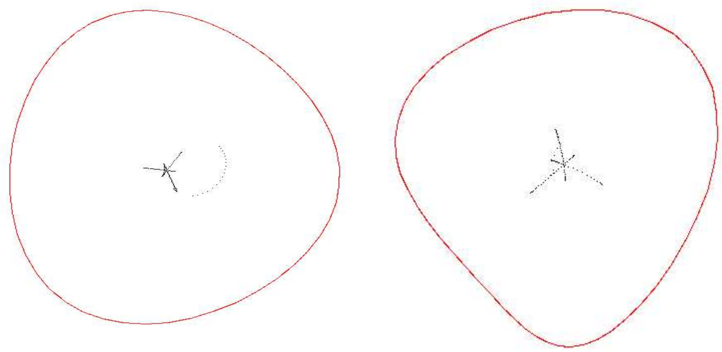


Figure 2.16: *Two AASS plots. Three branches of the AASS are clearly shown.*

2.7.5 Further Research

Question 1: How does the concept of the family of area functions defined on a curve γ generalise for

(i) $\mathbf{x}$ outside the γ, and

(ii) γ as a non-oval?

Question 2: How does the AASS capture the local affine symmetry of a plane curve? Does a straight line AASS imply that the curve is affine symmetric?

Question 3: How can we improve our method of plotting the AASS?

Question 4: How does the AASS fit with other affine-invariant symmetry sets?

2.8 Examples

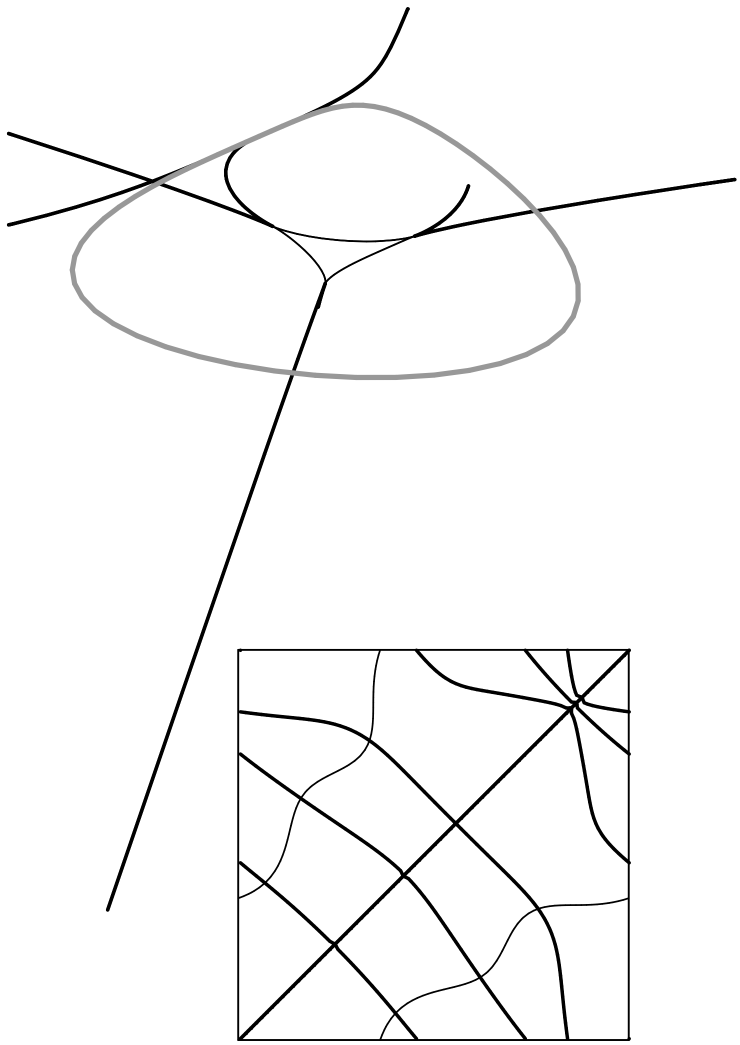


Figure 2.17: *The thick grey curve is the original curve, which has an inflexion. The thin dark curve is the MPTL, and the thick dark curve the truncated AESS. Since the curve is* only just *non-oval, it is difficult to determine what is happening on the AESS. Below the curve is the pre-sets for the AESS and MPTL, the thicker curve being the pre-AESS. The AESS near the inflexion corresponds with the* triple crossing *on the pre-AESS, towards the top right-hand corner of parameter-space.*

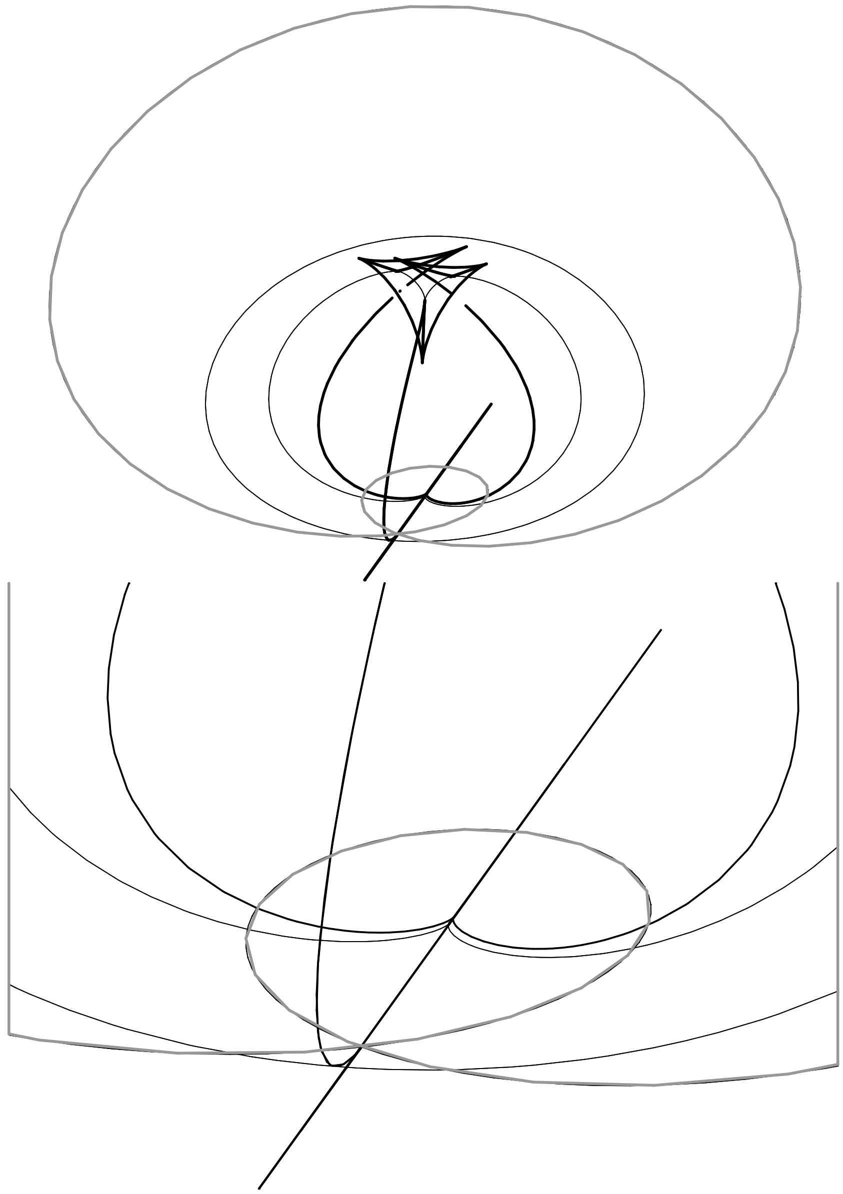


Figure 2.18: *Above is the untruncated AESS (thick dark curve) and MPTL (thin dark curve) for a non-simple curve (thick grey curve). Note that the AESS passes through the self-intersection point of the original curve. It is not easy to see what is happening, and so below is an enlargement of the area around the self-intersection.*

Chapter 3

Affine Distance Symmetry Sets

In this chapter we will consider an alternative affine-invariant symmetry set to the AESS, the subject of Chapter 2 and Chapter 4. This set is defined analogously to the Euclidean Symmetry Set, as was the AESS, but uses an alternative definition. This leads to a genuinely different affine-invariant symmetry set, which we will see can be expressed as (part of) the Full Bifurcation Set of the family of affine distance functions defined on a plane curve (parametrised by points in the plane) as introduced in §1.3.2.

Outline of Chapter 3

§**3.1:** We begin by introducing the *Euclidean Symmetry Set* (SS), defined in Definition 3.1.1 as the bifurcation set of the family of Euclidean distance functions. We briefly summarise some results concerning the local structure of this set from [BGG85].

§**3.2:** We introduce the *Affine Distance Symmetry Set* (ADSS), the affine-invariant analogue of the Symmetry Set as defined in Definition 3.1.1. This set was studied in [GS96], [GS98] for *ovals* only, and we present the conclusions here, along with some other useful results.

§**3.3:** We then extend this study by considering the ADSS in *non-oval* situations. Theorem 3.3.3 presents the full list of generic structures of the ADSS of a plane curve.

§3.4: The main part of this chapter concerns the study of 1-parameter families of Affine Distance Symmetry Sets, with the aim of classifying the transitions which may occur on the ADSS of a plane curve as it is deformed through a 1-parameter family. A full list of transitions on generic 1-parameter families of full bifurcation sets was obtained in [BG86], and furthermore the transitions which are realised on 1-parameter families of Euclidean Symmetry Sets were highlighted. In §§**3.5-3.9**, we carry out this same procedure for the ADSS. The conclusions are interesting and unexpected, exposing a distinct difference between those transitions which may occur on the ADSS of a family of ovals and those which may occur on the ADSS of any generic family of curves.

§3.10: We consider the structure of the ADSS at self-intersections on the original curve.

§3.11: This section contains numerous plots of the ADSS, to illustrate the results of §§3.5-3.9.

3.1 The Euclidean Symmetry Set (SS)

We begin by giving a brief exposition of the analogous definition of the Euclidean Symmetry Set. Let f denote the family of Euclidean distance (squared) functions defined on a curve $\gamma(t)$, given by

$$f(\mathbf{x}, t) \equiv \|\mathbf{x} - \gamma(t)\|^2 = (\mathbf{x} - \gamma(t)) \cdot (\mathbf{x} - \gamma(t)),$$

where $\mathbf{x} \in \mathbb{R}^2$ is the family parameter. We have the following:

Definition 3.1.1. *The* **Euclidean Symmetry Set** (SS) *of a simple, smooth plane curve is the closure of the locus of points* $\mathbf{x} \in \mathbb{R}^2$ *on (at least) two Euclidean normals and equidistant from the corresponding points on the curve.*

This definition of the Euclidean Symmetry Set is entirely equivalent to that given in Definition 1.1.1. To rephrase this in terms of the Euclidean distance function, we consider a curve $\gamma(t)$, where t denotes the Euclidean arclength parameter. Then a point $\mathbf{x} \in \mathbb{R}^2$ lies on the SS of $\gamma(t)$ if and only if there exist two distinct points $\gamma(t_1), \gamma(t_2)$ such that the Euclidean distance function f has

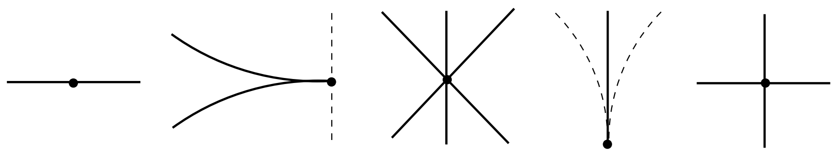
$$f(\mathbf{x}, t_1) = f(\mathbf{x}, t_2), \text{ and } f'(\mathbf{x}, t_1) = f'(\mathbf{x}, t_2) = 0,$$

or if $\mathbf{x}$ is the limit of such points, where $'$ (prime) denotes $\partial/\partial t$. In this case, we say that f has two distinct critical points *on the same level* (that is, the corresponding values of f at these critical points are equal), or, in the limit, that the first *three* derivatives of f vanish for some parameter value t. The set of points $\mathbf{x}$ for which this holds forms the *Levels Bifurcation Set* of the Euclidean distance function f, the closure of the set of $\mathbf{x} \in \mathbb{R}^2$ for which f has two distinct critical points on the same level. It is well-known that the Euclidean *evolute* of a plane curve γ is the set of points $\mathbf{x}$ for which f has a *degenerate* critical point, that is for which there exists some point $\gamma(t_1)$ such that $f'(\mathbf{x}, t_1) = f''(\mathbf{x}, t_1) = 0$, and the evolute and the Symmetry Set together form the *Full Bifurcation Set* of the family of Euclidean distance functions defined on the curve $\gamma(t)$ and parametrised by points in the plane. It is thus more natural to study the Symmetry Set and the evolute together, using the powerful and well-developed theory of bifurcation sets. In this way, the local structure of the Symmetry Set of a generic plane curve, as (part of) a full bifurcation set, can be deduced, and the possible local structures of the SS are listed as follows:

Theorem 3.1.2 ([BGG85]). *The five possible local structures for the SS (and evolute) of a generic plane curve are shown in Figure 3.1 as (i)* smooth *SS; (ii)* cusp *on the SS (at a smooth point of the evolute); (iii)* triple crossing *on the SS; (iv)* endpoint *of the SS (at a cusp of the evolute); (v)* crossing *on the SS.*

Notation: We use a variant of Arnold's A_k notation to express the *singularity types* of each of the local structures of the SS, denoting the type of singularity that the Euclidean distance function f exhibits:

- A_1^2-points occur when f has two distinct ordinary (or A_1) critical points on the same level;



(i) A_1^2 (ii) A_1A_2 (iii) A_1^3 (iv) A_3 (v) A_1^2/A_1^2

Figure 3.1: *(i)-(v) illustrate the possible local structures of the SS (with the evolute shown dashed), along with their corresponding singularity types.*

- A_1A_2-points occur when f has two critical points on the same level, one of which is a degenerate (or A_2) critical point;
- A_1^3-points occur when f has three critical points on the same level;
- A_3-points occur when three derivatives of f vanish for some parameter value; and finally,
- A_1^2/A_1^2 is used to denote the fact that we have a point which lies on two separate A_1^2 branches of the SS. Thus an A_1^2/A_1^2-point occurs when f has two distinct critical points on the same level, and two other distinct critical points on the same level, with the '/' denoting the fact that these levels need not be the same.

Remark 3.1.3. *The last case is inherently different from the preceeding cases, since an A_1^2/A_1^2-point is at the centre of two distinct circles tangent to the curve, whereas the other types correspond to centres of single circles with varying degrees of contact with the curve. The A_1^2/A_1^2-point can be thought of as the accidental intersection of two smooth (A_1^2) SS segments.*

3.2 Introducing the Affine Distance Symmetry Set

Constructing an *affine-invariant* symmetry set analogous to the Euclidean Symmetry Set defined begins with the concept of *affine distance.* In §1.3.2, we introduce the family of affine distance functions, d, defined on a curve $\gamma(s)$ as

$$d(\mathbf{x}, s) = [\mathbf{x} - \gamma(s), \gamma'(s)],$$

where $\mathbf{x} \in \mathbb{R}^2$ is the family parameter. The analogous symmetry set in the affine case is defined as follows:

Definition 3.2.1. *The* **Affine Distance Symmetry Set** (ADSS) *of a simple, smooth plane curve is the closure of the locus of points* $\mathbf{x} \in \mathbb{R}^2$ *on (at least) two affine normals and affine-equidistant from the corresponding points on the curve.*

For a curve $\gamma(s)$, a point $\mathbf{x} \in \mathbb{R}^2$ is on the Affine Distance Symmetry Set of $\gamma(s)$ if and only if there exist two different points $\gamma(s_1), \gamma(s_2)$ such that

$$d(\mathbf{x}, s_1) = d(\mathbf{x}, s_2), \text{ and } d'(\mathbf{x}, s_1) = d'(\mathbf{x}, s_2) = 0,$$

or if $\mathbf{x}$ is the limit of such points. The ADSS of a curve $\gamma(s)$ is the *Levels Bifurcation Set* of the family of affine distance functions defined on $\gamma(s)$ and parametrised by points in the plane, that is, the closure of the set of points $\mathbf{x} \in \mathbb{R}^2$ for which d has two distinct critical points on the same level. This can be considered as part of the *Full Bifurcation Set* along with the affine evolute, which is of course defined to be the closure of the set of points $\mathbf{x} \in \mathbb{R}^2$ for which d has a degenerate singularity, that is, $\mathbf{x} \in \mathbb{R}^2$ such that there exists s_1 with $d'(\mathbf{x}, s_1) = d''(\mathbf{x}, s_1) = 0$. We use this bifurcation set structure of the ADSS to deduce the local structure of the ADSS of a generic plane curve, in the same way that we can deduce the structure of the SS of any generic plane curve. The results are entirely analogous, and are set out in the §3.2.5 for oval curves, and extended in §3.3 to include non-oval curves.

3.2.1 The *ADSS Condition* and the pre-ADSS

Proposition 3.2.2 (ADSS Condition [GS98]). *Given a smooth, simple curve* $\gamma(s)$*, the necessary and sufficient condition for distinct* s_1, s_2 *to give a point of the ADSS is*

$$[\gamma(s_1) - \gamma(s_2), \gamma''(s_1) - \gamma''(s_2)] = 0,$$

where affine-arclength parametrisation is assumed.

Proof. If s_1 and s_2 contribute to the ADSS, then

$$\begin{aligned} & d'(\mathbf{x}_0, s_1) = d'(\mathbf{x}_0, s_2) = 0, && (3.1)\\ \iff & [\mathbf{x}_0 - \gamma(s_1), \gamma''(s_1)] = [\mathbf{x}_0 - \gamma(s_2), \gamma''(s_2)] = 0, \\ \iff & \mathbf{x}_0 - \gamma(s_1) = \lambda_1 \gamma''(s_1), && (3.2)\\ & \text{and } \mathbf{x}_0 - \gamma(s_2) = \lambda_2 \gamma''(s_2), && (3.3) \end{aligned}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$, and

$$\begin{aligned} & d(\mathbf{x}_0, s_1) = d(\mathbf{x}_0, s_2), \\ \iff & [\mathbf{x}_0 - \gamma(s_1), \gamma'(s_1)] = [\mathbf{x}_0 - \gamma(s_2), \gamma'(s_2)], \\ \iff & [\lambda_1 \gamma''(s_1), \gamma'(s_1)] = [\lambda_2 \gamma''(s_2), \gamma'(s_2)], \\ \iff & \lambda_1 = \lambda_2 = \lambda \text{ say, using (3.2), (3.3) and } [\gamma'(s_i), \gamma''(s_i)] = 1. \end{aligned}$$

Thus (3.2) and (3.3) tell us that

$$\begin{aligned} & \gamma(s_1) + \lambda\gamma''(s_1) = \gamma(s_2) + \lambda\gamma''(s_2), \\ \iff & \gamma(s_1) - \gamma(s_2) = \lambda(\gamma''(s_2) - \gamma''(s_1)), && (3.4)\\ \iff & [\gamma(s_1) - \gamma(s_2), \gamma''(s_1) - \gamma''(s_2)] = 0, \end{aligned}$$

as required. □

This expression picks out pairs of parameter values (s_1, s_2) which contribute to the ADSS, and provides us with what we will call the *'ADSS Condition'*. It leads to the following, which will be of use to us presently.

Corollary 3.2.3. *If two points $\gamma(s_1), \gamma(s_2)$ contribute point $\mathbf{x}_0$ to the ADSS of a smooth, simple plane curve γ, then*

$$\gamma(s_1) - \gamma(s_2) = -d_0(\gamma''(s_2) - \gamma''(s_1)),$$

where d_0 is the common affine distance from $\mathbf{x}_0$ to the curve γ through the points $\gamma(s_1)$ and $\gamma(s_2)$ (that is, $d_0 \equiv [\mathbf{x}_0 - \gamma(s_i), \gamma'(s_i)]$).

Proof. From above:

$$\begin{aligned} d_0 &= [\mathbf{x}_0 - \gamma(s_1), \gamma'(s_1)], \\ &= [\lambda\gamma''(s_1), \gamma'(s_1)], \text{ using } (3.2), \\ &= -\lambda, \text{ since } [\gamma'(s_1), \gamma''(s_1)] \equiv 1. \end{aligned}$$

Thus the result follows from (3.4). □

The locus of the points picked out by Proposition 3.2.2, as a set of points in parameter-space, is called the *pre-ADSS*. Note that we have made the implicit assumption that the curve $\gamma(s)$ is an *oval*, since affine-arclength is not defined at an inflexion point on the curve. To apply this definition of the ADSS to any generic plane curve, we must consider carefully how this condition should be interpreted at double tangents and inflexions: §3.2.2 and §3.2.3 contain discussions of these cases.

3.2.2 The *ADSS Condition* at a double tangent

Consider two curve segments $\gamma_1(s_1)$, $\gamma_2(s_2)$, parametrised by affine-arclength, and having a double tangent at $s_1 = s_2 = 0$. Then the vectors $\gamma_1(0) - \gamma_2(0)$, $\gamma_1'(0)$ and $\gamma_2'(0)$ are in the same direction (being along the double tangent), and thus

$$[\gamma_1(0) - \gamma_2(0), \gamma_1'(0)] = [\gamma_1(0) - \gamma_2(0), \gamma_2'(0)] = 0.$$

Then the ADSS Condition holds at $s_1 = s_2 = 0$ if and only if

$$[\gamma_1'(0), \gamma_1''(0) - \gamma_2''(0)] = 0,$$

which in turn hold if and only if $[\gamma_1'(0), \gamma_2''(0)] = 1$, using the fact that $[\gamma_1'(0), \gamma_1''(0)] \equiv 1$. Now $\gamma_1'(0)$ and $\gamma_2'(0)$ are in the same direction, and so $\gamma_1'(0) \equiv a\gamma_2'(0)$ for some $a \in \mathbb{R}$. Substituting this into the above, we see that

$$a[\gamma_2'(0), \gamma_2''(0)] = 1,$$

and hence $a \equiv 1$, since we know that $[\gamma_2'(0), \gamma_2''(0)] = 1$. Thus the ADSS Condition can hold at a double tangent if and only if the affine tangents at the points of contact of the curve with the double tangent are *identical*. This is a non-generic occurence, and so *a double tangent does not contribute to the ADSS of a generic plane curve.* (This is in contrast to the AESS, where the double tangent *always* contributes to the AESS, which is shown a smooth curve tangential to the double tangent. See Proposition 2.5.4.)

3.2.3 The ***ADSS Condition*** at inflexions

To apply the ADSS Condition (given in Proposition 3.2.2) to situations involving inflexional curve segments, we must consider the following cases.

The ***ADSS Condition*** at a single ordinary inflexion

Consider a single inflexional curve segment γ as shown in Figure 3.2, with the inflexion at $\gamma(0)$. Either side of the inflexion, the affine normals are in opposite directions, and this means that the affine distance from the intersection of these normals to each of the corresponding points must be of opposite sign. Hence points on either side of the inflexion point cannot contribute to the ADSS. (This is in contrast to the AESS, which is shown to approach an ordinary inflexion tangentially and have an endpoint there. See Proposition 2.5.3.)

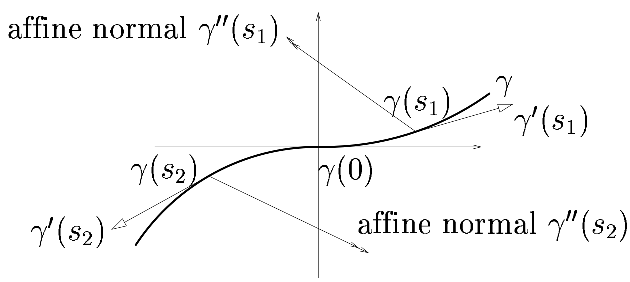


Figure 3.2: *The ADSS Condition at a single ordinary inflexion.*

The ***ADSS Condition*** where the tangent at an ordinary inflexion cuts the curve again

Consider two curve segments $\gamma_1(s_1)$ and $\gamma_2(s_2)$, with an ordinary inflexion at $\gamma_1(0)$ (see Figure 3.3), and the tangent line there cutting γ_2 at $\gamma_2(0)$. As s_1

tends to 0, $\gamma_1''(s_1)$ tends to a vector of infinite length in the direction of the inflexional tangent to γ_1 at $\gamma_1(0)$ (see §1.3.1). Since $\gamma_2''(0)$ is finite for $s_2 = 0$, we see that the direction of vector $\gamma_1''(s_1) - \gamma_2''(s_2)$ approaches the direction of the inflexional tangent as s_1, s_2 tend to zero, which is in the direction of vector $\gamma_1(0) - \gamma_2(0)$. Hence the ADSS Condition is satisfied for $s_1 = s_2 = 0$. The ADSS point is at curve point $\gamma_2(0)$, since this point is at affine distance *zero* from the inflexion (see §1.3.2) and along the affine normal there, and also at distance zero along the affine normal to γ_2 at $\gamma_2(0)$. The structure of the ADSS local to this point is deduced in §3.3.1.

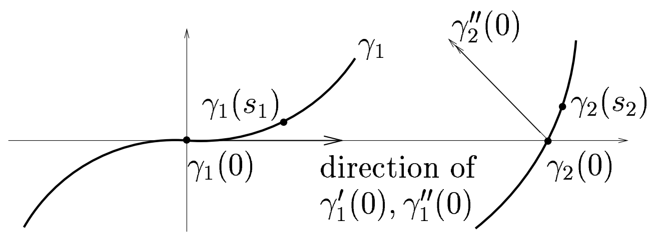


Figure 3.3: *The ADSS point is on the γ_2 curve segment, at the point $\gamma_2(0)$ where the inflexional tangent crosses.*

The *ADSS Condition* where two inflexional tangents intersect

The reason why this situation should contribute to the ADSS is better explained using the idea of a *'4+4 conic pair'*, developed in §3.2.5. The local structure of the ADSS in this situation is derived in §3.3.2.

3.2.4 From the pre-ADSS to the ADSS

Once we have found the pre-ADSS for a given curve, the next task is to map this set of points in parameter-space to the ADSS itself. Suppose two points $\gamma(s_1)$ and $\gamma(s_2)$ contribute a point to the ADSS of a curve γ, that is, (s_1, s_2) lies on the pre-ADSS. Bracketing both sides of the expression in Corollary 3.2.3 with $\gamma''(s_1)$, we can deduce that

$$-d_0 = \frac{[\gamma(s_1) - \gamma(s_2), \gamma''(s_1)]}{[\gamma''(s_2), \gamma''(s_1)]}.$$

The ADSS point $\mathbf{x}_0$ is given by

$$\mathbf{x}_0 = \gamma(s_1) - d_0\gamma''(s_1).$$

Thus, once we have found a pair of points $\gamma(s_1), \gamma(s_2)$ which satisfy the ADSS condition, then we can plot the corresponding ADSS point $\mathbf{x}_0$ using the mapping

$$\mathbf{x}_0 = \gamma(s_1) + \frac{[\gamma(s_1) - \gamma(s_2), \gamma''(s_1)]}{[\gamma''(s_2), \gamma''(s_1)]}\gamma''(s_1). \tag{3.5}$$

This enables us to produce computer plots of the ADSS using, for example, the graphics package [LSMP]. For a given curve γ (expressed parametrically), the zeros of the equation in Proposition 3.2.2 are found to give the pre-ADSS, and for each point (s_1, s_2) of the pre-ADSS we use the map (3.5) to plot the ADSS itself. Some examples of these plots are found in §3.2.5, §3.11, and throughout Chapter 6.

3.2.5 The local structure of the ADSS

For a point $\mathbf{x}$ to lie on the ADSS of a smooth plane curve γ it must lie at the intersection of *two* distinct affine normals to the curve, and *affine-equidistant* from the corresponding curve points. Geometrically, this is equivalent to $\mathbf{x}$ being at the *common centre of two distinct conics having 4-point contact with the curve and sharing the same affine radius.* Such a pair of conics we will refer to as a *'4+4 conic pair'*.

It is with reference to this geometric interpretation of the ADSS that we shall phrase our results concerning the local structure of the ADSS of a generic plane curve. We begin by stating the following result from [GS98], which assumes that plane curve γ is an oval.

Theorem 3.2.4 ([GS98]). *A point* $\mathbf{x}$ *lies on the ADSS of a plane curve* γ *if there exist two (or more) distinct conics with common centre* $\mathbf{x}$ *having* $\geq$ 4*-point contact with* γ *in two (or more) distinct points, and having the same affine radius. Locally, the Affine Distance Symmetry Set of a generic plane curve* γ *at such a point* $\mathbf{x}$ *is:*

- **smooth** *when both conics have exactly 4-point contact with* γ*;*
- *an* **ordinary cusp** *when one of the conics has 5-point contact with* γ *(*$\mathbf{x}$ *is then on the affine evolute of* γ *too, at a smooth point of it);*
- *an* **endpoint** *when* $\mathbf{x}$ *is the centre of a 6-point contact conic, that is, a conic tangent to* γ *at a sextactic point: the endpoint is then in a cusp of the affine evolute;*
- *a* **triple crossing** *when there are* three *conics centred at* $\mathbf{x}$ *having equal affine radius and 4-point contact with* γ*.*

We will use a variant of Arnold's A_k notation to express the type of singularity that the affine distance function has at a point $\mathbf{x}$. For a generic plane curve, we should expect the distance function to have A_1^2 singularities, that is, to have two A_1 singularities with the same value of d. We refer to such a point $\mathbf{x}$ for which this is true as an A_1^2-point, and the first part of Theorem 3.2.4 above then says that the ADSS of a plane curve is smooth at an A_1^2-point. We should also expect to find isolated A_1A_2-points, where the distance function has two singularities on the same level, one of which is a degenerate singularity. The second part of Theorem 3.2.4 says that the ADSS generically has an ordinary cusp at these points. Similarly, we should also expect to find isolated A_3-points, where three derivatives of the distance function vanish, and these are the endpoints of the ADSS as mentioned in the third part of Theorem 3.2.4. Finally, we should expect to find isolated A_1^3-points, and by Theorem 3.2.4 this gives us a triple crossing on the ADSS, formed by taking the three pairs of A_1 singularities in turn to give us three smooth (A_1^2) branches of the ADSS. This is the complete series of singularities that we should expect to observe on the ADSS of a generic oval: a formal reason for this is outlined in [BGG85] (p.169). Note that do not include the A_1^2/A_1^2 case here, since this is simply the accidental intersection of two branches of the ADSS and of no interest to this local analysis.

In §3.3 we extend this classification to include *any* generic plane curve. Furthermore, in §3.4 this analysis will be extended to the study of 1-parameter families of Affine Distance Symmetry Sets, that is, we will consider the transitions that may occur on the ADSS of a plane curve as it is deformed through a 1-parameter family.

Before moving on to the analysis of non-oval situations and the study of 1-parameter families of the ADSS, we state a result which will be of use presently. We will refer to this result as the *Concurrent Tangents Condition* for the ADSS.

Proposition 3.2.5 (Concurrent Tangents Condition, [GS98]). *Suppose two points* $\gamma(s_1)$, $\gamma(s_2)$ *contribute point* $\mathbf{x}$ *to the ADSS of a curve* γ, *parametrised by affine-arclength* s. *Then the tangent line to the ADSS at* $\mathbf{x}$ *is in the direction* $\gamma'(s_1) - \gamma'(s_2)$, *and concurrent with the corresponding tangent lines at* $\gamma(s_1), \gamma(s_2)$. *(See Figure 3.4.)*

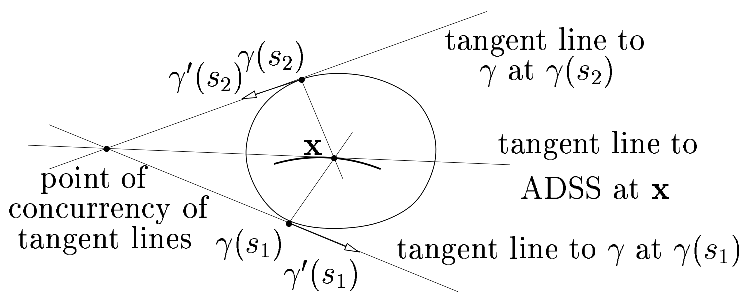


Figure 3.4: *Illustration of the* Concurrent Tangents Condition *for the ADSS.*

Remark 3.2.6. *Proposition 3.2.5 is reminiscent of Proposition 2.4.5, the Concurrent Tangents Condition for the AESS, which concerns the direction of the tangent line to the AESS.*

3.3 The local structure of the ADSS for non-ovals

We now consider the two generic occurrences on a *non-oval* plane curve which contribute points to the ADSS, and which are not covered by the result of Theorem 3.2.4.

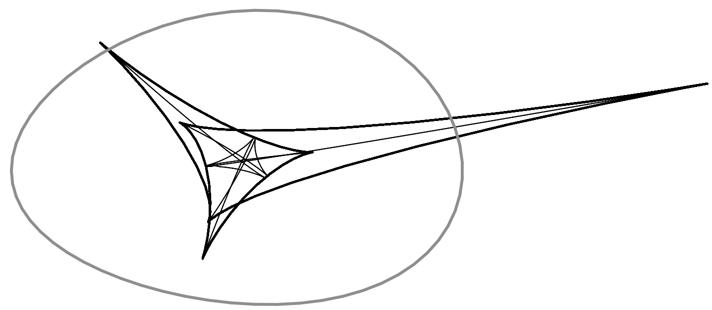


Figure 3.5: *The ADSS of an oval. The grey curve is the oval, the thicker dark curve the affine evolute, and the thinner dark curve the ADSS. Note that the cusps of the ADSS lie on the affine evolute, and the endpoints of the ADSS lie at the cusps of the affine evolute.*

3.3.1 Inflexional tangent cuts curve

Consider two curve segments γ_1 and γ_2 given by

$$\begin{aligned}\gamma_1(s) &= (X_1(s), Y_1(s)) = (s, a_3 s^3 + a_4 s^4 + \ldots),\\ \gamma_2(t) &= (X_2(t), Y_2(t)) = (c + b_1 t + b_2 t^2 + \ldots, t).\end{aligned}$$

For parameter values $(s,t) = (0,0)$, we have an inflexional tangent to γ_1 at $\gamma_1(0)$ cutting the curve segment γ_2 at $\gamma_2(0)$.

We begin by finding the equation for the pre-ADSS for this situation. The pre-ADSS is defined to be the set of parameter pairs (s,t) for which

$$[\gamma_1(s) - \gamma_2(t), \gamma_1''(s) - \gamma_2''(t)] = 0.$$

Since we are assuming that neither s nor t are affine-arclength parameters along the respective curve segments, we must use the formulae from §1.3.3. The above becomes (omitting the parameters s, t)

$$\left[\gamma_1 - \gamma_2, k_1^{-2/3}\ddot{\gamma}_1 - \frac{1}{3}\dot{k}_1 k_1^{-5/3}\dot{\gamma}_1 - k_2^{-2/3}\ddot{\gamma}_2 + \frac{1}{3}\dot{k}_2 k_2^{-5/3}\dot{\gamma}_2\right] = 0.$$

If we multiply the right-hand vector by $k_1^{5/3}k_2^{5/3}$, disregarding the fact that

$k_1(0) = 0$, then we get

$$\left[\gamma_1 - \gamma_2, k_1 k_2^{5/3}\ddot{\gamma}_1 - \frac{1}{3}\dot{k}_1 k_2^{5/3}\dot{\gamma}_1 - k_1^{5/3} k_2 \ddot{\gamma}_2 + \frac{1}{3}\dot{k}_2 k_1^{5/3}\dot{\gamma}_2\right] = 0, \tag{3.6}$$

and this defines the pre-ADSS. Calculation shows that

$$k_1^{5/3} = (6a_3)^{5/3}\overline{s}\left(1 + \frac{10a_4}{a_3}s + \ldots\right),$$

where we have denoted $s^{5/3}$ by $\overline{s}$. We will also denote $k_1^{5/3}$ by K_1 and $k_2^{5/3}$ by K_2. Expanding (3.6) can rewrite it as

$$c_1 = c_2\overline{s},$$

where c_1, c_2 are functions of s, t given by

$$\begin{aligned}
c_1 &= (X_1 - X_2)\left(k_1 k_2^{5/3}\ddot{Y}_1 - \frac{1}{3}\dot{k}_1 k_2^{5/3}\dot{Y}_1\right) - (Y_1 - Y_2)\left(k_1 k_2^{5/3}\ddot{X}_1 - \frac{1}{3}\dot{k}_1 k_2^{5/3}\dot{X}_1\right), \\
c_2 &= (6a_3)^{5/3}\left(1 + \frac{10a_4}{a_3}s + \ldots\right)\left((X_1 - X_2)\left(k_2\ddot{Y}_2 - \frac{1}{3}\dot{k}_2\dot{Y}_2\right)\right) \\
&\qquad -(Y_1 - Y_2)\left(k_2\ddot{X}_2 - \frac{1}{3}\dot{k}_2\dot{X}_2)\right).
\end{aligned}$$

This is the equation of the pre-ADSS, and it defines *precisely* the same set of points as the equation $c_1^3 = c_2^3 s^5$. Calculation shows that the lowest terms in the equation $c_1^3 - c_2^3 s^5 = 0$ are s^5 and t^3 (this is true under the generic condition that the inflexional tangent is not along the affine normal to γ_2 at $\gamma_2(0)$) – if we set $weight(s) = 1/5$, $weight(t) = 1/3$, then all other terms in this expression are of weight > 1). Thus we can parametrise the pre-ADSS by

$$s = u^3,\ t = c_5 u^5 + c_6 u^6 + c_7 u^7 + \mathcal{O}(s^8).$$

We can substitute this into the equation for the pre-ADSS to find that

$$c_5 = -\frac{3(9a_3^2 b_2)^{1/3} b_3 c}{b_2^2},\ c_6 = -15a_3 c,\ c_7 = 0.$$

Hence we have shown that the pre-ADSS has a (3, 5)-singularity in this situation (see Figure 3.6(a)).

We now use the standard formula for mapping the pre-ADSS to the ADSS itself, as outlined in §3.2.4. Calculation shows that the ADSS curve (X, Y) is given by

$$(X(u), Y(u)) = \left(c - 2 \times 3^{5/3} a_3^{2/3} b_2^{1/3} c u^5 + \ldots, -15 a_3 c u^6 + \ldots\right).$$

Thus the ADSS in this situation is singular (see Figure 3.6(b) for a schematic illustration of this singularity).

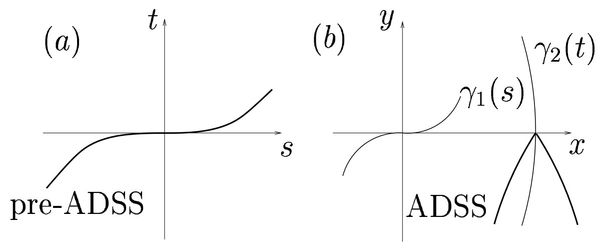


Figure 3.6: *Schematic representations of the pre-ADSS and the ADSS for the case where an inflexional tangent cuts the curve again. See* §3.3.1.

Remark 3.3.1.

(i) Note that the singularity is higher than we might expect, being a (5, 6)*-singularity rather than a* (3, 4)*-singularity (a swallowtail). However, although this situation is highly singular, it is stable in this setting. The* (5, 6)*-singularity will not vanish under perturbation of the original curve.*

(ii) Phrasing this in terms of conic pairs, this situation corresponds to the case of the 6+4 conic pair comprising the repeated line-pair of the inflexional tangent to γ_1 *and the tangent to* γ_2 *at the point where the inflexional tangent crosses* γ_2*, both counted twice.*

3.3.2 Two inflexional tangents meet

Consider two curve segments γ_1 and γ_2 given by

$$\gamma_1(s) = (X_1(s), Y_1(s)) = (s, a_3s^3 + a_4s^4 + \ldots),$$
$$\gamma_2(t) = (X_2(t), Y_2(t)) = (t, d + b_1t + b_3t^3 + \ldots),$$

where we will assume that $a_3b_1b_3d \neq 0$. For parameter values $(s,t) = (0,0)$, we have inflexions on γ_1 and γ_2, and the inflexional tangents meet at $(-d/b_1, 0)$.

The pre-ADSS is given by

$$\left[\gamma_1 - \gamma_2, k_1^{-2/3}\ddot{\gamma}_1 - \frac{1}{3}\dot{k}_1k_1^{-5/3}\dot{\gamma}_1 - k_2^{-2/3}\ddot{\gamma}_2 + \frac{1}{3}\dot{k}_2k_2^{-5/3}\dot{\gamma}_2\right] = 0.$$

In this case both k_1 and k_2 are zero at $s = t = 0$. We use the same sleight-of-hand as in §3.3.1 to remove these zeros, and after some calculation we are able to show that the initial terms of the equation of the pre-ADSS are given by

$$a_3^3b_3^5d^3s^5 \text{ and } a_3^5b_3^3d^3t^5.$$

Thus we deduce that the pre-ADSS has a *smooth* branch passing through the origin in parameter-space, and we can write t as a function of s local to $s = t = 0$ as follows:

$$t = \frac{(a_3^2b_3^3)^{1/5}}{b_3}s + \mathcal{O}(s^2)$$

We now use the standard formula for mapping the pre-ADSS to the ADSS itself, as outlined in §3.2.4. Calculation shows that the ADSS (X,Y) itself is given by

$$(X,Y) = \left(e_0 + e_2s^2 + e_3s^3 + \ldots, f_2s^2 + f_3s^3 + \ldots\right),$$

where

$$e_0 = -\frac{d}{b_1},$$

$$
\begin{aligned}
e_2 &= \frac{15d}{b_1^2}\left(a_3 - \frac{(a_3^2 b_3^3)^{2/5}}{b_3}\right),\\
e_3 &= -4\frac{a_4 d}{b_1^2} + 12\frac{a_4 b_3^{1/5} d}{a_3^{1/5} b_1^2} + 2\frac{a_3^{6/5}}{b_1 b_3^{1/5}} + 16\frac{a_3}{b_1} - 18\frac{a_3^{4/5} b_3^{1/5}}{b_1} - 8\frac{a_3^{6/5} b_4 d}{b_1^2 b_3^{6/5}},\\
f_2 &= \frac{15a_3 d}{b_1} s^2,\\
f_3 &= 4\left(4a_3 - \frac{a_4 d}{b_1}\right).
\end{aligned}
$$

Thus the ADSS exhibits an ordinary cusp at $(-d/b_1, 0)$, where the inflexional tangents meet, provided that $e_2 f_3 - e_3 f_2 \neq 0$, which will hold generically. See Figure 3.7.

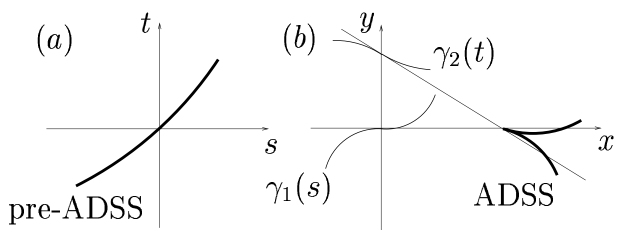


Figure 3.7: *Schematic representations of the pre-ADSS and the ADSS for the case where two inflexional tangents meet. See* §3.3.2.

Remark 3.3.2. *Phrasing this in terms of conics pairs, this situation corresponds to the case of the 4+4 conic pair comprising the* line-pair *of inflexional tangents, both counted twice. It is in fact a 6+6 conic pair, with centre at the intersection of the tangents. It is interesting to note that this 6+6 conic pair leads to a less singular ADSS than the 6+4 conic pair corresponding to the 'inflexional tangent cuts curve' situation considered in §3.3.1.*

We have now classified all the generic structures that may appear on the ADSS of a generic plane curve.

Theorem 3.3.3. *A point* $\mathbf{x}$ *lies on the ADSS of a plane curve* γ *if there exists a 4+4 conic pair with common centre* $\mathbf{x}$*. Locally, the Affine Distance Symmetry Set of a generic plane curve* γ *at such a point* $\mathbf{x}$ *is:*

- *one of the normal forms listed in Theorem 3.2.4 if the conic pair is non-degenerate.*

- *an* **ordinary cusp** *if the conic pair is the repeated line-pair comprising two inflexional tangents to* γ *counted twice;*
- *a* $(5,6)$**-singularity** *if the conic pair is the repeated line-pair comprising an inflexional tangent to* γ *together with the tangent to* γ *at a point where this inflexional tangent cuts the curve again, both counted twice.*

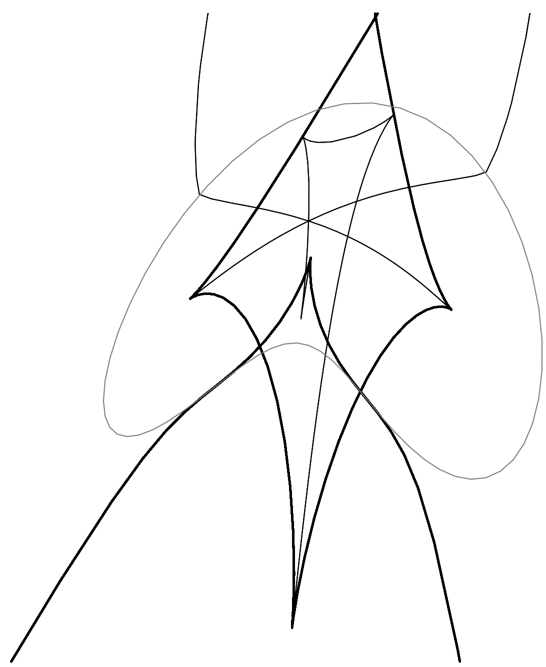


Figure 3.8: *An [LSMP] plot of the ADSS of a non-oval. The lightest curve is the non-oval plane curve, the thicker dark curve the (truncated) affine evolute, and the thinner dark curve the ADSS. Note that: two cusps of the ADSS lie on the affine evolute, the other cusp occuring at the point where the two inflexional tangents to the curve meet; the endpoints of the ADSS lie at the cusps of the evolute; the evolute inflects the curve at inflexions of the latter; the ADSS exhibits a singularity where the inflexional tangents to the curve cut the curve again. (The structure of the truncated evolute at the top of the picture is a crossing,* not *a cusp.)*

3.4 Families of Affine Distance Symmetry Sets

The singularities we should expect to observe on a 1-parameter family of Affine Distance Symmetry Sets, as well as those listed in §3.2.5, are A_1^4, $A_1^2A_2$, A_1A_3, A_2^2 and A_4.

In the study of 1-parameter families of Euclidean Symmetry Sets in [BG86], a full list of all the possible transitions that may occur on the full bifurcation set of a generic family of functions is obtained using the well-known theory of discriminants. For reference, Figure 3.9 illustrates the transitions that will be considered in this thesis. In [BG86], it is shown that, although two distinct transitions are generically possible for (most of) the various singularity types, the specific geometrical situation rules out some of the transitions from actually occurring on the Symmetry Set. For example, in the A_1^4 case, two transitions are generally possible for generic full bifurcation sets, but only one (labelled $A_1^4(a)$) may actually occur (and has been observed) on the SS.

We would like to carry out a similar analysis of the transitions on 1-parameter families of Affine Distance Symmetry Sets, that is, we would like to classify the transitions which may actually occur on the ADSS of a smooth plane curve as this curve is deformed through a 1-parameter family. The details of the method used are contained in [BG86]. Here, we will highlight the important aspects of the theory by working through the simplest situation, the A_1^4 case, in detail. Readers who require a more analytical exposition of the methods used in the next section are encouraged to read the article [BG86].

The conclusions can be summarised as follows.

Theorem 3.4.1. *The transitions of Figure 3.9 that may occur on the Affine Distance Symmetry Set of a* generic oval *are precisely those transitions which may occur on the Euclidean Symmetry Set of a generic plane curve, namely* $A_1^4(a)$, $A_1^2A_2(a)$, $A_1A_3(a)$, $A_2^2(a)$, $A_2^2(b)$ *and* A_4 *as illustrated in Figure 3.9.*

Theorem 3.4.2. *The transitions of Figure 3.9 that may occur on the ADSS of a* generic plane curve *are* $A_1^4(a)$, $A_1^4(b)$, $A_1^2A_2(a)$, $A_1A_3(a)$, $A_1A_3(b)$, $A_1^2A_2(b)$, $A_2^2(a)$, $A_2^2(b)$ *and* A_4 *as illustrated in Figure 3.9.*

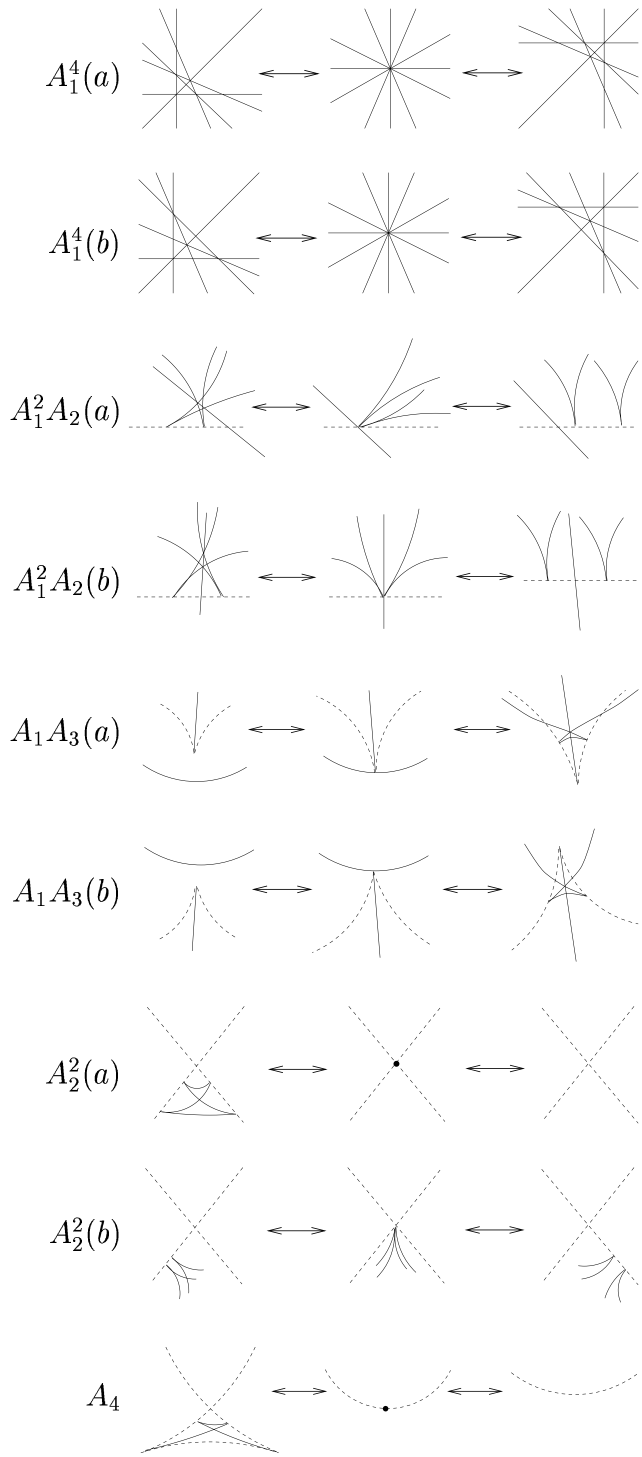


Figure 3.9: *Some generic transitions that occur on 1-parameter families of full bifurcation sets in the plane (from [BG86]).*

3.5 The A_1^4 transitions

In this section, we will use the simplest case, that of the A_1^4 transitions, to illustrate the methods of [BG86] by which we will to classify the transitions that may occur on 1-parameter families of Affine Distance Symmetry Sets.

We begin by considering the *standard multi-versal unfolding* of an A_1^4 singularity, given by

$$G\colon \mathbb{R}^{(4)} \times \mathbb{R}^3 \to \mathbb{R},$$

where $\mathbb{R}^{(4)}$ denotes the set of parameters t_1, t_2, t_3, t_4, $\mathbb{R}^3$ denotes the (y_1, y_2, y_3)-space of unfolding parameters, and multi-versal unfolding G is given by the four unfoldings

$$\begin{aligned} G_1&\colon (t_1, \mathbf{y}) \mapsto t_1^2 + y_1, \\ G_2&\colon (t_2, \mathbf{y}) \mapsto t_2^2 + y_2, \\ G_3&\colon (t_3, \mathbf{y}) \mapsto t_3^2 + y_3, \\ G_4&\colon (t_4, \mathbf{y}) \mapsto t_4^2, \end{aligned}$$

where $\mathbf{y}$ denotes the unfolding parameters (y_1, y_2, y_3). Following the method as outlined in [BG86], the first task is to find the *Big Bifurcation Set* (BBS) of standard unfolding G, which sits in $\mathbf{y}$-space. This object contains all the possible bifurcation sets in a neighbourhood of the A_1^4 singularity of which G is a multi-versal unfolding. The A_1^4-point itself sits at the origin in this space. The individual bifurcation sets can be recovered by *'slicing'* this BBS with (non-singular) surfaces near to the origin in $\mathbf{y}$-space: the intersection of the BBS with a surface will (locally) produce one such bifurcation set. We then consider a generic family of these *'slicing surfaces'* passing through the origin, which gives us a family of bifurcation sets passing through the A_1^4-point, precisely the transitions we wish to study.

The BBS comprises subsets of $\mathbf{y}$-space corresponding to the A_1^2-sets of G, defined to be the set of values of $t_i (i = 1, 2, 3, 4)$ for which G (considered as a function of t_i, with parameters $\mathbf{y}$) has an A_1^2-singularity. Now G has an A_1^2-singularity if and only if any two of its defining functions are singular and have the same value. The set of values of $t_i (i = 1, \ldots, 4)$ for which G has an

A_1^2-singularity will be called the A_1^2-set.

$\partial G_i/\partial t_i = \partial G_j/\partial t_j = 0$	$G_i = G_j$	A_1^2-sets of the BBS
$t_1 = t_2 = 0$	$t_1^2 + y_1 = t_2^2 + y_2$	$y_1 = y_2$
$t_1 = t_3 = 0$	$t_1^2 + y_1 = t_3^2 + y_3$	$y_1 = y_3$
$t_1 = t_4 = 0$	$t_1^2 + y_1 = t_4^2$	$y_1 = 0$
$t_2 = t_3 = 0$	$t_2^2 + y_2 = t_3^2 + y_3$	$y_2 = y_3$
$t_2 = t_4 = 0$	$t_2^2 + y_2 = t_4^2$	$y_2 = 0$
$t_3 = t_4 = 0$	$t_3^2 + y_3 = t_4^2$	$y_3 = 0$

The six equations listed in the right-hand column define six planes in $\mathbf{y}$-space, and these six planes constitute the BBS for the standard A_1^4-singularity (see Figure 3.10). These planes intersect in the lines given in (3.7), and these are the 1-dimensional strata of the BBS.

$$\left.\begin{aligned} y_1 = y_2 = y_3, \\ y_1 = y_2 = 0, \\ y_1 = y_2, y_3 = 0, \\ y_1 = y_3 = 0, \\ y_1 = y_3, y_2 = 0, \\ y_2 = y_3, y_1 = 0, \\ y_2 = y_3 = 0. \end{aligned}\right\} \tag{3.7}$$

The 2-dimensional strata of the BBS are the A_1^2 subsets of the BBS listed in the table. We will call a plane through the origin in $\mathbb{R}^3_{\mathbf{y}}$ a *bad plane* if it contains the limit of tangent spaces to a stratum of the BBS at points tending to the origin. Our task is to find all of these bad planes, as we wish to avoid them as tangent planes to the slicing surfaces at the origin, since these bad planes correspond to non-generic slices. Now the tangent plane to a slicing surface at the origin in $\mathbf{y}$-space is given by

$$a_1y_1 + a_2y_2 + a_3y_3 = 0,$$

which corresponds to a function with linear part $h = a_1y_1 + a_2y_2 + a_3y_3$ on the BBS. The special nature of the BBS in this case is important when finding these bad planes, since the 1- and 2-dimensional strata of the BBS are lines

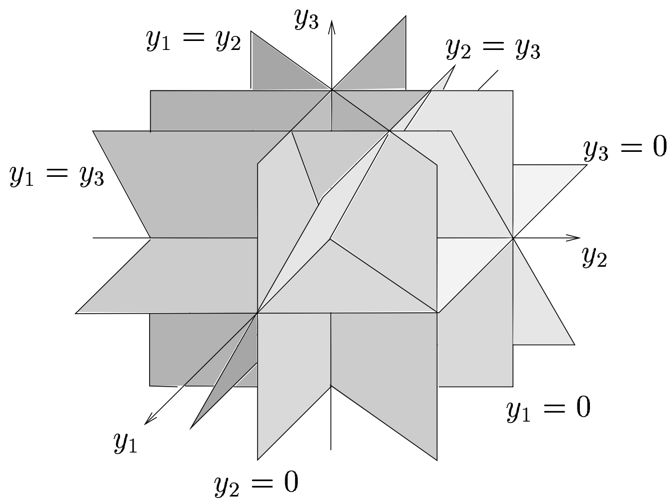


Figure 3.10: *The Big Bifurcation Set for the standard* A_1^4 *singularity.*

and planes respectively, and the limit of the tangent space to each stratum at points tending towards the origin is the corresponding line or plane itself. Thus a bad plane is one which contains any of the 1- or 2-dimensional strata of the BBS. However, it is clear that any plane which contains one of the latter will automatically contain one of the former, and thus we can see that the set of bad planes will be precisely those which contain a line in (3.7). A plane $a_1y_1 + a_2y_2 + a_3y_3 = 0$ is represented by a point $(a_1 \colon a_2 \colon a_3) \in \mathbb{R}P^2$. Then, for example, the plane *contains* the line $y_1 = y_2 = y_3$ if and only if $a_1 + a_2 + a_3 = 0$, and contains the line $y_1 = y_2 = 0$ if and only if $a_3 = 0$, and so on for all the lines in (3.7). In this way, we get a collection of lines

$$\left.\begin{aligned} a_1 + a_2 + a_3 &= 0, \\ a_3 &= 0, \\ a_1 + a_2 &= 0, \\ a_2 &= 0, \\ a_1 + a_3 &= 0, \\ a_2 + a_3 &= 0, \\ a_1 &= 0. \end{aligned}\right\} \tag{3.8}$$

in $\mathbb{R}P^2$ whose points represent bad planes. The lines in (3.8) constitute the set of all bad planes, which we denote by Δ in $\mathbb{R}P^2$, shown in Figure 3.11. The components of $\mathbb{R}P^2 - \Delta$ represent collections of normals to planes which, as kernels of $dh(0)$, give stratified C^0-equivalent functions h, that is, each component in the region swept out by normals to planes giving C^0-equivalent families of sections. For remarks on *stratified C^0 equivalence*, and a discussion of why this is the correct equivalence to use here, the reader is referred to [BG86] (p.199) and [Bru86].

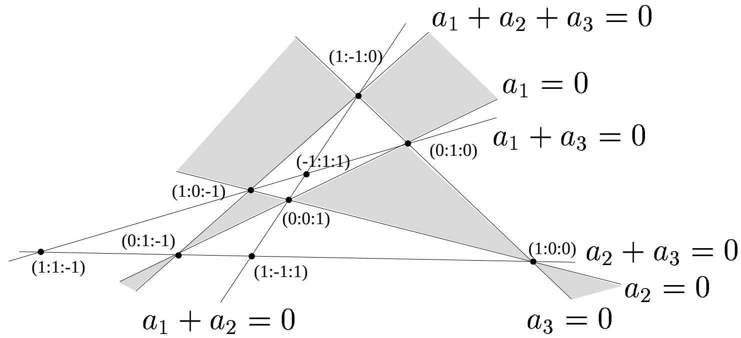


Figure 3.11: *The shaded regions have* $a_1a_2a_3(a_1 + a_2 + a_3) < 0$.

Calculation shows that the shaded regions of Figure 3.11 give one type of section, and the non-shaded regions give another. We are able to deduce the following:

Proposition 3.5.1 (A_1^4 condition). *A point* $(a_1 : a_2 : a_3)$ *is in a shaded/unshaded region of Figure 3.11 depending upon whether*

$$a_1a_2a_3(a_1 + a_2 + a_3),$$

is negative/positive, and the corresponding full bifurcation set exhibits a transition of type $A_1^4(a)/A_1^4(b)$.

The *'A_1^4 condition'* is found by multiplying the equations of the lines, noting that the function changes sign when crossing any of the lines. It gives us a means of distinguishing between the two different possible transitions for an A_1^4 singularity of a generic function. Note that, as a family of parallel planes passes through the origin, the configuration of the intersections of the planes with the BBS stays the same (due to the symmetrical structure of the BBS, illustrated in Figure 3.10).

3.5.1 Interpretation of A_1^4 condition

We now link this general analysis of a multi-versal unfolding of an A_1^4 singularity to the family of affine distance functions on a curve γ, whose bifurcation sets give us the ADSS of γ.

Consider four simple, smooth plane curve segments $\gamma_1, \gamma_2, \gamma_3$ and γ_4, with parameters s_1, s_2, s_3 and s_4 respectively, which we assume to be the affine-arclength parameter along the corresponding curve segment. Suppose there exist four distinct non-degenerate conics C_1, C_2, C_3 and C_4 having four point contact with $\gamma_1, \gamma_2, \gamma_3$ and γ_4 at $\gamma_1(0), \gamma_2(0), \gamma_3(0)$ and $\gamma_4(0)$ respectively, and all sharing the same centre, $\mathbf{x}_0$, and affine radius, d_0. The common centre $\mathbf{x}_0$ of these conics is the A_1^4-point, the point at which the affine distance function defined on the curve has four singularities at the same level. The ADSS has six branches passing through $\mathbf{x}_0$, formed by taking pairs of curve segments $\gamma_1, \gamma_2, \gamma_3$ and γ_4 in turn, each pair contributing one branch to the ADSS.

Our first task is to link the family of affine distance functions to the standard unfolding G. We will consider four families of curve segments close to $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. With family parameter u, we will denote these segments as $\gamma_{i,u}(s_i) \equiv (X_{i,u}(s_i), Y_{i,u}(s_i))$, where the parameters s_i are all taken in a neighbourhood of zero. We will take $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, and denote by $\mathbf{x}_0$ the A_1^4-point on the ADSS. Then the family of affine distance functions on the family of curve segments consists of four germs

$$F_i\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2, (0, 0, \mathbf{x}_0) \to \mathbb{R},$$

given by

$$\begin{aligned} F_i(s_i, u, \mathbf{x}) &= [\mathbf{x} - \gamma_{u,i}(s_i), \gamma'_{u,i}(s_i)], \\ &= \begin{vmatrix} x_1 - X_{u,i}(s_i) & X'_{u,i}(s_i) \\ & \\ x_2 - Y_{u,i}(s_i) & Y'_{u,i}(s_i) \end{vmatrix}, \end{aligned}$$

where $'$ (prime) will always denote $\partial/\partial s_i$. Since we are assuming that each F_i is a multi-versal unfolding, then by the uniqueness of multi-versal unfoldings each of the unfoldings G_i in the standard multi-versal unfolding G can be induced from the affine distance functions F_i by

$$G_i(t_i, \mathbf{y}) = F_i(A_i(t_i, \mathbf{y}), B(\mathbf{y})) + C(\mathbf{y}), \text{ for } i = 1, 2, 3, 4, \tag{3.9}$$

where each $A_i \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is a germ at $(0, \mathbf{0})$, and B, C denote respectively the germs $B \colon (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R} \times \mathbb{R}^2, (0, \mathbf{x}_0))$ and $C \colon (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}, d_0)$.

$$\begin{array}{ccccccc}
\mathbb{R} \times \mathbb{R}^3 & \xrightarrow{G} & \mathbb{R} \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{h} & \mathbb{R} \\
{\scriptstyle (A_i \times B)} \downarrow & & {\scriptstyle (-C \times B)} \downarrow & & {\scriptstyle B} \downarrow & & \downarrow \text{identity} \\
\mathbb{R} \times \mathbb{R}^3 & \xrightarrow{F} & \mathbb{R} \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\pi_1} & \mathbb{R}
\end{array}$$

From the commutative diagram we see that $h = \pi_1 \circ B$, where π_1 denotes projection onto the first coordinate. Thus B_1 is the map h on the standard A_1^4-set (the BBS), which corresponds to the plane through the origin in $\mathbf{y}$-space representing the tangent plane to the surface with which we are slicing the BBS. This tangent plane thus corresponds to the kernel of the map h on the BBS, i.e.

$$\ker dB_1 \colon \mathbb{R}^3 \to \mathbb{R}, \text{ with matrix } \left.\left(\frac{\partial B_1}{\partial y_1}, \frac{\partial B_1}{\partial y_2}, \frac{\partial B_1}{\partial y_3} \right)\right|_{\mathbf{y}=\mathbf{0}}$$

Hence the kernel plane has equation

$$\left.\frac{\partial B_1}{\partial y_1}\right|_{\mathbf{0}} y_1 + \left.\frac{\partial B_1}{\partial y_2}\right|_{\mathbf{0}} y_2 + \left.\frac{\partial B_1}{\partial y_3}\right|_{\mathbf{0}} y_3 = 0.$$

We are now able to re-interpret the A_1^4 condition from Proposition 3.5.1.

Proposition 3.5.2. *The ADSS exhibits a transition of type $A_1^4(a)/A_1^4(b)$ depending on whether*

$$\frac{\partial B_1}{\partial y_1} \frac{\partial B_1}{\partial y_2} \frac{\partial B_1}{\partial y_3} \left(\frac{\partial B_1}{\partial y_1} + \frac{\partial B_1}{\partial y_2} + \frac{\partial B_1}{\partial y_3} \right)$$

is negative/positive respectively (where everything is evaluated at $\mathbf{y} = \mathbf{0}$).

To interpret this condition in terms of the affine distance function, we need some further analysis in order to link the function B_1 to our original

F_i, using the relationship given in (3.9). Considering the case $i = 1$ to begin with, we see that

$$\left(\frac{\partial G_1}{\partial t_1}\ \frac{\partial G_1}{\partial y_1}\ \frac{\partial G_1}{\partial y_2}\ \frac{\partial G_1}{\partial y_3}\right)\Bigg|_{(t_1,\mathbf{0})} = (2t_1\ 1\ 0\ 0),$$

$$= \left(\frac{\partial F_1}{\partial s_1}\ \frac{\partial F_1}{\partial u}\ \frac{\partial F_1}{\partial x_1}\ \frac{\partial F_1}{\partial x_2}\right)\Bigg|_{(A_1(t_1,\mathbf{0}),\mathbf{x}_0)} \times \left.\begin{pmatrix} \frac{\partial A_1}{\partial t_1} & \frac{\partial A_1}{\partial y_1} & \frac{\partial A_1}{\partial y_2} & \frac{\partial A_1}{\partial y_3} \\ 0 & \frac{\partial B_1}{\partial y_1} & \frac{\partial B_1}{\partial y_2} & \frac{\partial B_1}{\partial y_3} \\ 0 & \frac{\partial B_2}{\partial y_1} & \frac{\partial B_2}{\partial y_2} & \frac{\partial B_2}{\partial y_3} \\ 0 & \frac{\partial B_3}{\partial y_1} & \frac{\partial B_3}{\partial y_2} & \frac{\partial B_3}{\partial y_3} \end{pmatrix}\right|_{(t_1,\mathbf{0})} + \left(0\ \frac{\partial C}{\partial y_1}\ \frac{\partial C}{\partial y_2}\ \frac{\partial C}{\partial y_3}\right)\Bigg|_{\mathbf{0}}$$

where the second row comes from using the chain rule for derivatives. We can do the same for G_2, G_3 and G_4, which have the right-hand side of the first line as $(2t_2\ 0\ 1\ 0)$, $(2t_3\ 0\ 0\ 1)$ and $(2t_4\ 0\ 0\ 0)$ respectively. Now

$$\frac{\partial F_i}{\partial s_i}(0, \mathbf{x}_0) \equiv 0,$$

since by definition F_i has an A_1 singularity at $(0, \mathbf{x}_0)$. Also, we calculate that

$$\frac{\partial F_i}{\partial x_1} = Y'_{i,u}(s_i),\ \frac{\partial F_i}{\partial x_2} = -X'_{i,u}(s_i).$$

We may put $t_i = 0$, since only the 0-jets are required. For brevity, we will assume the shorthand way of denoting the matrices

$$JB = \left.\begin{pmatrix} \frac{\partial B_1}{\partial y_1} & \frac{\partial B_1}{\partial y_2} & \frac{\partial B_1}{\partial y_3} \\ \frac{\partial B_2}{\partial y_1} & \frac{\partial B_2}{\partial y_2} & \frac{\partial B_2}{\partial y_3} \\ \frac{\partial B_3}{\partial y_1} & \frac{\partial B_3}{\partial y_2} & \frac{\partial B_3}{\partial y_3} \end{pmatrix}\right|_{\mathbf{y}=\mathbf{0}} \quad \text{and } JC = \left(\frac{\partial C}{\partial y_1}\ \frac{\partial C}{\partial y_2}\ \frac{\partial C}{\partial y_3}\right)\Bigg|_{\mathbf{y}=\mathbf{0}}$$

Taking all of the G_i together leads to the system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} \\ \frac{\partial F_4}{\partial u} & \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\mathbf{x}_0)} \times JB \quad + \begin{pmatrix} JC \\ JC \\ JC \\ JC \end{pmatrix}$$

Subtracting the bottom row from each of the other rows we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial F_1}{\partial u} - \frac{\partial F_4}{\partial u} & \frac{\partial F_1}{\partial x_1} - \frac{\partial F_4}{\partial x_1} & \frac{\partial F_1}{\partial x_2} - \frac{\partial F_4}{\partial x_2} \\ \frac{\partial F_2}{\partial u} - \frac{\partial F_4}{\partial u} & \frac{\partial F_2}{\partial x_1} - \frac{\partial F_4}{\partial x_1} & \frac{\partial F_2}{\partial x_2} - \frac{\partial F_4}{\partial x_2} \\ \frac{\partial F_3}{\partial u} - \frac{\partial F_4}{\partial u} & \frac{\partial F_3}{\partial x_1} - \frac{\partial F_4}{\partial x_1} & \frac{\partial F_3}{\partial x_2} - \frac{\partial F_4}{\partial x_2} \\ \frac{\partial F_4}{\partial u} & \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\mathbf{x}_0)} \times JB \; + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\partial C}{\partial y_1} & \frac{\partial C}{\partial y_2} & \frac{\partial C}{\partial y_3} \end{pmatrix}$$

Substituting in our expressions for $\partial F_i/\partial x_1, \partial F_i/\partial x_2$, and deleting the last row, leads us to the final system, which is written in full:

$$I_3 = \begin{pmatrix} \frac{\partial F_1}{\partial u} - \frac{\partial F_4}{\partial u} & Y'_{1,0}(0) - Y'_{4,0}(0) & -X'_{1,0}(0) + X'_{4,0}(0) \\ \frac{\partial F_2}{\partial u} - \frac{\partial F_4}{\partial u} & Y'_{2,0}(0) - Y'_{4,0}(0) & -X'_{2,0}(0) + X'_{4,0}(0) \\ \frac{\partial F_3}{\partial u} - \frac{\partial F_4}{\partial u} & Y'_{3,0}(0) - Y'_{4,0}(0) & -X'_{3,0}(0) + X'_{4,0}(0) \end{pmatrix} \times \left.\begin{pmatrix} \frac{\partial B_1}{\partial y_1} & \frac{\partial B_1}{\partial y_2} & \frac{\partial B_1}{\partial y_3} \\ \frac{\partial B_2}{\partial y_1} & \frac{\partial B_2}{\partial y_2} & \frac{\partial B_2}{\partial y_3} \\ \frac{\partial B_3}{\partial y_1} & \frac{\partial B_3}{\partial y_2} & \frac{\partial B_3}{\partial y_3} \end{pmatrix}\right|_{\mathbf{y}=\mathbf{0}}$$

where I_3 denotes the (3×3) identity matrix. Now we only require the partial derivatives of B_1 with respect to y_1, y_2 and y_3, in order to interpret the expression in Proposition 3.5.2. These partial derivatives are present in the top row of JB, and so using the above expression, we can find them in terms of their cofactors in the other matrix. For example, using β to denote

$\det(JB)$,

$$\left.\frac{\partial B_1}{\partial y_1}\right|_{\mathbf{0}} = \beta \det \begin{pmatrix} Y'_{2,0}(0) - Y'_{4,0}(0) & -X'_{2,0}(0) + X'_{4,0}(0) \\ \\ Y'_{3,0}(0) - Y'_{4,0}(0) & -X'_{3,0}(0) + X'_{4,0}(0) \end{pmatrix},$$

and similarly for $\partial B_1/\partial y_2, \partial B_1/\partial y_3$ evaluated at $\mathbf{y} = \mathbf{0}$. Since we are now considering only $\mathbf{y} = \mathbf{0}$ and $u = 0$, we will omit the subscripts. Now

$$\begin{aligned} \frac{\partial B_1}{\partial y_1} &= \beta \begin{vmatrix} Y'_2 - Y'_4 & -X'_2 + X'_4 \\ Y'_3 - Y'_4 & -X'_3 + X'_4 \end{vmatrix} \\ &= \beta \begin{vmatrix} -X'_3 + X'_4 & -X'_2 + X'_4 \\ Y'_3 - Y'_4 & Y'_2 - Y'_4 \end{vmatrix} \\ &= \beta \begin{vmatrix} X'_4 - X'_3 & X'_2 - X'_4 \\ Y'_4 - Y'_3 & Y'_2 - Y'_4 \end{vmatrix} \\ &= \beta[\gamma'_4 - \gamma'_3, \gamma'_2 - \gamma'_4]. \end{aligned}$$

Similarly, we find:

$$\begin{aligned} \frac{\partial B_1}{\partial y_2} &= -\beta[\gamma'_4 - \gamma'_3, \gamma'_1 - \gamma'_4], \\ \frac{\partial B_1}{\partial y_3} &= \beta[\gamma'_4 - \gamma'_3, \gamma'_2 - \gamma'_4]. \end{aligned}$$

These expressions have a *cyclic* nature, for example

$$\begin{aligned} \frac{\partial B_1}{\partial y_1} &= \beta[\gamma'_4 - \gamma'_3, \gamma'_2 - \gamma'_4], \\ &= \beta\left([\gamma'_2, \gamma'_3] + [\gamma'_3, \gamma'_4] + [\gamma'_4, \gamma'_2]\right), \\ &= \beta\left([\gamma'_4, \gamma'_2] - [\gamma'_2, \gamma'_2] - [\gamma'_4, \gamma'_3] + [\gamma'_2, \gamma'_3]\right), \\ &= \beta[\gamma'_4 - \gamma'_2, \gamma'_2 - \gamma'_3]. \end{aligned}$$

In this way, we can use the shorthand

$$\frac{\partial B_1}{\partial y_1} = \beta(2, 3, 4),$$

where

$$\begin{aligned}(i,j,k) &= [\gamma'_k - \gamma'_i, \gamma'_i - \gamma'_j],\\ &= [\gamma'_k - \gamma'_i, \gamma'_k - \gamma'_j] \ (\equiv (k,i,j) \ ),\\ &= [\gamma'_k - \gamma'_j, \gamma'_i - \gamma'_j] \ (\equiv (j,k,i) \ ),\end{aligned}$$

for pairwise distinct $i, j, k \in \{1,2,3,4\}$. Similarly,

$$\frac{\partial B_1}{\partial y_2} = -\beta(1,3,4) \equiv \beta(3,1,4),$$
$$\frac{\partial B_1}{\partial y_3} = \beta(1,2,4),$$

and the sum of all three expressions is

$$\begin{aligned}\frac{\partial B_1}{\partial y_1} + \frac{\partial B_1}{\partial y_2} + \frac{\partial B_1}{\partial y_3} &= \beta[\gamma'_4(0) - \gamma'_2(0), \gamma'_2(0) - \gamma'_3(0)]\\ &\qquad -\beta[\gamma'_4(0) - \gamma'_1(0), \gamma'_1(0) - \gamma'_3(0)]\\ &\qquad\qquad +\beta[\gamma'_4(0) - \gamma'_1(0), \gamma'_1(0) - \gamma'_2(0)],\\ &= \beta[\gamma'_1 - \gamma'_2, \gamma'_2 - \gamma'_3] \equiv \beta(1,2,3).\end{aligned}$$

Thus, omitting the positive power of β, we can denote the expression that we wish to interpret as $(1,2,3)\cdot(2,3,4)\cdot(4,3,1)\cdot(1,2,4)$, which can be rewritten 'cyclicly' itself as $-(1,2,3)\cdot(2,3,4)\cdot(3,4,1)\cdot(4,1,2)$.

Proposition 3.5.3 (A_1^4 condition for the ADSS). *The ADSS at an A_1^4-point exhibits a transition of the form $A_1^4(a)/A_1^4(b)$ depending on whether*

$$-(1,2,3)\cdot(2,3,4)\cdot(3,4,1)\cdot(4,1,2), \tag{3.10}$$

is negative/positive respectively, i.e. whether

$$-[\gamma'_1 - \gamma'_2, \gamma'_2 - \gamma'_3]\cdot[\gamma'_2 - \gamma'_3, \gamma'_3 - \gamma'_4]\cdot[\gamma'_3 - \gamma'_4, \gamma'_4 - \gamma'_1]\cdot[\gamma'_4 - \gamma'_1, \gamma'_1 - \gamma'_2],$$

is negative/positive respectively.

3.5.2 Interpreting the A_1^4 condition for ovals

The first thing to note is that the expression of Proposition 3.5.3 is independent of the ordering of the points $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, meaning that if we swap any two points and the corresponding affine tangent vectors, the expression remains unchanged. For example, if we swap the points γ_2 and γ_4, and the vectors γ_2' and γ_4', then the expression (3.10) becomes

$$\begin{aligned} & -(1,4,3)\cdot(4,3,2)\cdot(3,2,1)\cdot(2,1,4), \\ = \; & -(-(3,4,1))\cdot(-(2,3,4))\cdot(-(1,2,3))\cdot(-(4,1,2)), \\ = \; & -(3,4,1)\cdot(2,3,4)\cdot(1,2,3)\cdot(4,1,2), \\ = \; & -(1,2,3)\cdot(2,3,4)\cdot(3,4,1)\cdot(4,1,2). \end{aligned}$$

This means that we may assume that the points $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are ordered around the oval.

Notation: *We will use* $\mathbf{v}_{ij}$ *to denote the vector* $\gamma_i' - \gamma_j'$*, and* $\mathbf{u}_{ij}$ *to denote the vector* $\gamma_i - \gamma_j$*.*

Now we know from the Concurrent Tangents Condition (Proposition 3.2.5) that the vector $\mathbf{v}_{ij}$ is in the *direction* of the line passing through the ADSS point $\mathbf{x}_0$ and the intersection of the tangent lines at γ_i, γ_j. So the line in which

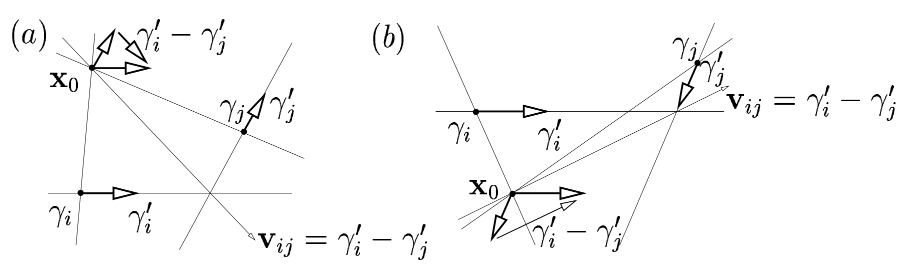


Figure 3.12: *Interpreting the* A_1^4 *condition for ovals.*

vector $\gamma_i' - \gamma_j'$ lies is known. However, we also require the *orientation* of this vector (i.e. its oriented direction), which depends on the relative orientations of γ_i', γ_j'. We note that if we were to reverse the orientation of vector γ_j' in Figure 3.12(a), then the direction of $\gamma_i' - \gamma_j'$ would change, but the direction of $\mathbf{v}_{ij}$ would remain the same, since it only depends upon the positions of $\mathbf{x}_0$ and the intersection of the tangent lines. However, $\mathbf{x}_0$ cannot then be the corresponding ADSS point, since the opposite sign to the affine distance from $\mathbf{x}_0$ to γ_j. For the distances to remain equal, we must move $\mathbf{x}_0$ to a position such as that shown in Figure 3.12(b), in which case the directions and orientations of $\mathbf{v}_{ij}$ and $\gamma_i' - \gamma_j'$ tally.

This leads us to the idea of *allowable* positions for the ADSS point $\mathbf{x}_0$, given the four corresponding points and the directions and orientations of the affine tangent vectors. Consider Figure 3.13(a), where we have four points and the corresponding four affine tangents oriented consistently around the curve. If these points are to contribute to an A_1^4-point $\mathbf{x}_0$, then $\mathbf{x}_0$ must be at an equal affine distance from them. Now the tangent lines associated to the four points divide the plane into six regions. However, only one of these regions contains *allowable* positions for $\mathbf{x}_0$, namely the central region shown shaded. For $\mathbf{x}_0$ in every other region, we cannot have $\mathbf{x}_0$ at the same affine distance from each of the four points, since the signs of the distances $d(\mathbf{x}_0, \gamma_i)$ will not all be the same. Thus we can define 'allowable' positions for $\mathbf{x}_0$, given four points and four affine tangent vectors. The allowable positions for $\mathbf{x}_0$ will be the region R where the affine distances from $\mathbf{x}_0 \in R$ to each of the points γ_i have the same sign. In fact, we will make the same definition for an ADSS point corresponding to any number of curve points.

Definition 3.5.4. *Given n points γ_i and n corresponding affine tangent vectors, the region of* **allowable positions** *for $\mathbf{x}_0$ will be the region R where the affine distances from the ADSS point $\mathbf{x}_0 \in R$ to each of the points γ_i have the same sign.*

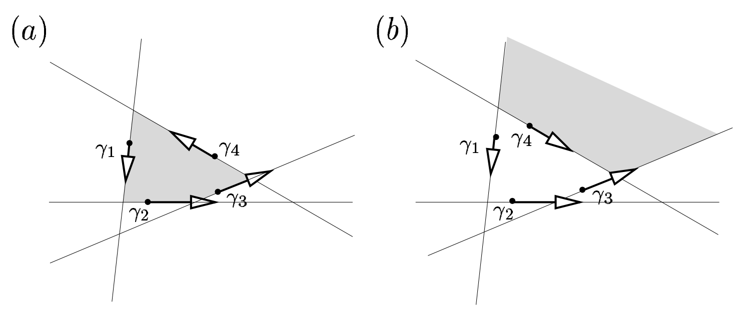


Figure 3.13: *Allowable positions for the ADSS point. See* §3.5.2.

As another example, in Figure 3.13(b), with the affine tangent vectors oriented as shown, the region of allowable positions for $\mathbf{x}_0$ is shown shaded. Thus for points $\gamma_1, \gamma_2, \gamma_3$ and γ_4 on an oval, with corresponding affine tangent vectors $\gamma_1', \gamma_2', \gamma_3'$ and γ_4', the only allowable position for A_1^4-point $\mathbf{x}_0$ will be in the region shown shaded in Figure 3.13(a). Hence for the oval case we

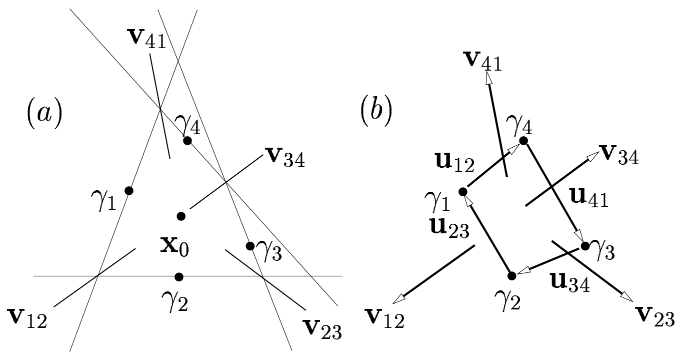


Figure 3.14: *(a) Using the Concurrent Tangents Condition of Proposition 3.2.5, we know the* directions *of the vectors* $\mathbf{v}_{ij} \equiv \gamma_i' - \gamma_j'$*, but not their* orientations*. (b) Using Corollary 3.5.7, we are able to deduce the orientations of the vectors* $\mathbf{v}_{ij}$ *from the vectors* $\mathbf{u}_{ij} \equiv \gamma_i - \gamma_j$.

know the directions of each of the lines $\mathbf{v}_{12}$, $\mathbf{v}_{23}$, $\mathbf{v}_{34}$ and $\mathbf{v}_{41}$, as shown in Figure 3.14(a). Furthermore, we are able to deduce the *orientations* of these vectors in the following way. First of all we require:

Lemma 3.5.5 (Oval Condition). *If* γ_i, γ_j *are two distinct points on an oval parametrised by affine-arclength, then*

$$[\gamma_i - \gamma_j, \gamma_i'] > 0,$$

where as usual ′ (prime) denotes derivative w.r.t. affine-arclength.

Proof. Since the affine-arclength parametrisation induces a counter-clockwise orientation on the oval, the result follows immediately. See Figure 3.15 for an illustration. □

Remark 3.5.6. *In essence, the Oval Condition captures the simple geometrical fact that on oval lies entirely to one side of any of its tangent lines.*

Two applications of Lemma 3.5.5 immediately gives us:

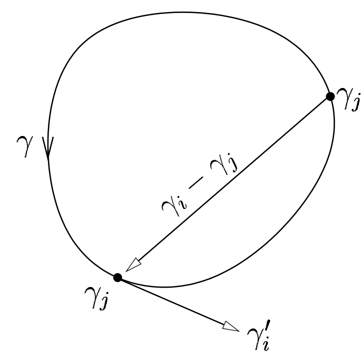


Figure 3.15: *Illustration of the Oval Condition. γ is parametrised by affine-arclength, which induces an anti-clockwise orientation (see* §1.3.4*).*

Corollary 3.5.7. *For any two distinct $i, j \in \{1, 2, 3, 4\}$,*

$$[\gamma_i - \gamma_j, \gamma_i' - \gamma_j'] > 0.$$

The above corollary completely defines the orientations of the vectors $\mathbf{v}_{ij}$. For example, Figure 3.14(a) illustrates the four points γ_1, γ_2, γ_3 and γ_4 along with the tangent lines at each point, the corresponding A_1^4-point $\mathbf{x}_0$, and the vectors $\gamma_i - \gamma_j$, which we denote $\mathbf{u}_{ij}$. (This situation is entirely general, since we have shown that the ordering of the points around the oval is immaterial.) The directions of the vectors $\mathbf{v}_{12}$, $\mathbf{v}_{23}$, $\mathbf{v}_{34}$ and $\mathbf{v}_{41}$ are known, and their orientations are deduced from Corollary 3.5.7, which says that

$$[\mathbf{u}_{ij}, \mathbf{v}_{ij}] > 0 \text{ for all distinct } i, j \in \{1, 2, 3, 4\}.$$

These orientations are illustrated in Figure 3.14(b), and from this it is clear that the expression

$$-(1,2,3)\cdot(2,3,4)\cdot(3,4,1)\cdot(4,1,2) \equiv -[\mathbf{v}_{12}, \mathbf{v}_{23}]\cdot[\mathbf{v}_{23}, \mathbf{v}_{34}]\cdot[\mathbf{v}_{34}, \mathbf{v}_{41}]\cdot[\mathbf{v}_{41}, \mathbf{v}_{12}],$$

always takes the same sign, namely negative. We deduce the following:

Proposition 3.5.8. *The transition $A_1^4(a)$ occurs on the ADSS of a family of ovals, but the transition $A_1^4(b)$ does not (see Figure 3.9).*

Remark 3.5.9. *In Figure 3.13 and Figure 3.14 the tangent lines through the four points are illustrated as forming a closed quadrangle. Similarly,* *in subsequent analysis when we consider the tangent lines at a set of three points, they will be illustrated as forming a closed triangle (see Figure 3.18 for example). In all cases, the arguments apply just as well when the quadrangle or triangle is open.*

3.5.3 Interpreting the A_1^4 condition for non-ovals

Disregarding the oval restriction, we use a process similar to that used in §3.5.2 to construct situations where expression (3.10) of Proposition 3.5.3 is positive. Without the oval condition, there is less restriction on the possible orientations of the affine tangents, and it can be seen that for any combination of orientations for these affine tangents, there is an allowable region for ADSS point $\mathbf{x}_0$. Figure 3.16 shows the six regions with a symbol denoting the sign of the expression (3.10) for $\mathbf{x}_0$ in that region. (The sign depends upon the fact that γ_1' is oriented as shown.) Thus we have shown that both A_1^4 transitions may occur on the ADSS of a generic plane curve.

Proposition 3.5.10. *The ADSS of a generic family of plane curves may exhibit transitions of type $A_1^4(a)$ and $A_1^4(b)$.*

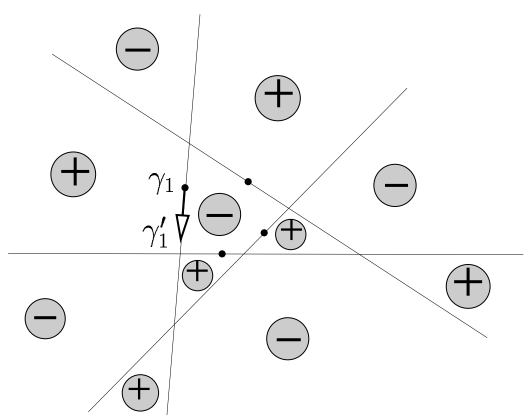


Figure 3.16: *The sign in each region denotes the sign of the expression (3.10) for ADSS point $\mathbf{x}_0$ in the corresponding region, predicting which A_1^4 transition will occur.*

3.6 The $A_1^2A_2$ transitions

Following the procedure outlined in §3.5 we deduce:

Proposition 3.6.1 ($A_1^2A_2$ condition for the ADSS). *The ADSS at an $A_1^2A_2$-point exhibits a transition of the form $A_1^2A_2(a)/A_1^2A_2(b)$ depending on whether*

$$-[\gamma_1' - \gamma_2', \gamma_1''] \cdot [\gamma_1' - \gamma_3', \gamma_1''], \tag{3.11}$$

is negative/positive respectively.

Note that this condition is independent of the ordering of γ_2 and γ_3, since once γ_1 is fixed, it is immaterial where γ_2 or γ_3 are situated on the oval. Thus we may assume that the points γ_1, γ_2 and γ_3 appear in that order.

3.6.1 Interpreting the $A_1^2A_2$ condition for ovals

For an oval, the allowable positions of the $A_1^2A_2$-point is as shown in Figure 3.17. The Oval Condition of Lemma 3.5.5 and Corollary 3.2.3 combine to give the condition,

$$d_0[\gamma_1' - \gamma_i', \gamma_1''] > 0, \tag{3.12}$$

for $i = 2, 3$, where d_0 is the common affine distance from the $A_1^2A_2$-point $\mathbf{x}_0$ to the points γ_i ($d_0 \equiv [\mathbf{x}_0 - \gamma_i, \gamma_i']$). In this case, since we have an A_2-singularity of the distance function at γ_1, we know that

$$d_0 \equiv [\mathbf{x}_0 - \gamma_1, \gamma_1'] \equiv -\frac{1}{\mu_1},$$

where μ_1 is the affine curvature of the curve at the point γ_1 (so $\mathbf{x}_0 = \gamma_1 + \frac{1}{\mu_1}\gamma_1''$). Thus $[\gamma_1' - \gamma_2', \gamma_1'']$ and $[\gamma_1' - \gamma_3', \gamma_1'']$ always have the same sign, and hence expression (3.11) is negative for any choice of $\gamma_1', \gamma_2', \gamma_3'$ (and $\gamma_1, \gamma_2, \gamma_3$), which tells us that the transition $A_1^2A_2(b)$ may not occur on the ADSS of a family of ovals. The transition $A_1^2A_2(a)$ *is* allowed, and has been observed using [LSMP].

Proposition 3.6.2. *The transition $A_1^2A_2(a)$ occurs generically on the ADSS of a family of ovals, but the transition $A_1^2A_2(b)$ does not.*

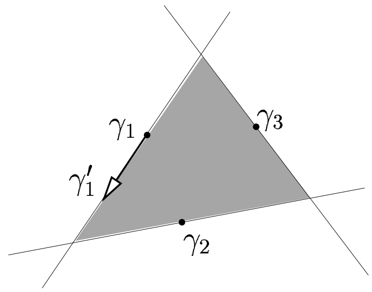


Figure 3.17: *The allowable region for* $\mathbf{x}_0$ *is shown shaded.*

3.6.2 Interpreting the $A_1^2A_2$ condition for non-ovals

Disregarding the condition that the curve is an oval, we are able to use a similar analysis to show that expression (3.11) may take positive values. As for the A_1^4 non-oval case, we see that there is an allowable region for ADSS point $\mathbf{x}_0$ for any orientations of the affine tangents, and for any combination of orientations for these affine tangents, there is an allowable region for ADSS point $\mathbf{x}_0$. Figure 3.18 shows the four regions with a symbol denoting the sign of the expression (3.11) for $\mathbf{x}_0$ in that region. (The sign depends upon the fact that γ_1' is oriented as shown.) Thus we have shown that the $A_1^2A_2(b)$ transition is not ruled out from occurring on the ADSS of a generic plane curve. Using [LSMP], we are able to observe this transition, and thus we have:

Proposition 3.6.3. *The ADSS of a generic family of plane curves may exhibit transitions of type* $A_1^2A_2(a)$ *and* $A_1^2A_2(b)$*.*

3.7 The A_1A_3 transitions

Following the procedure outlined in §3.5 we deduce:

Proposition 3.7.1 (A_1A_3 condition for the ADSS). *The ADSS at an* A_1A_3*-point exhibits a transition of type* $A_1A_3(a)/A_1A_3(b)$ *depending upon whether*

$$-\mu_2[\gamma_1' - \gamma_2', \gamma_2''] \tag{3.13}$$

is positive/negative respectively.

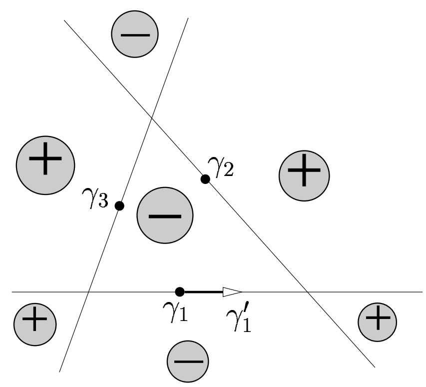


Figure 3.18: *The sign in each region denotes the sign of the expression (3.11) for ADSS point* $\mathbf{x}_0$ *in the corresponding region, predicting which* $A_1^2A_2$ *transition will occur.*

3.7.1 Interpretation of A_1A_3 condition for ovals

We will assume that our curve points γ_1 and γ_2 lie on the same oval, with corresponding affine tangents γ_1', γ_2', and thus we can use the Oval Condition of Lemma 3.5.5 and the corollary of the ADSS Condition in §3.2. Since we have an A_3 singularity of the affine distance function at γ_2, we know that the A_1A_3 ADSS point $\mathbf{x}_0$ can be expressed as $\mathbf{x}_0 \equiv \gamma_2 + \frac{1}{\mu_2}\gamma_2''$ (see Proposition 1.3.5), and the fact that γ_1 and γ_2 must be the same affine distance d_0 from $\mathbf{x}_0$ implies that $d_0 = -1/\mu_2$, and therefore $\mathbf{x}_0 \equiv \gamma_1 + \frac{1}{\mu_2}\gamma_1''$. We substitute this into the Oval Condition $[\gamma_1 - \gamma_2, \gamma_1'] > 0$ to get

$$\begin{aligned}
& \left[\frac{1}{\mu_2}(\gamma_2'' - \gamma_1''), \gamma_1'\right] > 0, \\
\iff \quad & \frac{1}{\mu_2}([\gamma_2'', \gamma_1'] + 1) > 0, \\
\iff \quad & \frac{1}{\mu_2}(1 - [\gamma_1', \gamma_2'']) > 0, \\
\iff \quad & \frac{1}{\mu_2}([\gamma_2' - \gamma_1', \gamma_2'']) > 0, \text{ since } [\gamma_2', \gamma_2''] = 1,
\end{aligned}$$

which proves that the expression (3.13) takes only positive values for ovals. Thus the transition $A_1A_3(b)$ *will not occur* on the ADSS of a family of ovals. The transition $A_1A_3(a)$ may occur, and has been observed using the graphics package [LSMP] (see §3.11), and thus we have:

Proposition 3.7.2. *The transition $A_1A_3(a)$ occurs generically on the ADSS of a family of ovals, but the transition $A_1A_3(b)$ does not.*

3.7.2 Interpretation of A_1A_3 condition for non-ovals

We will now show that, if we disregard the assumption that the points γ_1 and γ_2 lie on the same oval, then the expression (3.13) may take negative values. Consider Figure 3.19(a), where without loss of generality we have fixed γ_2,

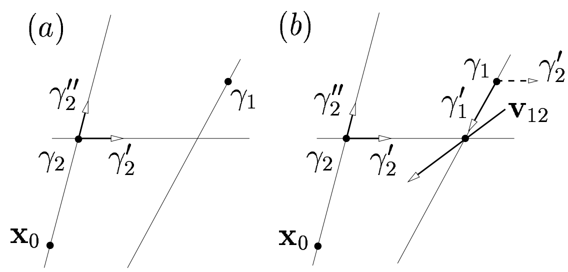


Figure 3.19: *Interpreting the A_1A_3 condition for non-ovals.*

γ_2' and $\mathbf{x}_0$, and also the point γ_1 and the corresponding tangent line through this point. Since $[\gamma_2', \gamma_2''] = 1$, we can deduce the direction and length of γ_2'' as shown. Then, since the γ_2 point corresponds with the A_3 singularity of the affine distance function, we know that $\mathbf{x}_0 \equiv \gamma_2 + \frac{1}{\mu_2}\gamma_2''$, and hence $\mu_2 < 0$. Also, since $\mathbf{x}_0$ must be the same affine distance from γ_1 as it is from γ_2, we can deduce that γ_1' has direction and length as shown in Figure 3.19(b), and from this it follows that $\mathbf{v}_{12} \equiv \gamma_1' - \gamma_2'$ has orientation as shown. (The Concurrent Tangent Condition tells us the direction of $\mathbf{v}_{12}$.) Thus

$$[\mathbf{v}_{12}, \gamma_2''] < 0,$$

and therefore (3.13) is negative. Note that in this case γ_1 and γ_2 cannot lie on the same oval with corresponding affine tangent vectors γ_1' and γ_2'.

Proposition 3.7.3. *The ADSS of a generic family of plane curves may exhibit transitions of type $A_1A_3(a)$ and $A_1A_3(b)$.*

We can follow this procedure to find the sign of expression (3.13) for $\mathbf{x}_0$ in each of the other regions in turn. Figure 3.20 shows the two regions with a symbol denoting the sign of the expression for $\mathbf{x}_0$ in that region.

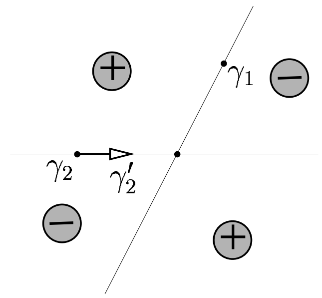


Figure 3.20: *The sign in each region denotes the sign of the expression (3.13) for ADSS point* $\mathbf{x}_0$ *in the corresponding region (assuming* γ_1, γ_1' *and* γ_2 *are fixed as shown), and thus predicts which* A_1A_3 *transition should occur.*

3.8 The A_2^2 transitions

Following the procedure outlined in §3.5 we deduce:

Proposition 3.8.1 (A_2^2 condition for the ADSS). *The ADSS at an* A_2^2*-point exhibits a transition of type* $A_2^2(a)/A_2^2(b)$ *depending upon whether*

$$-\frac{1}{\mu_1'\mu_2'}[\gamma_1 - \gamma_2, \gamma_1'] \cdot [\gamma_1 - \gamma_2, \gamma_2'], \tag{3.14}$$

is positive/negative respectively (assuming $\mu_1' + \mu_2' \neq 0$*).*

3.8.1 Interpretation of the A_2^2 condition

We express the A_2^2 Condition for the ADSS in this way since the two expressions $[\gamma_1 - \gamma_2, \gamma_1']$ and $[\gamma_1 - \gamma_2, \gamma_2']$ are familiar to us from the Oval Condition of Lemma 3.5.5. This result then tells us that, if we restrict our points γ_1 and γ_2 to lie on the same oval with corresponding affine tangents γ_1' and γ_2', then

$$[\gamma_1 - \gamma_2, \gamma_1'] \cdot [\gamma_1 - \gamma_2, \gamma_2'] < 0.$$

Proposition 3.8.2. *At an* A_2^2*-point on the ADSS of a family of ovals, the ADSS exhibits a transition of type* $A_2^2(a)/A_2^2(b)$ *depending on whether* $\mu_1'\mu_2'$ *is positive/negative respectively.*

Thus neither of the A_2^2 transitions are ruled out from occurring on the ADSS of a family of ovals, since there is no restriction on the sign that the product $\mu_1'\mu_2'$ may take. Thus, even for a family of ovals, *both* A_2^2 transitions may occur, the type of transition depending upon the sign of $\mu_1'\mu_2'$, with the added assumption that $\mu_1' + \mu_2' \neq 0$.

Proposition 3.8.3. *The ADSS of a generic family of plane curves may exhibit transitions of type $A_2^2(a)$ and $A_2^2(b)$.*

Remark 3.8.4. *This result is strikingly similar to the analogous A_2^2 situation for the Euclidean Symmetry Set. In this case, we recall that the two transitions are distinguished by the sign of the product $\kappa_1'\kappa_2'$, where κ_i denotes the Euclidean curvature at γ_i, with the added condition that $\kappa_1' + \kappa_2' \neq 0$. Again, both A_2^2 transitions occur on the SS of a generic family of plane curves.*

3.9 Conclusion

Thus Propositions 3.5.8, 3.6.2, 3.7.2, and 3.8.3 prove Theorem 3.4.1 and Propositions 3.5.10, 3.6.3, 3.7.3, and 3.8.3 prove Theorem 3.4.2. It should be added that there are other transitions that may occur on the ADSS of a generic family of plane curves. For example, a double tangent does not generically contribute to the ADSS of a single plane curve, but, as we showed in §3.2.2, it does contribute if the affine tangents to the curve at the corresponding points of contact are *identical*, and thus this situation could result in a transition on the ADSS different from those listed in Theorem 3.4.2. Similarly, there are numerous other situations which are generic for a family of plane curves, involving higher inflexions for example, which could also result in other transitions on the ADSS. Thus the list of transitions in Theorem 3.4.2 is by no means an exhaustive list of *all* transitions on 1-parameter families of Affine Distance Symmetry Sets.

3.10 The ADSS for non-simple curves

Until now, we have assumed that the curves for which we are finding the ADSS are *simple*, that is, have no self-intersections. We will now extend this by studying the local structure of the ADSS at a crossing on the original curve. Consider two smooth curve segments γ_1 and γ_2 given by

$$\gamma_1(s) = (s, f(s)), \quad \gamma_2(t) = (t, g(t)),$$

where

$$f(s) = a_2 s^2 + a_3 s^3 + \ldots, \quad g(t) = b_1 t + b_2 t^2 + b_3 t^3 + \ldots,$$

and where we will assume that $a_2 b_1 \neq 0$ (that is, that γ_1 and γ_2 cross transversally at the origin, with no inflexion on γ_1 there). The pre-ADSS is defined by solutions (s, t) to

$$[\gamma_1(s) - \gamma_2(t), \gamma_1''(s) - \gamma_2''(t)] = 0, \tag{3.15}$$

where $'$ (prime) denotes derivative w.r.t. the affine-arclength parameter along γ_1 and γ_2. Using $\gamma_i' = k_i^{-1/3}\dot{\gamma}_i$, where $\dot{}$ (dot) denotes derivative w.r.t. the corresponding parameter s or t, $k_1(s) = \ddot{f}(s)$, and $k_2(t) = \ddot{g}(t)$, then we can expand (3.15) near $s = t = 0$ as

$$\begin{vmatrix} s - t & -\frac{1}{3}\ddot{f}^{-5/3}\dddot{f} + \frac{1}{3}\ddot{g}^{-5/3}\dddot{g} \\ & \\ f(s) - g(t) & \ddot{f}^{1/3} - \frac{1}{3}\dot{f}\ddot{f}^{-5/3}\dddot{f} - \ddot{g}^{1/3} + \frac{1}{3}\dot{g}\ddot{g}^{-5/3}\dddot{g} \end{vmatrix} = 0$$

$$\iff \begin{vmatrix} s - t & -\frac{1}{3}\ddot{g}^{5/3}\dddot{f} + \frac{1}{3}\ddot{f}^{5/3}\dddot{g} \\ & \\ f(s) - g(t) & \ddot{f}^{2}\ddot{g}^{5/3} - \frac{1}{3}\dot{f}\ddot{g}^{5/3}\dddot{f} - \ddot{f}^{5/3}\ddot{g}^{2} + \frac{1}{3}\dot{g}\ddot{f}^{5/3}\dddot{g} \end{vmatrix} = 0.$$

Substituting in for f and g as power series in s and t we find that the equation of the pre-ADSS is

$$\left(2\left(a_2^2 b_2^{5/3} - a_2^{5/3} b_2^2\right)\right) s + \left(b_1 b_2^{5/3} a_3 - \left(a_2^2 b_2^{5/3} - a_2^{5/3} b_2^2\right)\right) t + \mathcal{O}(2) = 0.$$

Thus if $b_1 \neq 0$, then the pre-ADSS is always smooth, and the ADSS is generically a smooth curve.

Proposition 3.10.1. *The ADSS passes smoothly through the self-intersection points of a non-simple plane curve.*

Remark 3.10.2. *If $b_1 = 0$, then the segments γ_1 and γ_2 are tangent at the origin. The pre-ADSS however is still smooth, so long as $a_2 \neq b_2$which is the condition for the Euclidean curvatures of γ_1 and γ_2 to be different at the origin.*

3.11 Examples

The following plots were made using [LSMP]. In the case of the A_1A_3 transitions, we used [MAPLE] to calculate the equations of the local curve segments which correspond to an A_1A_3 transition, and also to predict which A_1A_3 transition should occur. Similarly, we may construct similar programs which predict which of the $A_1^2A_2$, A_1^4 and A_2^2 transitions occur, and output suitable curve segments which can then be put into [LSMP] in order to view the transitions.

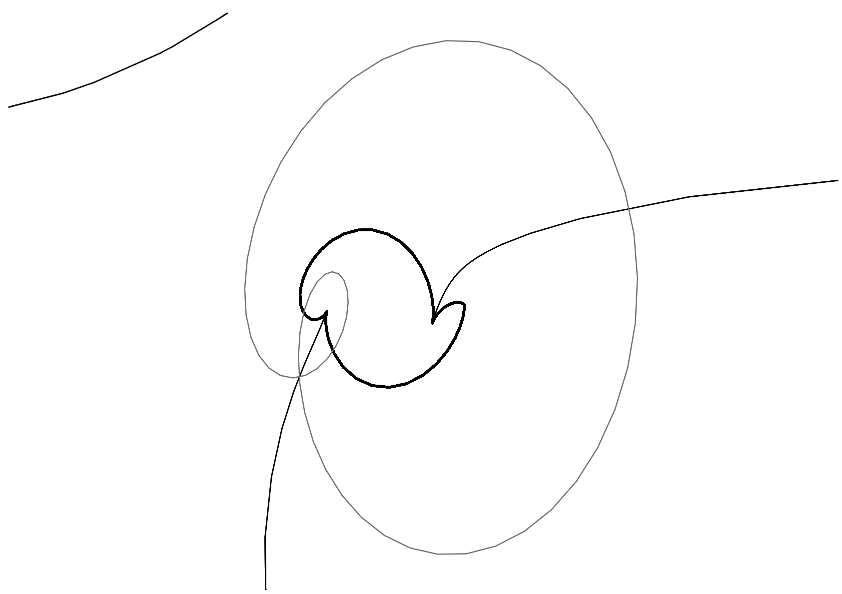


Figure 3.21: *An [LSMP] plot of the ADSS (thin dark curve) and the affine evolute (thick dark curve) for a non-simple plane curve (grey curve). Note that the ADSS passes through the self-intersection point of the curve.*

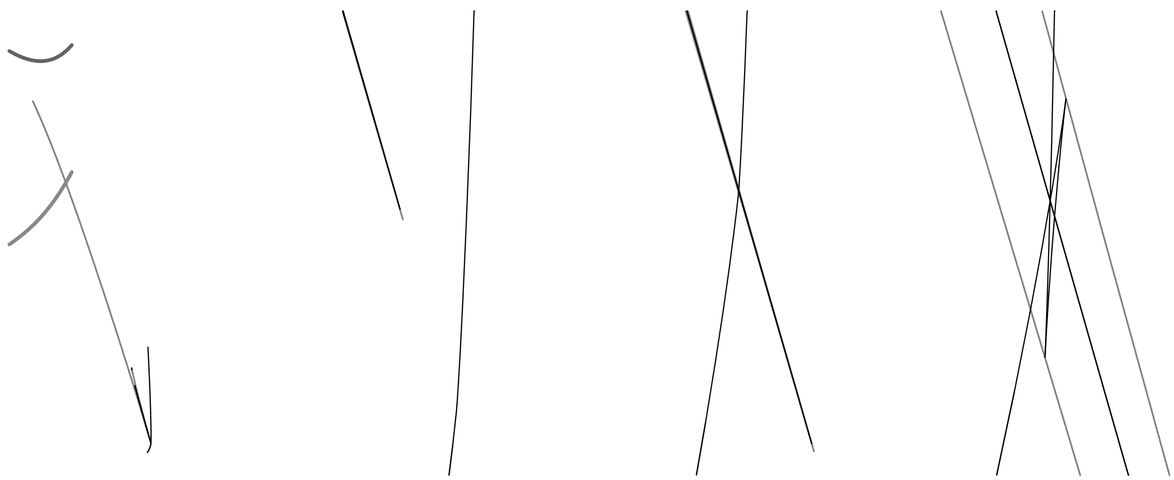


Figure 3.22: *On this page is shown an $A_1A_3(a)$ transition. From left to right: The two local curve segments are shown as thick curves, the two ADSS branches as thin dark curves, and the affine evolute as a thin grey curve; we can see the end of one branch of the ADSS in the cusp of the affine evolute approaching the other branch of the ADSS; the ADSS branches then intersect and a swallowtail is formed, as shown enlarged in the right-hand plot. Figure 3.23 overleaf shows another $A_1A_3(a)$ transition.*

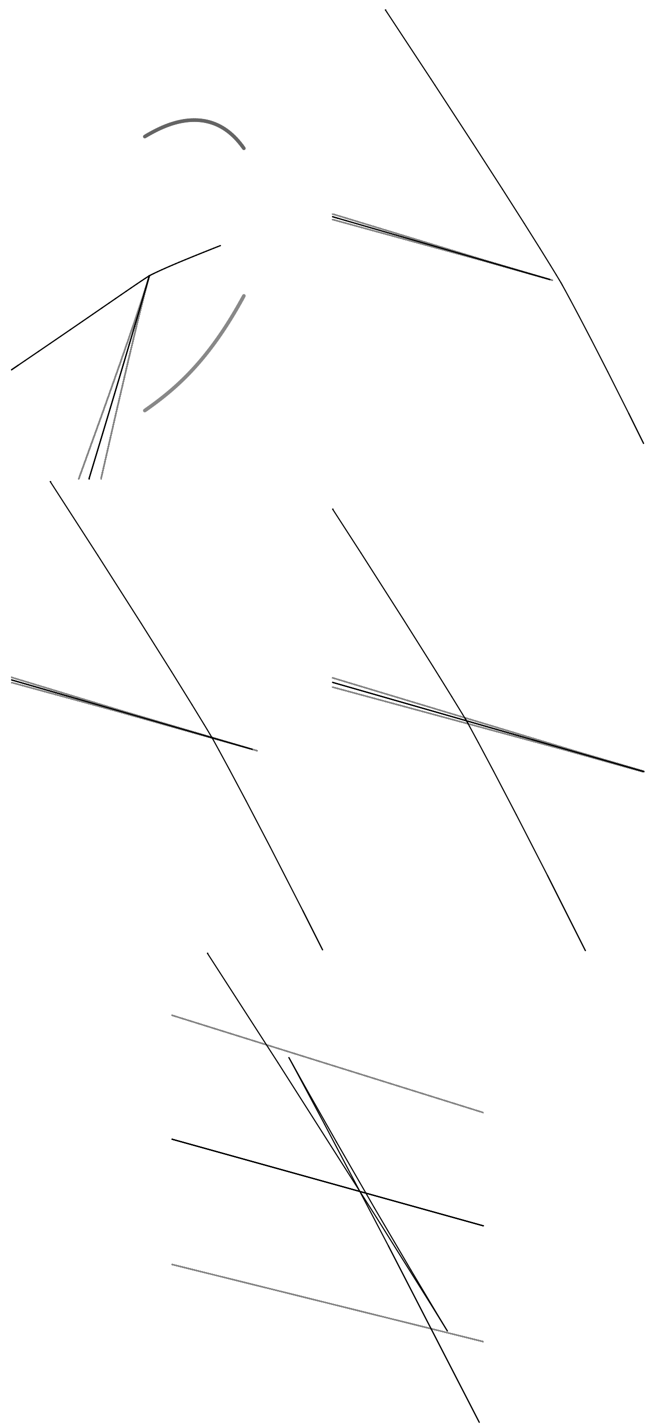


Figure 3.23: *An* $A_1A_3(a)$ *transition.*

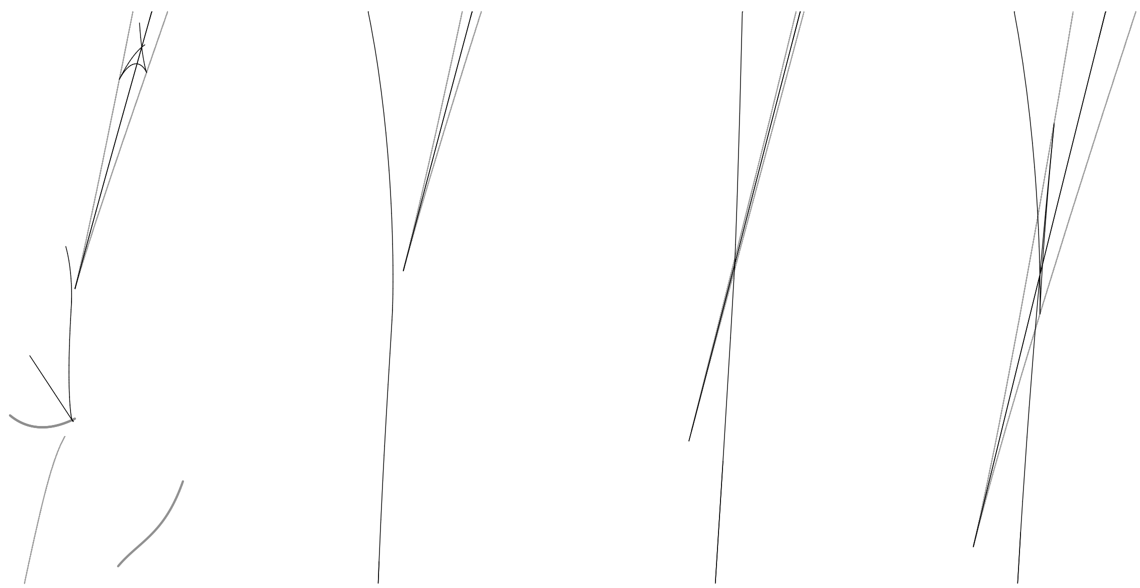


Figure 3.24: *An* $A_1A_3(b)$ *transition. From left to right: The two local curve segments are shown as thick grey curves, the two ADSS branches as thin dark curves, and the affine evolute as a thin grey curve. We are interested in the locality where the end of one branch of the ADSS in the cusp of the affine evolute approaches another branch of the ADSS; the ADSS branches then intersect and a swallowtail is formed, as shown enlarged in the right-hand plot. Note that the two local curve segments cannot be contained in the same oval curve.*

Chapter 4

The Reconstruction Problem For The AESS

4.1 Introduction

In [BG86], the following problem connected with the Symmetry Set (SS) is considered. Suppose we are given a smooth curve segment which we know to be the SS of some original curve (not necessarily closed – it may consist of two or more disjoint segments). We ask the question: *What additional information do we require to enable us to reconstruct the original curve?* We will refer to this as *The Reconstruction Problem* for the SS. The answer is that we must be given the radii of the individual bitangent circles, that is, we require a radius function which, for each point of the SS, tells us the radius of the bitangent circle at that point. The original curve is then reconstructed as the envelope of these circles.

We may rephrase this in the following way: given a curve segment S, parametrised by t, at each point of S there is a 1-parameter family of circles with this point as centre. Geometrically, this parameter is the *radius* of the circle. If we were to choose this parameter as a smooth function of t, say radius $r \equiv r(t)$, then this would define uniquely each of the circles centred on S, and by construction the envelope of these circles would be a curve having S as its Symmetry Set. We will keep this in mind as we begin to study the analogous Reconstruction Problem for the Affine Envelope Symmetry Set.

Remark 4.1.1. *As an aside, this geometrical construction can be used to justify a statement concerning the 'infinitesimal axes of reflexional symmetry' stated in §2.1.2. For a given curve S, consider a fixed member C of a 1-parameter family of circles with centre on S. The envelope of this family is a curve having S as its Symmetry Set, and the characteristic points on C will clearly be symmetric in the tangent line L to S at the centre of C, that is, the points will be reflexions of each other in L, and it follows that the midpoints of the chord joining these two points, and the intersection of the tangent lines to C at these two points, will lie on L, as claimed.*

We use the definition of the AESS as given in Definition 2.1.1, as opposed to the *dual of the discriminant of the midline map* definition (see Definition 2.4.6), since it supplies us with a more geometric approach. The problem we will consider is:

The Reconstruction Problem: *Given a smooth curve segment E, construct a smooth plane curve which has E as its AESS.*

For a curve segment E, at each point of E there is a 3-parameter family of conics having this point as centre. If the segment E is parametrised by t then we would like to choose these three parameters as smooth functions of t, so as to uniquely define a conic at each of the points of E. However, the envelope of these conics would in general have only 2-point contact with each of its constituent conics, and thus the envelope curve would not have E as its AESS. In the Reconstruction Problem for the AESS, we must choose the three parameters as functions of t in such a way that the conics thus defined have 3-point contact with their envelope curve. Then, by construction, the envelope curve *will* have E as its AESS. This is a more complicated procedure than the analogous problem for the SS, and so instead we take a sideways step and consider the following connected, but simpler, problem.

The Semi-Reconstruction Problem (SRP): *Given a smooth curve segment γ_1, and a smooth curve segment E, what choice do we have in constructing a corresponding smooth curve segment γ_2 such that the composite curve $\gamma_1 \cup \gamma_2$ has E as its AESS?*

The answer to this question is found at the end of §4.3. Here we outline how we will tackle the Semi-Reconstruction Problem for the AESS. Consider the 2-parameter family of conics $C(s,t)$ having 3-point contact with a given smooth plane curve segment γ_1 (parametrised by t) and having centre on a smooth plane curve segment E (parametrised by s). Taking s as a function of t defines a *1-parameter sub-family* of this 2-parameter family. The envelope of this family will contain γ_1, and each of the constituent curves of the sub-family will have 3-point contact with γ_1, that is, γ_1 will be a 3-*point contact envelope* segment (see Definition 1.5.1). The problem we will consider is this:

How do we choose $s \equiv s(t)$ (in a neighbourhood of $t = t_0$) in such a way that the resulting envelope also has 3-point contact with its constituent curves away from γ_1?

Outline of Chapter 4

§**4.2**: We develop some general analysis concerning the construction of 3-point contact envelopes from general 2-parameter families of curves.

§**4.3**: We apply the theory developed in §4.2 to the Semi-Reconstruction Problem for the AESS. We will show that, given two smooth curve segments γ_1 and E, and two corresponding points on these two curves, we can, at least locally, find an *unique* curve segment γ_2 such that the composite curve $\gamma_1 \cup \gamma_2$ has E as its AESS. This answers the SRP for the AESS.

§**4.4**: Here we consider an example of the Semi-Reconstruction Problem for the AESS as a straight line segment, using [MAPLE].

§**4.5**: Finally, we use the ideas developed in §§4.2-§4.3 to solve the Reconstruction Problem for the AESS.

4.2 3-Point Contact Envelopes

Consider a general 2-parameter family of smooth plane curves given by

$$F(x, y, s, t) = 0,$$

where s, t are the family parameters. Take a 1-parameter sub-family defined by $s \equiv s(t)$ for t near t_0, giving $F(x, y, s(t), t) = 0$ as our 1-parameter family. Its envelope (x, y) is given by solving

$$\begin{cases} F(x, y, s(t), t) = 0, \\ \frac{\partial}{\partial t}\{F(x, y, s(t), t)\} = F_s(x, y, s(t), t)s'(t) + F_t(x, y, s(t), t) = 0. \end{cases}$$

for $x = x(t), y = y(t)$ close to $x_0 \equiv x(t_0)$, $y_0 \equiv y(t_0)$, where $'$ (prime) denotes d/dt. We denote $s(t_0)$ by s_0, and $s'(t_0)$ by s'_0. Suppose that this envelope is smooth at $t = t_0$, that is, $(x'(t_0), y'(t_0)) \neq (0, 0)$. Then our envelope satisfies

$$F(x(t), y(t), s(t), t) = 0, \tag{4.1}$$

$$F_s(x(t), y(t), s(t), t)s'(t) + F_t(x(t), y(t), s(t), t) = 0. \tag{4.2}$$

Expressions (4.1) and (4.2) are identities in t, and thus we can differentiate them w.r.t. t. Differentiating (4.1) w.r.t. t gives us

$$\begin{aligned} &F_x(x(t), y(t), s(t), t)x'(t) + F_y(x(t), y(t), s(t), t)y'(t) \\ &\quad + F_s(x(t), y(t), s(t), t)s'(t) + F_t(x(t), y(t), s(t), t) = 0, \end{aligned} \tag{4.3}$$

which holds for all t, and then (4.2) tells us that

$$F_x(x(t), y(t), s(t), t)x'(t) + F_y(x(t), y(t), s(t), t)y'(t) = 0, \tag{4.4}$$

for all t. Differentiating (4.4) w.r.t. t we get

$$\begin{aligned} &F_{xx}(x(t), y(t), s(t), t)x'(t)^2 + 2F_{xy}(x(t), y(t), s(t), t)x'(t)y'(t) \\ &\quad + F_{yy}(x(t), y(t), s(t), t)y'(t)^2 + F_{xs}(x(t), y(t), s(t), t)x'(t)s'(t) \\ &\quad + F_{ys}(x(t), y(t), s(t), t)y'(t)s'(t) + F_{xt}(x(t), y(t), s(t), t)x'(t) \\ &\quad + F_{yt}(x(t), y(t), s(t), t)y'(t) + F_{tt}(x(t), y(t), s(t), t) = 0, \end{aligned} \tag{4.5}$$

and differentiating (4.2) w.r.t. t we get

$$\begin{aligned}
F_{xs}(x(t), y(t), s(t), t)x'(t)s'(t) + F_{ys}(x(t), y(t), s(t), t)y'(t)s'(t) \\
+F_{ss}(x(t), y(t), s(t), t)s'(t)^2 + 2F_{st}(x(t), y(t), s(t), t)s'(t) \\
+F_{xt}(x(t), y(t), s(t), t)x'(t) + F_{yt}(x(t), y(t), s(t), t)y'(t) \\
+F_{tt}(x(t), y(t), s(t), t) = 0. \qquad (4.6)
\end{aligned}$$

Now initial curve $F(x, y, s_0, t_0) = 0$ which intersects the envelope where

$$F(x(t), y(t), s(t), t) = 0. \qquad (4.7)$$

We measure the contact between the initial curve and the envelope by finding conditions on the vanishing at $t = t_0$ of the derivatives of (4.7) w.r.t. t. Differentiating (4.7) we find that the condition for $\geq$ 2-point contact between the initial curve and the envelope at $t = t_0$ is that

$$F_x(x(t), y(t), s_0, t_0)x'(t) + F_y(x(t), y(t), s_0, t_0)y'(t) = 0, \qquad (4.8)$$

holds at $t = t_0$, and this is true by (4.4). (Of course, we should expect *at least* 2-point contact between the initial curve and the envelope, by construction.) Furthermore, for $\geq$ 3-point contact between the initial curve and the envelope we require that the derivative of (4.8) w.r.t. t vanishes at $t = t_0$:

$$\begin{aligned}
F_{xx}(x(t), y(t), s_0, t_0)x'(t)^2 + 2F_{xy}(x(t), y(t), s_0, t_0)x'(t)y'(t) \\
+F_{yy}(x(t), y(t), s_0, t_0)y'(t)^2 + F_x(x(t), y(t), s_0, t_0)x''(t) \\
+F_y(x(t), y(t), s_0, t_0)y''(t) = 0,
\end{aligned}$$

and this holds at $t = t_0$ if and only if

$$\begin{aligned}
F_{xs}(x_0, y_0, s_0, t_0)x_0's_0' + F_{sy}(x_0, y_0, s_0, t_0)y_0's_0' \\
+F_{xt}(x_0, y_0, s_0, t_0)x_0' + F_{yt}(x_0, y_0, s_0, t_0)y_0' = 0, \qquad (4.9)
\end{aligned}$$

using the expression (4.5) evaluated at $t = t_0$. Thus the 2-point contact condition (4.8) (evaluated at $t = t_0$) and the 3-point contact condition (4.9) together are

$$\left.\begin{aligned} F_x x_0' + F_y y_0' = 0 \\ (F_{xs}s_0' + F_{xt})\, x_0' + (F_{ys}s_0' + F_{yt})\, y_0' = 0 \end{aligned}\right\}$$

where everything is evaluated at (x_0, y_0, s_0, t_0). This system has $(x_0', y_0') = (0,0)$ as one solution. However, if we know that the envelope $(x(t), y(t))$ is non-singular at $t = t_0$, then the *3-point contact envelope condition* becomes

$$\begin{vmatrix} F_x & F_y \\ F_{xs}s' + F_{xt} & F_{ys}s' + F_{yt} \end{vmatrix} = 0,$$

where everything in the above determinant is evaluated at (x_0, y_0, s_0, t_0). This is the condition for the initial curve to have 3-point contact with the envelope curve at $t = t_0$. If we require that every member of the family of curves has 3-point contact with the envelope curve, then we simply require this determinant condition to hold for all $(x(t), y(t), s(t), t)$ in a neighbourhood of $t = t_0$. Thus we have:

Proposition 4.2.1 (3-Point Contact Envelope Condition).
Let $F(x, y, s, t) = 0$ be a 2-parameter family of smooth plane curves in (x, y)-space, parametrised by s, t. Suppose we define a 1-parameter sub-family of this family by taking s as a smooth function of t, and suppose that the envelope $(x, y) = (x(t), y(t))$ of this sub-family $F(x, y, s(t), t) = 0$ is smooth near some point $(x_0, y_0) \equiv (x(t_0), y(t_0))$. Then, in a neighbourhood of $t = t_0$, this envelope has 3-point contact with each of its constituent curves if, for all values of t close to t_0,

$$\begin{vmatrix} F_x & F_y \\ F_{xs}s' + F_{xt} & F_{ys}s' + F_{yt} \end{vmatrix} = 0, \tag{4.10}$$

where everything in the above determinant is evaluated at $(x(t), y(t), s(t), t)$.

We will use the 3-Point Contact Envelope Condition of Proposition 4.2.1 in the following way:

- Take a 1-parameter sub-family of $F(x, y, s, t) = 0$ by choosing $s \equiv s(t)$, with an arbitrary initial correspondence (s_0, t_0) between the parameters. Call the fixed curve $F(x, y, s_0, t_0) = 0$ the '*initial curve*'.

- Suppose the 1-parameter family defined by $s(t)$ in a neighbourhood of (s_0, t_0) in parameter-space has a smooth envelope segment (it may have many such segments). Then Proposition 4.2.1 tells us that the initial curve has 3-point contact with its envelope segment if the determinant condition stated holds at $t = t_0$.

- This determinant condition in effect tells us the value of $s'(t_0)$ we must have in order for there to be a 3-point contact between the initial curve and a segment of its envelope. There may be numerous such values, just as there may be numerous possible 3-point envelope segments.

- This leads us to a *vector field* (perhaps many-valued) in parameter-space. Through each point (s_0, t_0), we acquire at least one direction, namely s'_0, in which we must move in order to create a 3-point contact envelope.

The creation of this vector field will give a solution for each (s_0, t_0), and the local solution(s) in a neighbourhood of each (s_0, t_0) may be constructed from the corresponding integral curve(s).

4.3 Solving The SRP For The AESS

We now apply the results of §4.2 to the Semi-Reconstruction Problem (SRP) for the AESS. Suppose we are given a smooth curve segment γ_1, parametrised by t, and a corresponding smooth AESS segment E, parametrised by s. Let

$$C(x, y, s, t) = 0,$$

be the equation of the 2-parameter family of conics having 3-point contact with γ_1 at $\gamma_1(t)$ and having centre on the AESS at $E(s)$. We take a 1-parameter sub-family by choosing $s \equiv s(t)$ in a neighbourhood of some $t = t_0$.

Since we are assuming that both γ_1 and E are smooth, there is no natural restriction on which value of s should correspond to $t = t_0$. Thus we are able to choose any s_0 to correspond with t_0, that is, we may choose $s_0 \equiv s(t_0)$. This amounts to choosing an *initial correspondence* between two points on the curve segments γ_1 and E. We will assume an initial correspondence is given between the two curve segments γ_1 and E, and consider the problem of finding a suitable γ_2 segment. The choice of initial correspondence is equivalent to choosing the '*initial conic* given by $C(x, y, s_0, t_0) = 0$.

Notation: *For brevity, we will use superscript '0' to denote evaluation at* $s = s_0, t = t_0$. *For example, the initial conic* $C(x, y, s_0, t_0) = 0$ *will be denoted* $C^0 = 0$, *and* $C_s(x, y, s_0, t_0) = 0$ *will be denoted* $C_s^0 = 0$, *and so on.*

The envelope (x, y) of the family is given by the system

$$\left.\begin{aligned} C(x, y, s(t), t) = 0 \\ C_s(x, y, s(t), t)s'(t) + C_t(x, y, s(t), t) = 0 \end{aligned}\right\} \tag{4.11}$$

where $'$ (prime) denotes derivative with respect to t. We solve this for $x = x(t)$, $y = y(t)$ for t close to t_0. We will assume that the envelope curve $(x(t), y(t))$ is smooth near $t = t_0$, that is $(x'(t_0), y'(t_0)) \neq (0, 0)$. Proposition 4.2.1 tells us that, in a neighbourhood of $t = t_0$, this envelope has 3-point contact with each of its constituent curves if, for all values of t close to t_0, if

$$\begin{vmatrix} C_x & C_y \\ C_{sx}s' + C_{tx} & C_{sy}s' + C_{ty} \end{vmatrix} = 0. \tag{4.12}$$

From §1.5.3, we can interpret this geometrically: the 3-point contact envelope condition says that, at a solution (x_0, y_0) of the system (4.11) for which (4.12) holds, the curves in (4.11) are tangential. Now, for a given t_0,

- $C^0 = 0$ is a conic (the '*initial conic*');
- $C_s^0 s_0' + C_t^0 = 0$ is a pencil of conics (the '*initial pencil*') with parameter s_0'.

We know that γ_1 itself forms part of the 3-point contact envelope, that is $C^0 = 0$ and any member of the pencil $C_s^0 s_0' + C_t^0 = 0$ are tangent at $\gamma_1(t_0)$. Thus we are looking for values of s_0' for which the conics $C^0 = 0$ and $C_s^0 s_0' + C_t^0 = 0$ are *bitangent*.

Proposition 4.3.1. *The initial conic is bitangent to a member of the initial pencil for precisely two values of s'_0, one of which is always zero.*

Proof. Consider two smooth curve segments $\gamma(t) \equiv (t, \alpha(t))$ and $E(s) \equiv (s, \beta(s))$ where

$$\alpha(t) = \alpha_2 t^2 + \alpha_3 t^3 + \dots,$$
$$\beta(s) = \beta_0 + \beta_1 s + \beta_2 s^2 + \dots,$$

and where we assume $\alpha_2\beta_0 \neq 0$. Without loss of generality, we take $t_0 = 0$ and $s(t_0) \equiv s_0 = 0$, and by an affine transformation of the plane we may assume that $E(0)$ lies on the y-axis. Consider conic C given by

$$C(x, y, s, t) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where a, b, c, f, g, h are all homogeneous functions of s, t. We know that C has its centre on the AESS at the point $E(s)$, and 3-point contact with γ at the point $\gamma(t)$. From these two properties, we are able to derive the equation of the initial conic, which is

$$2\alpha_2\beta_0 x^2 + y^2 - 2\beta_0 y = 0. \tag{4.13}$$

and the equation of the initial pencil, which is

$$\left(-\frac{\beta_1}{\beta_0}y^2 - 4\alpha_2 xy\right) s'_0 + \left(-\frac{3\alpha_3}{\alpha_2}y^2 + \frac{6\alpha_3\beta_0}{\alpha_2}y\right) = 0. \tag{4.14}$$

Note that y is a factor of $C^0_s s'_0 + C_t = 0$ for all s'_0, and thus the initial pencil is reducible in this coordinate system, with each member being tangent to $C^0 = 0$ at the origin.

We now ask: *What values of s'_0 make $C^0_s s'_0 + C^0_t = 0$ tangent to $C^0 = 0$ somewhere else?* That is, when is the other line factor of the reducible conic tangent to $C^0 = 0$? Now we can write $C^0_s s'_0 + C^0_t = 0$ as $T_0 L = 0$, where $T_0 = 0$ is the tangent to $C^0 = 0$ at the origin (the line $y = 0$), and $L = 0$ is the pencil of lines (parametrized by s'_0) through the point $(-\beta_1/2\alpha_2, 2\beta_0)$. A simple calculation shows that precisely two of the lines in this pencil are tangent to $C^0 = 0$. They are:

- the line $y = 2\beta_0$, parallel to $T_0 = 0$, which corresponds to the solution $s_0' = 0$, when the tangents to the initial conic are parallel.

- the line
$$2\beta_1 x + \left(\frac{\beta_1^2}{2\alpha_2\beta_0} - 1\right) y + 2\beta_0 = 0,$$
which meets line $T_0 = 0$ at $(-\beta_0/\beta_1, 0)$. This corresponds to the solution $s_0' = -\frac{3\alpha_3\beta_1}{2\alpha_2^2}$, which is generally non-zero. We note that this line meets $T_0 = 0$ in the same point that $T_0 = 0$ meets the tangent to the AESS at $E(0)$, and hence this solution satisfies the condition on the concurrency of the tangents to the two curve segments and the tangent to the AESS (see Proposition 2.4.5 for details).

□

Thus we have, in parameter space, two vector fields: one is 'degenerate', being always along the direction parallel to the t-axis, and the other is in the required direction which leads to a 3-point contact envelope. This solves the problem by defining a vector field in (s, t)-space, and the local solutions are given by the integral curves of this vector field, which are the lines $s \equiv$ *constant*, along with the other curves $s = s(t)$ (with $s_0' = -\frac{3\alpha_3\beta_1}{2\alpha_2^2}$) corresponding to '*proper*' solutions.

In §2.5.9, we showed that the AESS has a cusp at the centre of a 3+3 conic with parallel tangents to the curve at the points of contact, and therefore solution $s_0' = 0$ corresponds to a non-smooth AESS point. Hence we conclude that at each (s_0, t_0) in parameter-space there is an *unique* value of s_0' which we must choose in order for the family thus defined to have a *smooth* 3-point contact envelope. In conclusion, this result says the following:

Solution to Semi-Reconstruction Problem: *Given two smooth curve segments γ_1 and E, and two corresponding points on these two curves (an* initial correspondence*), we can (at least locally) find an unique γ_2 such that the composite curve $\gamma_1 \cup \gamma_2$ has E as its AESS.*

This answers the question posed in the Semi-Reconstruction Problem: the information that we require is the *initial correspondence* between a point on

γ_1 and a point on E, and once that has been chosen, there is an unique 'other side' γ_2 with the property that $\gamma_1 \cup \gamma_2$ has E as its AESS. It is enlightening to restate the conclusion in the following way:

Proposition 4.3.2. *Given smooth curve segments γ_1, E there is a 1-parameter family of curve segments γ_2 such that $\gamma_1 \cup \gamma_2$ has E as its AESS.*

4.4 Example

We would now like to use the results of §4.3 to re-examine an example from [GS96] where we are given a straight line AESS segment E, one side of our original curve, γ_1, and we wish to reconstruct the other side γ_2 such that the AESS of $\gamma_1 \cup \gamma_2$ is E.

Suppose we are given a smooth curve segment $\gamma_1(s) = (X(s), Y(s))$, and a straight line AESS segment E which we will take to be along the x-axis. Suppose also that E is the AESS of γ_1 and some other curve segment γ_2. We then ask: *How can we derive an expression for γ_2?*

We construct the other side $\gamma_2 = (U(s), V(s))$ such that the AESS of $\gamma_1 \cup \gamma_2$ is E using the fact that the tangent line to E at each point is the x-axis, and that the x-axis contains the midpoint of the chord joining $\gamma_1(s)$ and $\gamma_2(s)$, and the intersection of the tangent lines to γ_1 and γ_2 at $\gamma_1(s)$ and $\gamma_2(s)$. The tangent to γ_1 at $\gamma_1(s)$ meets the x-axis at the point with x-coordinate $(XY' - X'Y)/Y'$, where we have omitted the variable s; the tangent to the other side γ_2 at the corresponding point $\gamma_2(s)$ meets the x-axis at $(UV' - U'V)/V'$, and therefore we have that

$$\frac{X(s)Y'(s) - X'(s)Y(s)}{Y'(s)} = \frac{U(s)V'(s) - U'(s)V(s)}{V'(s)}, \tag{4.15}$$

holds for all values of s. Now the midpoint of the chord joining $\gamma_1(s)$ and $\gamma_2(s)$ lies on the x-axis, so that

$$Y(s) + V(s) = 0, \tag{4.16}$$

for all s. This gives $Y'(s) + V'(s) = 0$ for all s, and substituting into (4.15) we get

$$X(s)Y'(s) - X'(s)Y(s) = U(s)Y'(s) - U'(s)Y(s).$$

We will assume that $Y(s) \neq 0$ for all s, since we do not want to allow γ_1 to cross the AESS segment, and we will also assume that $Y'(s) \neq 0$ for all s, so that the tangent line to γ_1 is never parallel to the AESS: these assumptions will not affect the result. Under these assumptions, the above can be rewritten as

$$\frac{d}{ds}\left(\frac{X(s)}{Y(s)}\right) = \frac{d}{ds}\left(\frac{U(s)}{V(s)}\right),$$

which implies that

$$X(s) = U(s) + \lambda Y(s), \text{ for some } \lambda \in \mathbb{R}. \tag{4.17}$$

Thus the other side $\gamma_2(s) = (U(s), V(s))$ is given by

$$\begin{pmatrix} U(s) \\ V(s) \end{pmatrix} = \begin{pmatrix} 1 & -\lambda \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix}.$$

Thus there is, as predicted, a 1-parameter family of other sides γ_2, and each of these γ_2 segments if affine symmetric with γ_1 about the x-axis, since the matrix of the affine transformation which links $(U(s), V(s))$ to $(X(s), Y(s))$ is an affine reflexion. The parameter corresponds to the choice of initial correspondence between a point of γ_1 and a point of E

Remark 4.4.1. *With reference to the geometrical interpretation of the AESS in* §2.1.2*, the fact that the curve segments* γ_1 *and* γ_2 *are affine symmetric follows from the observation that each pair of corresponding points of* γ_1 *and* γ_2 *are locally affine symmetric about the tangent to the AESS at the corresponding point, and since each of these tangents are identically the same, these axes of infinitesimal symmetry constitute a global axis of affine symmetry.*

4.5 The Reconstruction Problem

Returning to the Reconstruction Problem of §4.1, we use the analysis of §4.3 to tell us what freedom we will have in the construction of the original curve

γ. The solution to the Semi-Reconstruction Problem says that, given a fixed segment γ_1 of γ, and an AESS segment E, then there is a 1-parameter family of other sides γ_2 such that $\gamma_1 \cup \gamma_2$ has E as its AESS. Thus, given a smooth curve segment E, suppose we choose a curve segment γ_1. By 'degree of freedom' arguments, there will generically be a smooth 1-parameter family of conics centred on E and having 3-point contact with γ_1. If we choose a point γ_2^0 in the plane (away from γ_1, E), then there will be an unique curve segment γ_2 through γ_2^0 such that $\gamma_1 \cup \gamma_2$ has E as its AESS. This choice of point represents the choice of initial conic as discussed earlier.

We can thus state:

Proposition 4.5.1. *Given a smooth curve segment E, choose a smooth curve segment γ_1 and a point γ_2^0 away from γ_1, E. Then there is an unique curve γ containing γ_1 and γ_2^0 having E as its AESS.*

Chapter 5

The Reconstruction Problem For The ADSS

5.1 Introduction

In this chapter we consider the Reconstruction Problem for the ADSS.

The Reconstruction Problem: *Given a smooth curve segment D, construct a smooth plane curve which has D as its ADSS.*

We begin by refreshing our memory as to the definition of the ADSS, which we will state as:

Definition 5.1.1. *The ADSS of a smooth plane curve γ is the closure of the locus of centres of pairs of conics having 4-point contact with γ at two distinct points and sharing the same centre and affine radius.*

We choose this definition of the ADSS, as opposed to the 'bifurcation set' definition, because of its geometric nature. We will be interested, not in the critical points of the affine distance function, but in these pairs of 4-point contact conics. As for the AESS in Chapter 4, we will first of all consider a simpler connected problem. The idea is that, given two smooth curve segments γ_1 and D, we construct another segment of smooth curve, say γ_2, in such a way that the composite curve $\gamma_1 \cup \gamma_2$ has D as its ADSS. It should be noted that this other side of the curve will not be unique (see Remark 5.2.1 at the end of §5.2.1). In effect, this other side γ_2 will be recovered as part of the envelope of the pairs of conics (as in Definition 5.1.1) centred on D, where these conics are chosen in such a way as to have a 4-point contact envelope. (Of course, the other part of the 4-point contact envelope of conic pairs will contain the curve segment γ_1.) We state this simpler Reconstruction Problem as follows:

The Semi-Reconstruction Problem (SRP): *Given a smooth curve segment γ_1, and a smooth curve segment D, what choice do we have in constructing a corresponding smooth curve segment γ_2 such that the composite curve $\gamma_1 \cup \gamma_2$ has D as its ADSS?*

5.2 The SRP for the ADSS

Consider two smooth curve segments γ_1 and D, and suppose that parameter t is the affine-arclength parameter along γ_1. At each point $\gamma_1(t)$, there exists an unique conic $\mathcal{K}(t) = 0$, with affine radius $\sigma(t) = [\gamma_1(t) - D(t), \gamma_1'(t)]$ having 4-point contact with γ_1 at $\gamma_1(t)$ and having centre on D at $D(t)$. Now the affine

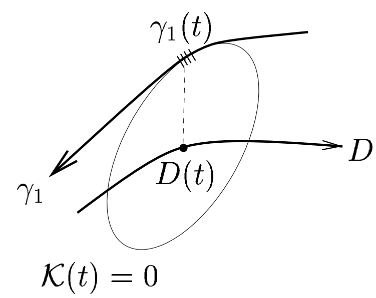


Figure 5.1: *Two smooth curve segments γ_1, D, and non-degenerate conic $\mathcal{K}(t) = 0$ having 4-point contact with γ_1 at $\gamma_1(t)$ and centre at $D(t)$. The dashed line represents the affine normal to curve segment γ_1 at $\gamma_1(t)$.*

normal to γ_1 at $\gamma_1(t)$ 'transfers' parameter t onto the ADSS segment D, since (at least locally) there is an unique point of D lying on a given affine normal to γ_1 (see Figure 5.1). Thus we have a natural correspondence between points of γ_1 and points of D. Of course, t is *not* the affine-arclength parameter along D. For a fixed t_0, there is a 2-parameter family of conics having centre at $D(t_0)$ and radius $\sigma(t_0)$. Allowing t to vary in a neighbourhood of t_0, we have a 3-parameter family of conics

$$C(x, y, t, u, v) = 0, \tag{5.1}$$

having centre at $D(t)$ and radius $\sigma(t)$, where u and v represent the two other family parameters. We define a 1-parameter sub-family of this 3-parameter family of conics by taking $u \equiv u(t)$ and $v \equiv v(t)$ in a neighbourhood of $t = t_0$. The envelope of this 1-parameter sub-family will, for arbitrary choices of $u \equiv u(t), v \equiv v(t)$, have normal 2-point contact with each of its constituent curves. In general, an envelope of conics will have four distinct envelope segments having 2-point contact with its constituent curves, since 'consecutive' conics intersect in four distinct points. We must choose $u \equiv u(t)$ and $v \equiv v(t)$ in such a way that *three* of these intersection points coincide to fom a 4-point contact envelope segment (of course, there will remain another 2-point contact envelope segment). The problem we will consider is this:

How do we choose $u \equiv u(t), v \equiv v(t)$ (in a neighbourhood of $t = t_0$) in such a way that the resulting envelope has 4-point contact with its constituent curves away from γ_1?

In this way, we construct our other side γ_2, which will (under some minor conditions outlined in §5.2.1) have been constructed so that $\gamma_1 \cup \gamma_2$ has D as its ADSS.

5.2.1 Geometrical conditions on the choice of the *Initial Conic*

We will call the conic $C(x, y, t_0, u_0, v_0) = 0$ the '*initial conic*', and denote it by $C^0 = 0$, where $u_0 \equiv u(t_0), v_0 \equiv v(t_0)$ are arbitrarily chosen under some geometrical conditions outlined below. We will assume throughout that the initial conic is an ellipse or an hyperbola.

We require the Concurrent Tangents Condition of Corollary 3.2.5 to hold, that is, we require that the tangent line to γ_1 at $\gamma_1(t)$, the tangent line to D at $D(t)$, and the tangent line to the other side γ_2 (the segment that we are reconstructing) at the corresponding point are all concurrent. In the case of the initial conic $C^0 = 0$ being an ellipse, this implies that the two known tangents, to γ_1 and D at parameter value t, must intersect 'outside' $C^0 = 0$, in order for there to exist a tangent to $C^0 = 0$ through this intersection point. Thus the initial conic must be chosen such that the intersection of the tangent to γ_1 at $\gamma_1(t)$ and the tangent to D at $D(t)$ lies outside it. A similar geometrical condition applies to the choice of initial hyperbola.

We note that the conics $\mathcal{K}(t) = 0$ are always members of the 3-parameter family $C(x, y, t, u, v) = 0$. We would like to always choose u_0, v_0 such that the initial conic $C^0 = 0$ is different from $\mathcal{K}(t_0) = 0$, since we will be interested in studying the envelope of conics in some neighbourhood of our initial conic, which we will manipulate in order to be a 4-point contact envelope and therefore a suitable other side. However, if we choose initial conic to be $\mathcal{K}(t_0) = 0$, then this 'other side' will be identical to γ_1, and we will have failed to construct an appropriate γ_2.

Thus there is some restriction on the initial conic, and therefore some restriction on our choice of u_0, v_0. We will assume from now on that u_0 and v_0 have been chosen appropriately, so that a suitable other side can be constructed. Under this natural restriction on our choice of initial conic, we begin our attempt to study how a suitable γ_2 curve segment might be constructed.

Remark 5.2.1. *This (almost) arbitrary choice of initial values u_0, v_0 justifies our claim that the other side γ_2 will generally not be unique. The fact that we have two degrees of freedom in our choice of initial conic implies that, for fixed γ_1 and D, there is a 2-parameter family of suitable γ_2 curves for which $\gamma_1 \cup \gamma_2$ has D as its ADSS. We will show that, once the choice of initial conic is made, there are precisely two possible other sides γ_2.*

Outline of Chapter 5

§**5.3**: We relate the problem in hand, namely that of choosing $u \equiv u(t), v \equiv v(t)$ (locally, near $t = t_0$) so that the resulting envelope has *4-point* contact with its constituent curves, to finding *3-point* contact between the initial conic and a given *initial net* (2-parameter family) of conics, using results of §1.5. The parameters of the net of conics will be the values of $u'(t), v'(t)$ evaluated at $t = t_0$, which we will denote as u_0', v_0'.

§**5.4**: We set up a local coordinate system containing two curve segments γ_1 and D, and calculate explicit expressions for the initial conic and initial net in this coordinate system. We then consider the relationship between these objects, and explain how this leads to a solution to the SRP for the ADSS.

§**5.5**: We consider examples of the SRP for the ADSS, using [MAPLE]. In §5.5.1, we consider the situation where we have a straight line ADSS segment and a given γ_1 curve segment, and we construct a suitable γ_2 segment. In §5.5.2, we consider this same construction for a general ADSS segment.

§**5.6**: Finally, we use the ideas developed in §§5.3-5.4 to solve the Reconstruction Problem for the ADSS.

5.3 Choosing a 1-parameter family of conics

Suppose we have chosen suitable u_0, v_0 as discussed in §5.2.1, and furthermore have chosen u, v as functions of t in a neighbourhood of $t = t_0$. Then the envelope of the resulting 1-parameter family (in a neighbourhood of $t = t_0$) is defined by the intersections of the curves

$$\begin{cases} C(x,y,t,u(t),v(t)) = 0, \\ C_t(x,y,t,u(t),v(t)) + C_u(x,y,t,u(t),v(t))u'(t) + C_v(x,y,t,u(t),v(t))v'(t) = 0, \end{cases}$$

where the subscripts denote the partial derivative w.r.t. the corresponding variable t, u, v, and $'$ (prime) denotes derivative w.r.t. t.

Notation: *We will denote* $C_t(x,y,t_0,u_0,v_0) = 0$ *simply by* $C_t^0(x,y) = 0$ *or* $C_t^0 = 0$*, and similarly for* $C_u(x,y,t_0,u_0,v_0) = 0$*, etc.*

The envelope segment is given by solving the above system simultaneously (in a neighbourhood of $t = t_0$) for $x = x(t), y = y(t)$. We require a condition on the choice of $u \equiv u(t), v \equiv v(t)$ for $C^0 = 0$ to have 4-point contact with the envelope segment near $t = t_0$. From Proposition 1.5.4 and Corollary 1.5.8, which showed that $\geq$ 4-point contact envelopes occur when 'consecutive' conics have $\geq$ 3-point contact, we have the following:

Corollary 5.3.1. *Finding conditions on $u(t), v(t)$ at $t = t_0$ for the initial conic to have 4-point contact with the envelope segment at $t = t_0$ amounts to finding conditions for there to be 3-point contact between the curves*

$$\begin{cases} C(x, y, t_0, u_0, v_0) = 0, \\ C_t(x, y, t_0, u_0, v_0) + C_u(x, y, t_0, u_0, v_0)u'_0 + C_v(x, y, t_0, u_0, v_0)v'_0 = 0, \end{cases}$$

where $u'_0 \equiv u'(t_0), v'_0 \equiv v'(t_0)$.

Now the first of these curves is a *fixed conic*, the initial conic. The second expression is a *net of conics* with parameters u'_0, v'_0, and we will refer to this as the *initial net of conics*. Thus the SRP can be expressed as follows:

Problem: *How can we choose parameters u'_0, v'_0 such that the initial net $C^0_t + u'_0 C^0_u + v'_0 C^0_v = 0$ has 3-point contact with initial conic $C^0 = 0$?*

We answer the above Problem in §5.4, setting up a coordinate system containing a 3-parameter family of conics $C(t, u, v) = 0$, and finding explicit expressions for $C_t = 0, C_u = 0, C_v = 0$ at $t = t_0$ in this coordinate system. We will deduce that there are precisely two different ways in which to choose u'_0, v'_0 in order to get 3-point contact between the initial net, $C^0_t + u'_0 C^0_u + v'_0 C^0_v = 0$, and the initial conic, $C^0 = 0$. This will prove the following:

Proposition 5.3.2. *Consider the 3-parameter family of conics defined in (5.1). Suppose we take u, v to be functions of t in a neighbourhood of $t = t_0$. Then, for an arbitrary choice of u_0, v_0, there exists precisely two values for the pair $\{u'_0, v'_0\}$ such that the initial net $C^0_t + u'_0 C^0_u + v'_0 C^0_v = 0$ has 3-point contact with the initial conic $C^0 = 0$.*

Thus there are two distinct elements (that is, two distinct conics) in the initial net of conics that have 3-point contact with initial conic $C^0 = 0$ which implies that there are two values for the pair $\{u'_0, v'_0\}$ for which the initial conic has 4-point contact with its envelope. Hence we are able to state:

Proposition 5.3.3. *Given a smooth curve segment γ_1, a smooth curve segment D, and an initial conic (chosen under geometrical restrictions outlined in* §5.2.1*), we can construct* two *smooth curve segments γ_2 such that the composite curve $\gamma_1 \cup \gamma_2$ has D as its ADSS. (See Figure 5.2 for an illustration.)*

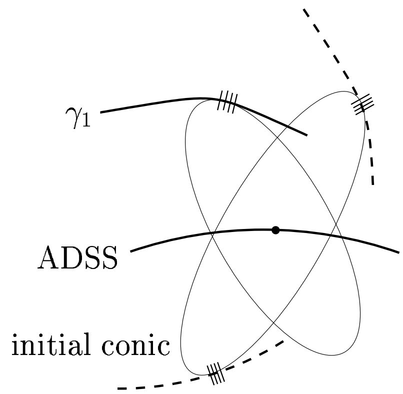


Figure 5.2: *The dashed curves represent the two possible other sides γ_2.*

Proposition 5.3.3 is the solution to the SRP for the ADSS: the *choice* we have is the *choice of initial conic*. Since there is a 2-parameter family of possible initial conics, and each choice leads to two 'other sides', we may state the following:

Corollary 5.3.4. *Given a smooth curve segment γ_1, and a smooth curve segment D, there are two distinct 2-parameter families of curve segments γ_2 with the property that $\gamma_1 \cup \gamma_2$ has D as its ADSS.*

5.4 Reconstructing the 'other side' γ_2

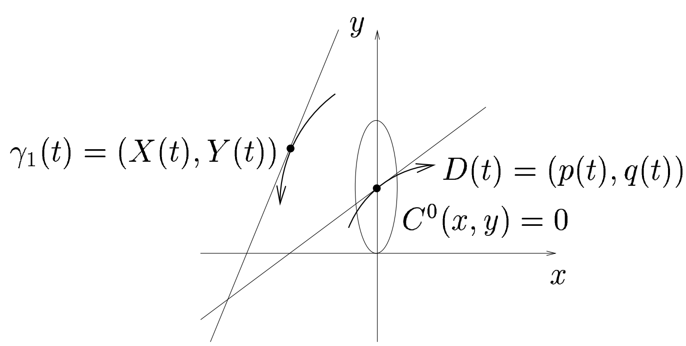


Figure 5.3: $C^0(x, y) = 0$ *is the initial conic, and the ADSS segment is $D(t)$, which passes through the centre of $C^0(x, y) = 0$ at parameter value $t = 0$.*

In this section, we prove Proposition 5.3.2. Consider the local coordinate system set up as illustrated in Figure 5.3, with initial conic $C^0(x, y) = 0$

centred at $(0,1)$ and tangent to the x-axis at the origin, and smooth curve segment $\gamma_1(t) = (X(t), Y(t))$. The ADSS segment $D(t) = (p(t), q(t))$ will pass through $(0,1)$ for parameter value $t = t_0 \equiv 0$. A short calculation shows that the equation of the initial conic $C^0(x,y) = 0$ in this coordinate system can be written as

$$x^2 + \sigma_0^3(y-1)^2 = \sigma_0^3, \tag{5.2}$$

where $\sigma_0 \equiv \sigma(0)$, and $\sigma(t)$ is the affine radius function (Definition 1.4.2). This is an ellipse if $\sigma_0 > 0$, in which case it is always real (since $(0,0)$ lies on it), and an hyperbola if $\sigma_0 < 0$. We will assume, without loss of generality, that $\sigma_0 \neq 0$.

In this coordinate system, we may also derive an expression for the 3-parameter family of conics having centre at $D(t)$ and affine radius $\sigma(t)$, using Proposition 1.4.5. With parameters u, v and t, this family is given by

$$x^2+2vxy+uy^2-2(p+vq)x-2(vp+uq)y+p^2+2vpq+uq^2-\sqrt{|\sigma|^3(u-v^2)} = 0, \tag{5.3}$$

where p, q and σ are functions of t. From this we calculate the equation for $C_t^0 = 0$, which is the line

$$-2p_1x - 2\sigma_0^3 q_1 y + 2\sigma_0^3 q_1 - \frac{3}{2}\sigma_0^2\sigma_1 = 0,$$

the equation $C_u^0 = 0$,which is the line-pair

$$(y-1)^2 - \frac{1}{2} = 0,$$

and the equation $C_v^0 = 0$ which is the line-pair

$$2x(y-1) = 0.$$

The initial net of conics is then given by $C_t^0 + u_0' C_u^0 + v_0' C_v' = 0$, where u_0', v_0' are the parameters we are trying to find. Writing $\sigma(t) = \sigma_0 + \sigma_1 t + \sigma_2 t^2 + \ldots$, we calculate that the intersection point (x_0, y_0) of the tangent to curve segment

$\gamma_1(t)$ at $\gamma_1(0)$ and the tangent to the ADSS segment $D(t)$ at $D(0)$ is

$$(x_0, y_0) = \left(-p_1 \frac{\sigma_0}{\sigma_1}, -q_1 \frac{\sigma_0}{\sigma_1} + 1\right). \tag{5.4}$$

By the assumptions made in §5.2.1, precisely two lines through the point (x_0, y_0) are tangent to the initial conic. The line through the points of contact of these tangents and the initial conic is the *polar line* of the point (x_0, y_0) w.r.t. the conic, and it has equation

$$xx_0 + \sigma_0^3 y y_0 - \sigma_0^3 (y + y_0) = 0.$$

Substituting the expressions for x_0 and y_0 from (5.4), we find that the polar line of the point (x_0, y_0) w.r.t. the initial conic is

$$-p_1 x - \sigma_0^3 q_1 y + \sigma_0^3 q_1 - \sigma_0^2 \sigma_1 = 0.$$

Note that this line is parallel to the line $C_t^0 = 0$.

By considering the cases of $\sigma_0 > 0$ and $\sigma_0 < 0$ in turn, we are able to measure the contact between the initial conic and the initial net of conics using the usual parametrisations $(x(t), y(t)) = (\sigma^{\frac{3}{2}} \cos t, 1 + \sigma^{\frac{3}{2}} \sin t)$ and $(x(t), y(t)) = (|\sigma|^{\frac{3}{2}} \sinh t, 1 + |\sigma|^{\frac{3}{2}} \cosh t)$ respectively. We are able to deduce that the initial conic and the initial net have 3-point contact at precisely the places where the polar line of the point (x_0, y_0) w.r.t. the initial conic intersects the initial conic. Since we know that, by the assumptions made in §5.2.1, the polar line of a finite point intersects the initial conic in two points, it follows that there are precisely two points on the initial conic where an element of the net has 3-point contact. We deduce that there exists precisly two elements of the initial net which have 3-point contact with the initial conic, and therefore two suitable pairs of values for $\{u_0', v_0'\}$.

Hence we have proved Proposition 5.3.2, which leads to a solution to the SRP for the ADSS. We now ask: *Is this solution geometrically plausible?* In §5.4.1, we answer this in the affirmative.

5.4.1 Geometrical interpretation

To interpret the result of the previous section, we use the Concurrent Tangents Condition of Proposition 3.2.5. Given a curve segment γ_1, and a curve segment D, then any other side γ_2 must satisfy the Concurrent Tangents Condition. Since γ_1 and D are fixed, this means that once we have chosen a point $\gamma_1(t_1)$ on γ_1 (and therefore $\mathbf{x} \equiv D(t_1)$ is fixed), then the intersection point i of the corresponding tangent lines to the curves at these points is fixed. Thus, upon choosing an initial conic, there are precisely two tangents to the initial conic passing through i. The points of intersection of the conic and the tangents will be at the intersection of the conic and the polar line of the conic with respect to i, as discussed (see Figure 5.4). These are precisely the points that the analysis of §5.4 tells us are the possible γ_2 points. Thus we know that the two solutions for the other side, γ_2 and $\bar{\gamma}_2$, will satisfy the Concurrent Tangents Condition. This geometrical interpretation justifies the solution to the SRP.

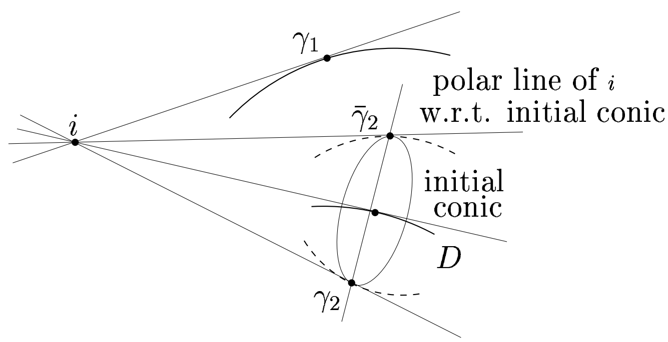


Figure 5.4: *Curve segments γ_2 and $\bar{\gamma}_2$ are the two possible 'other side' points corresponding to γ_1, meeting the initial conic at the points where the polar line of i meets the initial conic. The Concurrent Tangents Condition is satisfied for both γ_2 and $\bar{\gamma}_2$.*

5.5 Examples

We will now use the results of §§5.2-5.4 to analyse some examples where we are given an ADSS segment D, and one side of our original curve, γ_1, and we wish to reconstruct the other side γ_2 such that the ADSS of $\gamma_1 \cup \gamma_2$ is D.

5.5.1 The straight-line ADSS

This example was considered in [GS96]. Suppose we are given curve segment $\gamma_1(s) = (X(s), Y(s))$, where s is affine-arclength, and a straight-line ADSS segment which we will take to be the x-axis. We wish to construct another segment of curve $\gamma_2(s) = (U(s), V(s))$ such that the ADSS of $\gamma_1(s) \cup \gamma_2(s)$ is along the x-axis. Calculation shows that $\gamma_2(s)$ is given by solving the system

$$\begin{cases} Y^2 = V^2 + c, & \text{for arbitrary } c \in \mathbb{R}, \\ (XY' - X'Y)V' = (UV' - U'V)Y', \end{cases}$$

for (U, V) as functions of (X, Y), that is, for U, V as functions of s. The first equation gives us

$$YY' = VV', \text{ for all } s$$

and we substitute this (and $V^2 = Y^2 - c$) into the second equation to give

$$UYY' - U'(V^2 - c) = (XY' - X'Y)Y,$$

which can be re-written as

$$U\frac{d}{ds}\left(\frac{1}{Y}\right) + U'\left(\frac{1}{Y} - \frac{c}{Y^3}\right) = \frac{d}{ds}\left(\frac{X}{Y}\right). \tag{5.5}$$

Case $c = 0$

In this case (5.5) can be rewritten as

$$\frac{d}{ds}\left(\frac{U}{Y}\right) = \frac{d}{ds}\left(\frac{X}{Y}\right),$$

which has solution

$$\frac{U}{Y} = \frac{X}{Y} + \bar{c}, \text{ for some } \bar{c} \in \mathbb{R}.$$

Since we are assuming that $Y \neq 0$ (that is, that the γ_1 curve segment does not cross the x-axis near $s = 0$), we have solution

$$U(s) = X(s) + \bar{c}Y(s), \text{ for some } \bar{c} \in \mathbb{R},$$

which along with

$$V(s) = \pm\sqrt{Y(s)^2} = \pm Y(s)$$

defines the other side $\gamma_2(s) = (U(s), V(s))$. We write this in matrix form as

$$\begin{pmatrix} U(s) \\ V(s) \end{pmatrix} = \begin{pmatrix} 1 & \bar{c} \\ 0 & \pm 1 \end{pmatrix} \cdot \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix}.$$

This gives two 'other sides',

$$\begin{aligned} \gamma_2(s) &= \begin{pmatrix} 1 & \bar{c} \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix}, \\ \bar{\gamma}_2(s) &= \begin{pmatrix} 1 & \bar{c} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \bar{c} \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix}. \end{aligned}$$

Thus we are able to produce explicit expressions for the other sides (U, V) in the case $c = 0$, and we get two 1-parameter families of other sides: if $\bar{c}$ is fixed too, then there will be precisely two other sides $\gamma_2, \bar{\gamma}_2$. Figure 5.5 shows an example executed in [MAPLE] where we construct the two families $\gamma_2, \bar{\gamma}_2$ given a γ_1 curve segment and a straight line ADSS as shown.

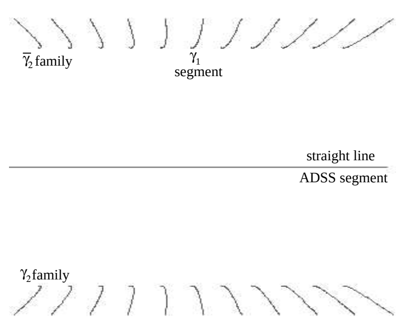


Figure 5.5: $c = 0$: *Given* γ_1 *and a straight line ADSS, we use [MAPLE] to construct the two 1-parameter families of curve segments* $\gamma_2, \bar{\gamma}_2$.

Remark 5.5.1. *Curve segment* γ_2 *is* affine symmetric *with* γ_1 *about the* x*-axis (the ADSS segment), since the matrix*

$$\begin{pmatrix} 1 & \bar{c} \\ 0 & -1 \end{pmatrix}$$

is an affine reflexion *in the x-axis (see §1.2.1). However, curve segment $\bar{\gamma}_2$ is* not *affine symmetric with γ_1 about the x-axis, but* is *affine symmetric with γ_2 about the x-axis.*

Case $c \neq 0$

For the case $c \neq 0$, we must solve the system

$$V(s)^2 = Y(s)^2 - c \tag{5.6}$$

$$U(s)\frac{d}{ds}\left(\frac{1}{Y(s)}\right) + U'(s)\left(\frac{1}{Y(s)} - \frac{c}{Y(s)^3}\right) = \frac{d}{ds}\left(\frac{X(s)}{Y(s)}\right) \tag{5.7}$$

for U, V as functions of X, Y, where parameter c is non-zero. There will still be two possible other sides, neither of which will be affine-symmetric with γ_1 about the x-axis. Suppose we fix c to be some non-zero number. Then (5.6) gives us two values for $V(s)$, namely $V(s) = \pm\sqrt{Y(s)^2 - c}$, and (5.7) is a first order differential equation for $U(s)$, the solution of which will introduce another arbitrary variable, say $\bar{c} \in \mathbb{R}$. We may solve this using an integrating factor, and we plot the solution curves $(U(s), V(s))$ for various values of the two parameters c and $\bar{c}$ using [MAPLE]. These parameters represent the two parameters that we may choose, as predicted, and once they are chosen, there will be precisely two other sides $\gamma_2, \bar{\gamma}_2$.

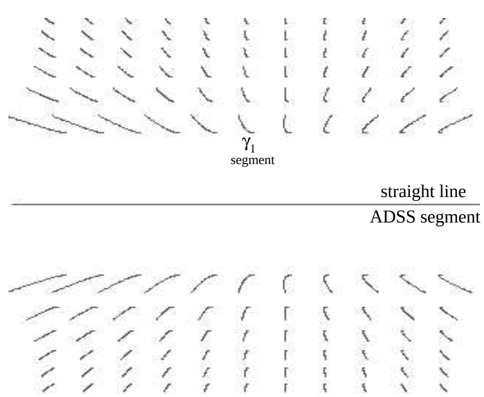


Figure 5.6: $c \neq 0$*: Given γ_1 and a straight line ADSS, we use [MAPLE] to construct the two 2-parameter families of curve segments $\gamma_2, \bar{\gamma}_2$. The γ_2 family lies below the ADSS segment, and the $\bar{\gamma}_2$ family lies above the ADSS segment.*

5.5.2 General ADSS

The analysis of the straight-line case of §5.5.1 can be generalised to consider the construction of the other side $\gamma_2 = (U, V)$ given curve segment $\gamma_1(s) = (X(s), Y(s))$ and a general ADSS segment $D(s) = (A(s), B(s))$. We begin by using the following geometrical facts, both of which follow from Corollary 3.2.5:

(i) The tangent lines to the two curve segments γ_1 and γ_2 at $\gamma_1(s)$ and $\gamma_2(s)$ meet on the tangent line to the ADSS at the corresponding point $D(s)$, that is

$$(X + \lambda X', Y + \lambda Y') = (A + \mu A', B + \mu B') = (U + \nu U', V + \nu V'),$$

for some $\lambda, \mu, \nu \in \mathbb{R}$. This gives us the condition

$$V'(\alpha X' + \beta(X - U)) + U'(\beta V + \delta Y' + \epsilon B'), \quad = \quad 0,$$

where

$$\begin{aligned} \alpha &= A'(Y - B) - B'(X - A), \\ \beta &= X'B' - Y'A', \\ \delta &= A'B - AB', \\ \epsilon &= XY' - X'Y. \end{aligned}$$

We can rewrite this condition as

$$V'(a - bU) + U'(bV + c) = 0, \tag{5.8}$$

where $a = \alpha X' + \beta X$, $b = \beta$, and $c = \delta Y' + \epsilon B'$.

(ii) The tangent to the ADSS at $D(s)$ is in the direction of $\gamma_1'(s) - \gamma_2'(s)$, where $'$ (prime) denotes derivative with respect to the affine-arclength parameter along the respective curve segments. We will assume that s is affine-arclength along neither γ_1 nor γ_2. Thus, with $k_1 = X'Y'' - X''Y'$

and $k_2 = U'V'' - U''V'$, we have

$$\begin{aligned} & \gamma_1' - \gamma_2' = \omega D', \text{ for some } \omega \in \mathbb{R} \\ \iff & k_1^{-1/3}(X', Y') - k_2^{-1/3}(U', V') = \omega(A', B'), \\ \iff & \frac{B'}{A'} = \frac{k_1^{-1/3}Y' - k_2^{-1/3}V'}{k_1^{-1/3}X' - k_2^{-1/3}U'}, \\ \iff & k_2 f = k_1(A'V' - B'U')^3, \end{aligned} \tag{5.9}$$

where f is the known function $(Y'A' - B'X')^3$, equal to $-\beta^3$.

Together, (5.8), (5.9) give the system

$$\left. \begin{array}{l} V'(a - bU) + U'(bV + c) = 0, \\ U'V'' - U''V'f = k_1(A'V' - B'U')^3, \end{array} \right\} \tag{5.10}$$

a system of ordinary differential equations that we wish to solve for $U(s), V(s)$ to give an other side $\gamma_2(s) = (U(s), V(s))$. We expect two arbitrary constants to appear in the solution to this system, and these are the two parameters as predicted in the solution to the SRP. Figure 5.7 shows [MAPLE] plots of the reconstruction of the other side, given curve segment γ_1 and ADSS segment D as shown.

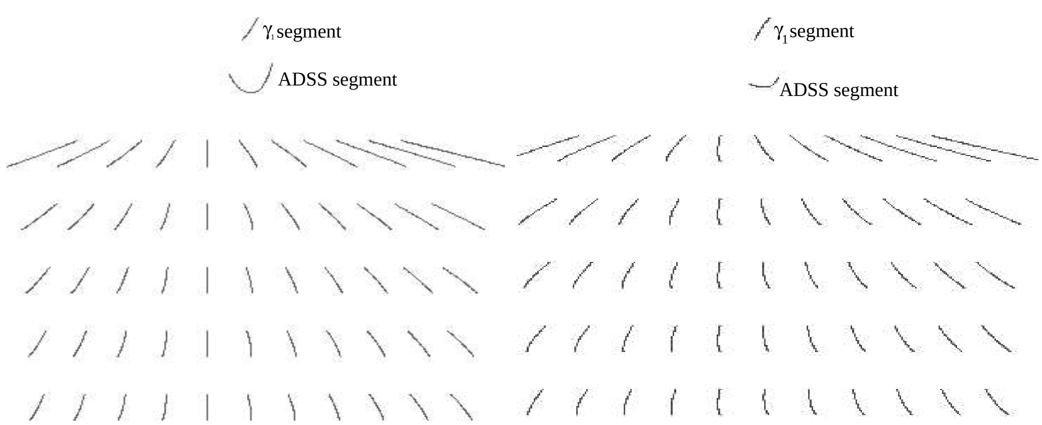


Figure 5.7: *Two examples of the reconstruction of the 'other side' given γ_1 and ADSS segment, implemented in [MAPLE].*

5.6 The Reconstruction Problem

We now return to the Reconstruction Problem as stated in §5.1, utilising the analysis contained in §§5.2-5.4. The Reconstruction Problem for the ADSS was stated as follows:

Given a smooth curve segment D, construct a smooth curve γ which has D as its ADSS.

Suppose we are given a smooth curve segment γ_1. By 'degree of freedom' arguments, there will generically be a smooth 1-parameter family of conics centred on D having 4-point contact with γ_1. Then, if we choose a point γ_2^0 in the plane (away from γ_1 and D) and a correspondence between γ_2^0 and a fixed point γ_1^0 of γ_1, then there will be an unique curve segment γ_2 passing through γ_2^0 such that $\gamma_1 \cup \gamma_2$ has D as its ADSS *and* such that the points γ_1^0 and γ_2^0 together give a point of the ADSS (that is, γ_1^0 and γ_2^0 *'correspond'*). The choice of correspondence between γ_1^0 and γ_2^0 is equivalent to choosing the direction of the affine normal at γ_2^0, or alternatively choosing the affine tangent at γ_2^0. These two choices represent the two parameters that appeared in the SRP.

Proposition 5.6.1. *Given a smooth curve segment D, choose any smooth curve γ_1, any point γ_2^0 away from γ_1 and D, and a correspondence between γ_2^0 and a point γ_1^0 of γ_1. Then there is an unique curve γ containing γ_1 and γ_2^0 having D as its ADSS* and *such that γ_1^0 and γ_2^0 correspond.*

Chapter 6

Affine-Invariant Symmetry Sets for a Union of Ellipses

6.1 Introduction

In [GBan93], a study of the Euclidean Symmetry Set was made for *piecewise-circular curves*, that is, for curves comprising segments of circles. The motivation for this study relates to the practicalities of plotting the symmetry sets on a computer. For a computer to be able to *'capture'* a curve, it must often approximate it using some simpler curves, for example line segments, circles or conics, and a small number of circular arcs can approximate an Euclidean curve to an acceptably high degree (see also [BanG87], [BanG94]). In this way, an approximation to an Euclidean plane curve is constructed by splining together segments of circles (or, exceptionally, straight lines) in such a way that the curve is sufficiently smooth at the joins. (In fact, we simply insist that the tangent line to the curve turns continuously.)

To carry out an analogous study for *affine-invariant* symmetry sets, we should consider curves which are constructed by splining together segments of conics. In this chapter, we will take the first step towards the study of the structure of the ADSS and the AESS for *piecewise-conic curves*, that is, for closed curves comprising segments of conics[1], joined in such a way that the resulting affine plane curve is sufficiently smooth. In practice, it may be

[1] Thank you to Prof.A.M.Bruckstein for originally suggesting this interesting problem.

possible to aproximate an affine plane curve to an acceptably high degree using ellipses only. However, in this thesis we will not consider piecewise-conic curves, or even piecewise-elliptical curves, but instead will consider the ADSS and AESS for a curve which is the union of *two whole ellipses.* The analysis of the possible structures on the ADSS and AESS of this composite curve will be the first small step towards studying the structure of the ADSS and AESS for truly piecewise-conic curves.

Outline of Chapter 6

§**6.2:** We begin our study of the ADSS and AESS of two complete ellipses E and U. We set up a suitable coordinate system, and begin the analysis common to both the ADSS and AESS cases.

§**6.3:** The analysis of §6.2 then diverges. We first of all consider the ADSS. In §6.3.1, we study the nature of the singularities of the ADSS of $E \cup U$, studying at the Morse singularities of the *pre-ADSS* and relating these phenomena on the pre-ADSS to the crossings and isolated points of the ADSS. We will see that, under certain conditions, the pre-ADSS of $E \cup U$ can contain horizontal or vertical line components, and this leads naturally to the *Collapse-Point Curve* of §6.3.4, which aids our understanding of the singularities of the ADSS of $E \cup U$.

§**6.4:** Here we consider the AESS of $E \cup U$ in the same set-up. We find the condition for the AESS to be an empty set, and interpret this geometrical condition with reference to the corresponding MPTL of $E \cup U$. In §6.4.3, we study in some detail the *'birth, death, marriage and divorce'* of components of the AESS, that is, how segments of AESS may be created or destroyed, join together or split apart. This leads to the *Singular-Point Curve*, which helps us to understand the singularities of the AESS of $E \cup U$.

§**6.5:** We conclude with a short discussion of possible further research.

6.2 The ADSS and AESS for a union of ellipses

Here we construct the coordinate system which we will use for the remainder of this chapter. This coordinate system will be common to the analysis of both the ADSS and the AESS. We will consider two complete ellipses in the affine plane, and attempt to construct the ADSS and AESS between them.

By an affine transformation, we are able to scale so that one of the ellipses is the unit circle, and make a rigid translation and rotation of the plane so that the other ellipse is centred at the origin, with axes along the x, y directions. No further simplification of this set-up can be made. We will denote the unit circle by U, and call the other ellipse E. Let us suppose that E has major and minor axes of length a and b respectively (so $a, b > 0$), and assume without loss of generality that $a > b$ (see Figure 6.1). We will always denote the centre of U by the coordinates (c, d). E and U may intersect. We

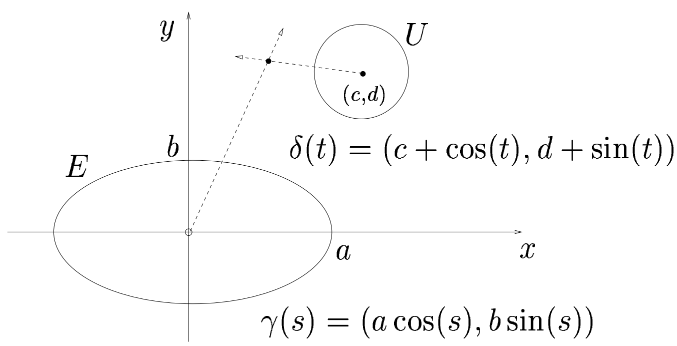


Figure 6.1: *The coordinate system set up in* §6.2. *Two affine normals to E and U are shown dashed.*

take E and U to be parametrised curves $\gamma(s)$ and $\delta(t)$ respectively, given by

$$\gamma(s) = (a \cos s, b \sin s),$$
$$\delta(t) = (c + \cos t, d + \sin t).$$

With $'$ (prime) denoting derivative w.r.t. affine-arclength, and $\dot{}$ (dot) denoting derivative w.r.t. the corresponding parameter s or t, we have

$$\begin{aligned}\dot{\gamma}(s) &= (-a\sin s, b\cos s),\\ \dot{\delta}(t) &= (-\sin t, \cos t),\\ \ddot{\gamma}(s) &= (-a\cos s, -b\sin s),\\ \ddot{\delta}(t) &= (-\cos t, -\sin t).\end{aligned}$$

Now using the identities of §1.3.3, we see that

$$\begin{aligned}\gamma'(s) &= (ab)^{-1/3}(-a\sin s, b\cos s),\\ \delta'(t) &= (-\sin t, \cos t).\end{aligned}$$

The analyses of the ADSS and AESS of $E \cup U$ now diverge. In §6.3, we continue the analysis of the ADSS of $E \cup U$, and in §6.4 we continue the corresponding analysis of the AESS of $E \cup U$.

6.3 The ADSS for $E \cup U$

We note first of all that E and U both contribute a single point, namely their respective centres, to the ADSS of $E \cup U$, since we can consider each of E and U to be a (repeated) conic having (at least) 4-point contact with $E \cup U$. Apart from these two points, the ADSS of $E \cup U$ will consist of points which are the common centre of two distinct conics with the same affine radius, one of which has (at least) 4-point contact with E, and the other having (at least) 4-point contact with U. Recall that we call such a pair of conics a *'4+4 conic pair'* (see §3.2.5).

We begin by deriving an equation for the ADSS of $E \cup U$. By the definition of the ADSS (see Definition 3.2.1), a point $(x, y) \in \mathbb{R}^2$ lies on the ADSS, corresponding to curve points $\gamma(s), \delta(t)$, if and only if (x, y) lies on both of the affine normals to γ, δ at $\gamma(s), \delta(t)$ respectively, and the affine distances from (x, y) to curve segments γ, δ at $\gamma(s), \delta(t)$ are equal. The affine normals to E and U at $\gamma(s)$ and $\delta(t)$ are in the direction of the radii of E and U at

$\gamma(s)$ and $\delta(t)$ respectively (see Figure 6.1). This gives us

$$(x, y) = (\lambda a \cos s, \lambda b \sin s) = (c + \mu \cos t, d + \mu \sin t). \tag{6.1}$$

Furthermore, the equal affine distances condition tells us that

$$[(x, y) - \gamma(s), \gamma'(s)] = [(x, y) - \delta(t), \delta'(t)],$$

which is

$$(ab)^{-1/3}(xb \cos s + ya \sin s - ab) = ((x - c) \cos t + (y - d) \sin t - 1). \tag{6.2}$$

Substituting $x = \lambda a \cos s, y = \lambda b \sin s$ into the left-hand side of expression (6.2), and $x = c + \mu \cos t$, $y = d + \mu \sin t$ into the right-hand side of (6.2), gives us

$$(ab)^{2/3}(\lambda - 1) = \mu - 1. \tag{6.3}$$

We also have the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2, \tag{6.4}$$

$$(x - c)^2 + (y - d)^2 = \mu^2. \tag{6.5}$$

Expressions (6.3), (6.4), and (6.5) define the ADSS, and together give us:

Proposition 6.3.1. *The equation of the ADSS of $E \cup U$ is*

$$(x - c)^2 + (y - d)^2 = \left((ab)^{2/3} \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} - ((ab)^{2/3} - 1) \right)^2.$$

When rationalised, it is algebraic *and of* degree 4 *in x, y.*

We are thus able to plot this ADSS for any values of a and b. In general, this is a smooth curve in two disjoint parts. Figure 6.2 shows a plot of the ADSS, shown as the thickest curve, and consisting of two smooth, closed curves, one outside and one inside E. Note that U is also entirely within E, which can only happen if we choose $a, b > 1$. As soon as U is allowed to intersect E, we should expect some more complicated behaviour near the intersection points. The ADSS can also be seen to exhibit crossings and isolated points for certain values of a, b. In §6.3.1 we explore these phenomena in detail.

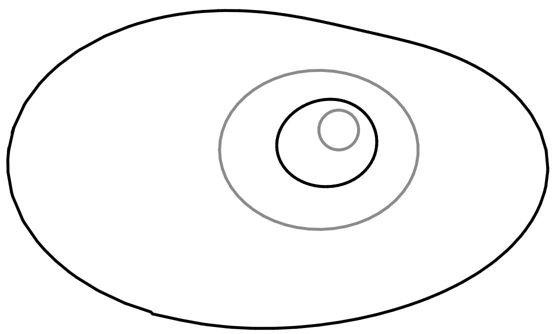


Figure 6.2: *The ellipse E centred at the origin and the unit circle U centred at $(c, d) = (1, 1)$ are shown as grey curves. The ADSS for $E \cup U$ is shown as the two darkest closed curves.*

6.3.1 Singularities of the ADSS of $E \cup U$

There are two possible ways to approach the study of the singularities of the ADSS of $E \cup U$. The first is to try to use the fact that the ADSS of a *generic* plane curve has a singularity (generically a cusp) when there exist two conics sharing the same centre and affine radius, having respectively 5- and 4-point contact with the curve (so-called *5+4-conic pairs*: see Theorem 3.2.4 for a statement of this result). However, we are not considering the ADSS of a *generic* plane curve, but instead the ADSS of a curve comprising two ellipses, E and U, and hence we must be very careful when attempting to use this result. Since five points determine a conic uniquely, for a conic to have 5-point contact with the curve $E \cup U$, it must be *identical* to one of E or U, and thus have '∞'-point contact with the corresponding conic. It is not possible for there to exist a conic pair having *exactly* 5+4 contact with $E \cup U$, and this confuses our interpretation of the resulting structure of the

ADSS of $E \cup U$. We will instead use the second approach, which involves studying the structure of the *pre-ADSS*, and relating it to the ADSS itself. This approach is a lot clearer, and can then be related to the above ideas about 5+4 conic pairs in a more transparent way.

6.3.2 Morse singularities of the pre-ADSS

Following on from §6.3, suppose we have ellipse E parametrised as $\gamma(s)$, and unit circle U parametrised as $\delta(t)$. Now we note that t is the affine-arclength parameter along δ, but that s is *not* the affine-arclength parameter along γ, since

$$\left[\frac{d}{ds}\left(\gamma(s)\right), \frac{d^2}{ds^2}\left(\gamma(s)\right)\right] = ab \neq 1 \text{ in general.}$$

However, if we reparametrise γ as

$$\gamma(s) = \left(a\cos\left(\frac{s}{(ab)^{1/3}}\right), b\sin\left(\frac{s}{(ab)^{1/3}}\right)\right),$$

then s is the affine-arclength parameter along γ. (See §1.3.1 for details. This parameterisation was used previously in the proof of Lemma 1.3.3.) Now the pre-ADSS is a subset of (s,t)-space defined by solutions (s,t) to the equation

$$[\gamma(s) - \delta(t), \gamma''(s) - \delta''(t)] = 0, \tag{6.6}$$

where $'$ (prime) denotes derivative w.r.t. the corresponding affine-arclength parameter along each of γ and δ. By Corollary 3.2.3, we can write

$$\gamma(s) - \delta(t) = -d_0(\delta''(t) - \gamma''(s)), \tag{6.7}$$

where d_0 is the common affine distance from ADSS point $\mathbf{x}$ to the curves through $\gamma(s)$ and $\delta(t)$.

There is a Morse singularity (that is, a *crossing* or an *isolated point*) on the pre-ADSS if and only if

$$\frac{\partial}{\partial s}\{(6.6)\} = \frac{\partial}{\partial t}\{(6.6)\} = 0,$$

for some s, t. This is equivalent to there existing s, t for which

$$\left.\begin{aligned} [\gamma'(s), \gamma''(s) - \delta''(t)] + [\gamma(s) - \delta(t), \gamma'''(s)] = 0, \\ [-\delta'(t), \gamma''(s) - \delta''(t)] + [\gamma(s) - \delta(t), -\delta'''(t)] = 0, \end{aligned}\right\} \tag{6.8}$$

From now on we will omit the parameters s and t. Now we also have, from §1.3.3, the identities $\gamma''' = -\mu_\gamma \gamma'$ and $\delta''' = -\mu_\delta \delta'$, where $\mu_\gamma \equiv [\gamma'', \gamma''']$ and $\mu_\delta \equiv [\delta'', \delta''']$ are the affine curvatures of γ and δ respectively. Since γ and δ are conics, we know that their affine curvatures are constant, and calculation shows that $\mu_\gamma \equiv (ab)^{-2/3}$ and $\mu_\delta \equiv 1$. Then we can rewrite (6.8) as

$$\left.\begin{aligned} [\gamma', \gamma - \delta](\mu_\gamma + \tfrac{1}{d_0}) = 0, \\ [\gamma - \delta, \delta'](\mu_\delta + \tfrac{1}{d_0}) = 0, \end{aligned}\right\} \tag{6.9}$$

So the pre-ADSS has a Morse singularity at (s, t) if and only if (6.6) and (6.9) hold. This can be split into three cases, and we find the pre-ADSS and ADSS for each case in turn.

Case (i): $[\gamma - \delta, \gamma'' - \delta''] = 0,$ **and** $\mu_\gamma = \mu_\delta = -\frac{1}{d_0}$

The ADSS point $\mathbf{x}$ can be written as

$$\mathbf{x} = \gamma + \frac{1}{\mu_\gamma}\gamma'' = \delta + \frac{1}{\mu_\delta}\delta'',$$

and thus lies at the centre of two *5*-point contact conics, since the affine radius of each of the conics is equal to the radius of affine curvature of γ and δ. The two 5-point contact conics must be identical to γ and δ, since they share a centre and have the same affine radius, which means that $\gamma \equiv \delta$. We conclude that, apart from this very degenerate case, there are no 5+5 conic pairs, and thus no corresponding Morse singularities on the pre-ADSS.

The ADSS in this case is the *isolated point* at the common centre of the 5+5 conic pair.

Case (ii)(a): $[\gamma - \delta, \gamma'' - \delta''] = 0, [\gamma', \gamma - \delta] = 0,$ **and** $\mu_\delta = -\frac{1}{d_0}$

The fact that $d_0 \equiv -1/\mu_\delta \equiv -1$ implies that one of the conics in the 4+4 conic pair has *5*-point contact with δ, and therefore is identical to δ (that is, identical to U). Then the ADSS point $\mathbf{x}$ lies at the centre of U. Thus, for some fixed s_0 and *any* t we have a *4+5* conic pair, and hence the whole *vertical line* is included in the pre-ADSS.

The parameter s_0 is fixed for a given a, b and U, having to fulfill the following conditions:

(i) s_0 must correspond to a point of γ where the affine normal passes through $\mathbf{x}$. There are precisely two affine normals to γ which pass through $\mathbf{x}$, and thus two possible values of s_0, as shown in Figure 6.3;

(ii) s_0 must also correspond to a point of γ at affine distance $d_0 \equiv -1$ from $\mathbf{x}$. This rules out one of the possible values, depending on the values of a and b and the position of $\mathbf{x}$;

(iii) finally, s_0 must correspond to a point of γ for which there exists real t such that

$$[\gamma'(s_0), \gamma(s_0) - \delta(t)] = 0.$$

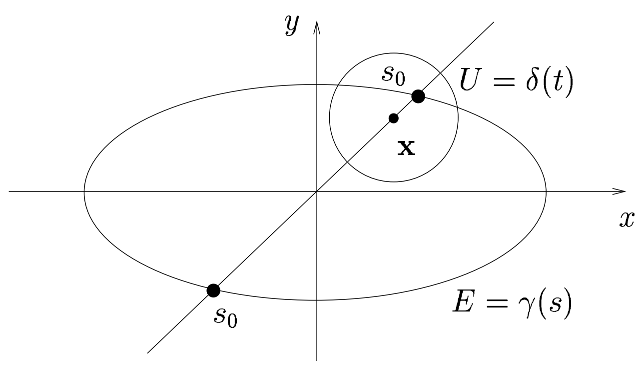


Figure 6.3: *The two possible s_0 lie on the common affine normal to both γ and δ through the point* $\mathbf{x}$.

Consider the following two situations:

1. **Figure 6.4:** Parameter s_0 such that (i) and (ii) hold is shown in Figure 6.4(a). We can see that in this case there exist two distinct pa-

rameter values $t = \{t_1, t_2\}$ for which (iii) holds, namely the parameter values of points of δ lying on the tangent line to γ at $\gamma(s_0)$. The pre-ADSS in this case is as shown in Figure 6.4(b), and consists of the vertical line (s_0, t) along with two smooth branches crossing it at points (s_0, t_1) and (s_0, t_2).

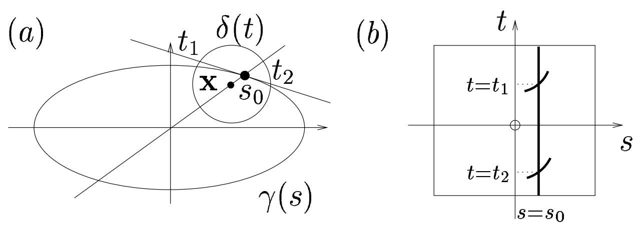


Figure 6.4: *(a) The unique parameter value s_0 such that (i) and (ii) hold is shown. We can see that there then exist two distinct $t = \{t_1, t_2\}$ such that (iii) holds. (b) The pre-ADSS in this case consists of the vertical line (s_0, t) along with two branches crossing it transversally at points (s_0, t_1) and (s_0, t_2).*

2. Figure 6.5: Parameter s_0 such that (i) and (ii) hold is shown in Figure 6.5(a). However, in this case, there exist no *real* parameter values t for which (iii) holds. The pre-ADSS is shown in Figure 6.5(b): it consists only of the vertical line (s_0, t), with no extra branches crossing it.

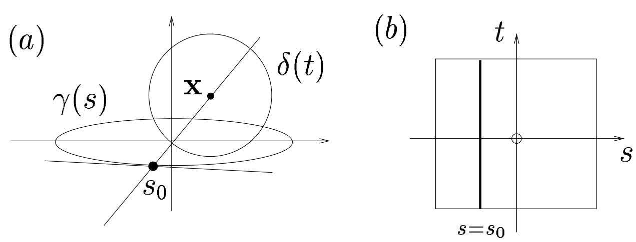


Figure 6.5: *(a) The unique parameter value s_0 such that (i) and (ii) hold is shown. We can see that there then exist no parameter values t for which (iii) holds, since the tangent line to γ at $\gamma(s_0)$ does not intersect δ. (b) The pre-ADSS in this case consists of the vertical line (s_0, t).*

Thus only case 1 leads to a Morse singularity on the pre-ADSS, in this case a pair of smooth branches crossing a vertical line. However, case 2 *is* still interesting to us, being as it also corresponds to a vertical line on the pre-ADSS: we return to these vertical lines in §6.3.3, when we define *'Collapse-Points'* on the ADSS, and in §6.3.4, when we consider the *'Collapse-Point Curve'*.

The structure of the ADSS in each of the two situations is as follows:

1: The vertical line (s_0, t) is mapped to a *single point* on the ADSS, namely the point $\mathbf{x}$ at the centre of the 5-point contact conic U. However, this is *not* an *isolated* point on the ADSS, since the two smooth branches of the pre-ADSS that cross the vertical line are mapped to two smooth branches of the ADSS passing through $\mathbf{x}$. Thus, in this case the ADSS is *two smooth branches* crossing transversally through the centre of U.

2: Again, the vertical line is mapped to the single point $\mathbf{x}$ on the ADSS at the centre of U. No other branches of the pre-ADSS cross the vertical line, and thus no other branches of the ADSS pass through $\mathbf{x}$. The ADSS in this case is an *isolated point.*

Case (ii)(b): $[\gamma - \delta, \gamma'' - \delta''] = 0, [\delta', \gamma - \delta] = 0,$ **and** $\mu_\gamma = -\frac{1}{d_0}$

The fact that $d_0 = -1/\mu_\gamma = -(ab)^{2/3}$ implies that one of the conics of the 4+4 pair has *5*-point contact with γ, and is therefore identical to γ (that is, identical to E). Then the ADSS point $\mathbf{x}$ lies at the centre of E. Thus, for some fixed t_0 and *any* s we have a *5+4* conic pair, and hence the whole *horizontal line* (s, t_0) is included in the pre-ADSS.

By symmetry, the analysis of this case mirrors that of Case (ii)(a). We consider these horizontal lines in §6.3.3.

Case (iii): $[\gamma - \delta, \gamma'' - \delta''] = 0,$ **and** $[\gamma', \gamma - \delta] = [\delta', \gamma - \delta] = 0$

Geometrically, this is the situation where γ', δ' and $\gamma - \delta$ are parallel, the *double tangent* situation. Then $\gamma' = \lambda\delta'$ for some $\lambda \in \mathbb{R}$, and the conditions above give us

$$\begin{aligned} & [\gamma' - \delta', \gamma'' - \delta''] = 0, \\ \iff\ & 2 - [\delta', \gamma''] - [\gamma', \delta''] = 0, \text{ using } [\gamma', \gamma''] = [\delta', \delta''] = 1, \\ \iff\ & 2 - \lambda - \frac{1}{\lambda} = 0, \end{aligned}$$

using $\gamma' = \lambda\delta'$ and $[\gamma', \gamma''] = [\delta', \delta''] = 1$, and this holds if and only if $\lambda = 1$. Thus the affine tangents to γ and δ at $\gamma(s)$ and $\delta(t)$ must be *identical* for these points to contribute to the ADSS in this case. (Note that we came to a similar conclusion in §3.2.2.) Since U is the unit circle, we deduce that the affine tangent to γ at such a point $\gamma(s)$ must be unit. Now suppose we find a point $\gamma(s)$ on γ having unit affine tangent (see Figure 6.6). Then, for any position of unit circle U touching the tangent line to γ at $\gamma(s)$, such that the corresponding affine tangent δ' is in the same direction as $\gamma'(s)$, the conditions above are satisfied and the pre-ADSS exhibits a Morse singularity. Thus, for any (c, d) lying on a line parallel (at Euclidean distance 1) to a tangent line to γ at a point where γ has unit affine tangent (and for which γ' and δ' are in the same direction), we will have a Morse singularity on the pre-ADSS.

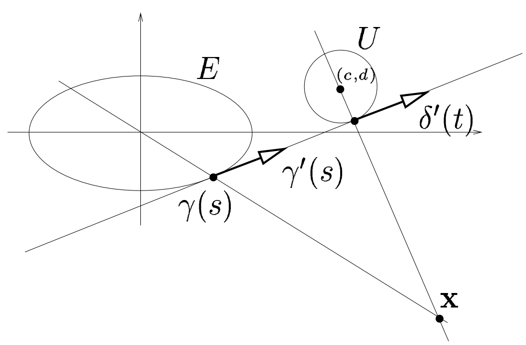


Figure 6.6: *Unit circle U is tangent to the tangent line to γ at $\gamma(s)$, and the affine tangents $\gamma'(s)$ and $\gamma'(t)$ are identical.*

Of course, it is possible that there exists no point of γ with unit affine tangent. Since such a point will have Euclidean curvature of 1, then the necessary and sufficient condition that there exists such a point is that the value 1 lies between the maximum and minimum values of the Euclidean curvature κ_γ of γ, that is,

$$\min(\kappa_\gamma) = \frac{b}{a^2} \leq 1 \leq \frac{a}{b^2} = \max(\kappa_\gamma),$$

since κ_γ is continuous and attains its bounds at its (Euclidean) vertices. Since we are assuming $a > b > 0$, the necessary and sufficient condition for there to exist a point of γ having unit affine tangent is

$$ab < 1 \text{ and } a^2 \geq b, \quad \text{or } ab > 1 \text{ and } a \geq b^2.$$

Remark 6.3.2. *By symmetry arguments, γ will generically have zero or four points with unit affine tangent, and two such points in the special case when the Euclidean curvature of γ at a vertex is* 1.

The pre-ADSS in this situation will have an isolated point if and only if the expression

$$\frac{\partial^2}{\partial s^2}\{(6.6)\}\frac{\partial^2}{\partial t^2}\{(6.6)\} - \left(\frac{\partial^2}{\partial s \partial t}\{(6.6)\}\right)^2 \tag{6.10}$$

is positive. Differentiating the expressions in (6.9) we get

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\{(6.6)\} &= [\gamma'', \gamma - \delta]\left(\mu_\gamma + \frac{1}{d_0}\right), \text{ since } \mu'_\gamma = 0 \\
\frac{\partial^2}{\partial t^2}\{(6.6)\} &= -[\delta'', \gamma - \delta]\left(\mu_\delta + \frac{1}{d_0}\right), \text{ since } \mu'_\delta = 0 \\
\frac{\partial^2}{\partial s \partial t}\{(6.6)\} &= [\gamma', \delta'](\mu_\gamma - \mu_\delta) = 0, \text{ since } [\gamma', \delta'] = 0.
\end{aligned}$$

Then expression (6.10) is

$$-[\gamma'', \gamma - \delta][\delta'', \gamma - \delta]\left(\mu_\gamma + \frac{1}{d_0}\right)\left(\mu_\delta + \frac{1}{d_0}\right),$$

which becomes

$$-d_0^2[\gamma'', \delta'']^2\left(\mu_\gamma + \frac{1}{d_0}\right)\left(\mu_\delta + \frac{1}{d_0}\right),$$

using $\gamma - \delta = d_0(\gamma'' - \delta'')$. Thus the pre-ADSS exhibits an isolated point if and only if

$$\left(\mu_\gamma + \frac{1}{d_0}\right)\left(\mu_\delta + \frac{1}{d_0}\right) < 0.$$

Geometrically, this corresponds to the common radius of the 4+4 conic pair (which is equal to $-1/d_0$) lying between μ_γ and μ_δ, that is, there is an isolated point on the pre-ADSS if

- $ab > 1$ and
$$(ab)^{-2/3} < -\frac{1}{d_0} < 1,$$
- $ab < 1$ and
$$1 < -\frac{1}{d_0} < (ab)^{-2/3}.$$

Summary

In Cases (ii)(a) and (ii)(b), we are able to relate the occurrence of vertical and horizontal line components of the pre-ADSS to the existence of 5+4 conic pairs (which in fact are ∞+4 conic pairs), and in this way we are able to link the method of studying the singularities of the ADSS via the singularities of the pre-ADSS to the other method mentioned in §6.3.1, which was to apply the generic result that $5+4$ conic pairs lead to singularities of the ADSS. In these cases, we see that it is possible for smooth branches of the pre-ADSS to cross these vertical and horizontal lines, and this leads to crossings on the ADSS. Thus we conclude that these cases correspond to a combination of 'isolated' points and 'crossings' on the ADSS that may occur at the same point. In §6.3.3 we aim to understand these situations more fully.

Cases (i) and (iii) concern 'genuine' isolated points and crossings on the ADSS. Case (i) is trivial, since it only occurs when E and U share the same centre. Case (iii) is more interesting: under certain conditions (that is, for certain values of a and b) it will not occur; otherwise, it leads to a set of four lines for which, as the centre of U crosses one of the lines, the pre-ADSS undergoes an isolated point or a crossing transition, as does the ADSS. We consider this situation again in §6.3.3.

6.3.3 Collapse-Points

The study of the Morse singularities on the pre-ADSS in §6.3.2 suggests a suitable approach to the study of the singularities of the ADSS for $E \cup U$. Referring to Cases (ii)(a) and (ii)(b), we saw that horizontal and vertical lines on the pre-ADSS are mapped to single points of the ADSS, that point being the centre of the corresponding 5-point contact conic, and these 5-point contact conics arise when the common affine distance from the ADSS point to the corresponding points of the ellipse and the circle is equal to the radius of curvature of either the ellipse or the circle. However, we will avoid the temptation to label the image of these horizontal and vertical lines as *'isolated points'*. It is true that the whole line is mapped to a single point, at the centre of the corresponding 5-point contact conic, but other branches of the ADSS may pass through the point, as we have seen in Case (ii)(a) of §6.3.1. Instead, we will label the image of horizontal or vertical line components of the pre-ADSS as *'Collapse-Points'*, since they arise, not from isolated points of the pre-ADSS, but from horizontal or vertical lines of the pre-ADSS which *collapse* to a single point of the ADSS.

Definition 6.3.3. *A* **Collapse-Point** *on the ADSS of $E \cup U$ is the image of a horizontal or vertical line component of the pre-ADSS of $E \cup U$.*

Thus 5+4 (or 4+5) conic pairs lead to horizontal or vertical lines on the pre-ADSS, which are mapped to *Collapse-Points* of the ADSS.

6.3.4 The *Collapse-Point Curve*

Collapse-Points signal situations in which segments of the AESS are created or destroyed, join together or split apart. We now consider the question: *When do Collapse-Points occur on the ADSS?* We will consider the set of centres (c, d) which lead to a Collapse-Point on the ADSS.

Definition 6.3.4. *The* **Collapse-Point Curve** (CPC) *of $E \cup U$ is the locus of points in (c, d)-space which, as centres of the unit circle U, lead to Collapse-Points on the ADSS.*

To rephrase this, the CPC is *the locus of points (c, d) for which the pre-ADSS exhibits a horizontal or vertical line component.* We will consider in turn the contribution of the vertical and horizontal line components of the pre-ADSS to the CPC.

Vertical line components of the pre-ADSS

From Case(ii)(a) of §6.3.1, we know that vertical line components of the pre-ADSS occur when the ADSS Condition holds *and* the common affine distance from the corresponding ADSS point to γ and δ is equal to $-1/\mu_\delta$, which is fixed at -1. Thus we deduce that the component of the CPC corresponding to vertical line components of the pre-ADSS is the affine parallel to E at affine distance -1, since this is the locus of points at affine distance -1 from E along each affine normal to E. Calculation shows that this affine parallel in (c, d)-space is given by the equation

$$\frac{c^2}{a^2} + \frac{d^2}{b^2} = \left(1 - (ab)^{-2/3}\right)^2, \tag{6.11}$$

which is generally an ellipse, as expected. Thus if the centre (c, d) of U is chosen such that (6.11) holds, then the pre-ADSS of $E \cup U$ exhibits a vertical line component, and the ADSS of $E \cup U$ exhibits a Collapse-Point at the centre of U. Figure 6.7 contains [LSMP] plots illustrating this component of the CPC, along with the pre-ADSS in this case where the vertical line components can clearly be seen.

Horizontal line components of the pre-ADSS

Similarly, from Case(ii)(b) of §6.3.1, we know that horizontal line components of the pre-ADSS occur when the ADSS Condition holds *and* the common affine distance from the corresponding ADSS point to γ and δ is equal to $-1/\mu_\gamma$, which is fixed at $-(ab)^{2/3}$. In this case, we find that the component of the CPC corresponding to horizontal line segments on the pre-ADSS is the circle with radius $(ab)^{2/3} - 1$ centred at the origin, and is given by

$$c^2 + d^2 = ((ab)^{2/3} - 1)^2. \tag{6.12}$$

Thus if the centre (c, d) of U is chosen such that (6.12) holds, then the pre-ADSS of $E \cup U$ contains a horizontal line component, and the ADSS of $E \cup U$

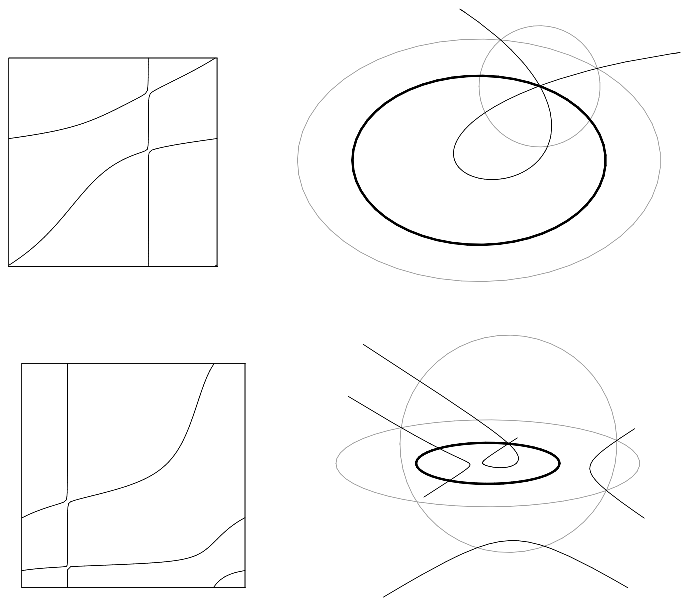


Figure 6.7: *The pre-ADSS's are on the left. The CPC corresponding to vertical lines on the pre-ADSS is shown as the darkest curve, the ADSS itself the thinner dark curve, and the ellipses are shown grey. The Collapse-Point is obscured 'beneath' a crossing on the ADSS. (Note that the ADSS has been truncated in both of these plots.)*

exhibits a Collapse-Point at the centre of E. Figure 6.8 contains [LSMP] plots illustrating this component of the CPC, along with the pre-ADSS in this case where the horizontal line components can clearly be seen.

Taking expressions (6.11) and (6.12) together we thus have:

Proposition 6.3.5. *The* **Collapse-Point Curve** *in* (c,d)*-space consists of two curves, which we call CPC1,CPC2 given by*

$$\begin{aligned} \mathit{CPC1} &: \frac{c^2}{a^2}+\frac{d^2}{b^2}=\left(1-(ab)^{-2/3}\right)^2, \\ \mathit{CPC2} &: c^2+d^2=((ab)^{2/3}-1)^2. \end{aligned}$$

We can now easily plot the CPC for any given a, b. Figure 6.9 shows an example.

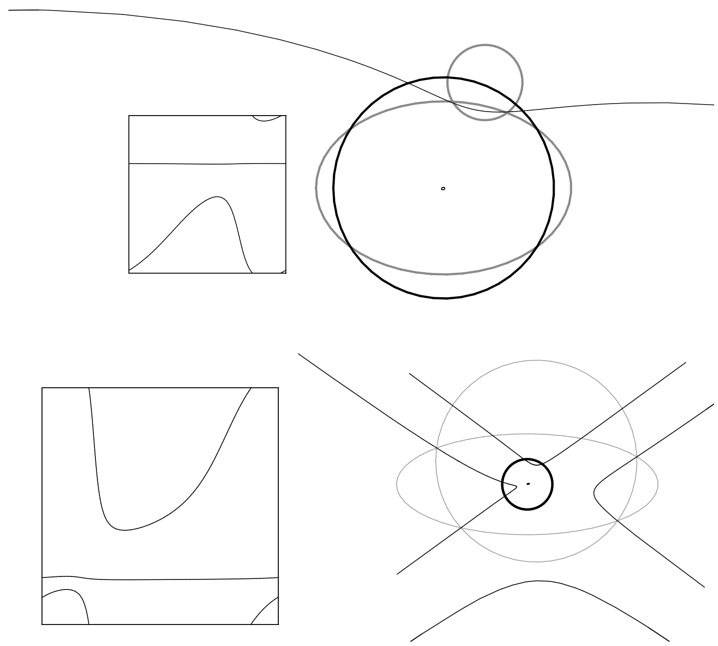


Figure 6.8: *On the left are the pre-ADSS's, the horizontal lines clearly visible. On the right, the corresponding plots of the ADSS (thin dark curve), along with ellipse E and circle U (grey curves) and the CPC corresponding to the horizontal line components of the pre-ADSS. The Collapse-Points themselves are visible at the centres of E.*

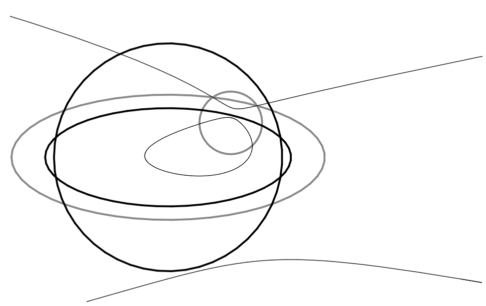


Figure 6.9: *The thicker dark curves are the two components of the CPC for* $a = 5, b = 2, c = 2, d = 1.1$. *The thinner dark curve is the (truncated) ADSS.*

Other Morse singularities on the pre-ADSS

We now briefly consider what we might term 'genuine' Morse singularites on the pre-ADSS, as opposed to those considered above which correspond to smooth branches crossing a vertical or horizontal line component of the pre-ADSS. In Case (iii) of the analysis of the pre-ADSS in §6.3.1, we saw that the pre-ADSS of $E \cup U$ exhibits a Morse singularity when there exists a line tangent to both E and U with the further condition that the affine tangent to E at the point of contact of this double tangent is unit and in the same direction as the affine tangent to U at the corresponding point of contact. It follows that if we can find a unit affine tangent to E, then any position of unit circle U tangent to the tangent line at this point of E (such that the affine tangents at the points of contact are in the same direction) will lead to a Morse singularity on the pre-ADSS.

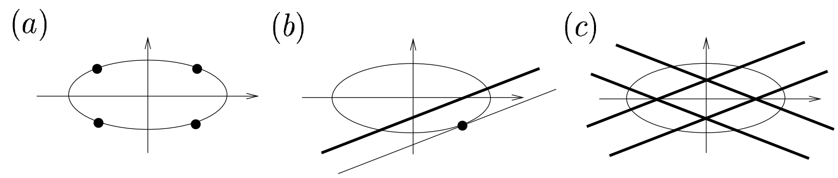


Figure 6.10: *(a) Four points of E have Euclidean curvature equal to unity, and thus unit affine tangents. (b) Consider one of these points. One of the parallels to the tangent line to E at distance* 1 *is shown. When the centre of U lies on this line, the ADSS of $E \cup U$ exhibits a Morse singularity. (c) All such lines described in (b).*

In this way, we are able to locate Morse singularities on the pre-ADSS, and thus on the ADSS itself: first, find all points of γ with unit affine tangent (that is, points of γ where the Euclidean curvature is equal to 1) – see Figure 6.10(a); draw the tangent lines to γ at these points – see Figure 6.10(b); draw the Euclidean parallel at distance 1 to each of these lines, on the same side of the line as E – see Figure 6.10(c). These four lines contain the set of points which, as centres (c, d) of unit circle U lead to a Morse singularity on the pre-ADSS, and thus the corresponding Morse singularity on the ADSS. See Figures 6.11 and 6.12 for [LSMP] plots of this situation.

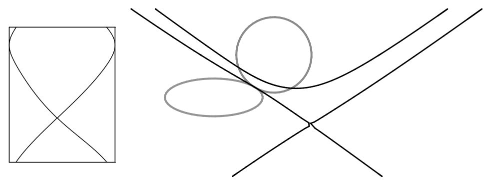


Figure 6.11: *The dark curve is the (truncated) ADSS for the ellipse and circle, shown in grey, for* $a = 1.3, b = 0.5, c = 1.6, d = 1.126$ *(so* $ab < 1, a^2 > b$*). On the left is the corresponding pre-ADSS.*

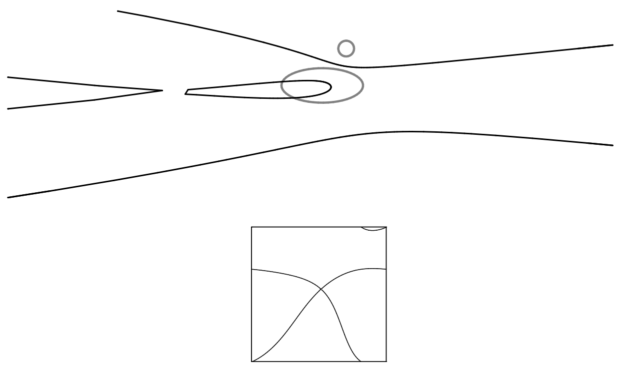


Figure 6.12: *(above) The dark curve is the (truncated) ADSS for the ellipse and circle, shown in grey, for* $a = 5.2, b = 2.2, c = 3.0491, d = 4.7$ *(so* $ab > 1, a > b^2$*). (below) The corresponding pre-ADSS.*

6.4 The AESS (and MPTL) for a union of ellipses

We now turn to consider the AESS $\cup$ MPTL for a curve comprising two complete ellipses. We use the same coordinate system as set out in §6.2, illustrated in Figure 6.1. We will assume throughout, without loss of generality, that $0 < b < a$.

The analysis of this situation began in §6.2, since it follows the same line as the analogous study of the ADSS. We now continue this analysis.

6.4.1 When is the AESS of $E \cup U$ non-empty?

As in the case of the ADSS for $E \cup U$, we begin by noting that E and U both contribute a single point, namely their respective centres, to the AESS of $E \cup U$, since we can consider each of E and U to be a conic having (at least) 3-point contact with $E \cup U$. Apart from these two points, the AESS of $E \cup U$ will consist of points which are centres of a conic having 3-point contact with E and 3-point contact with U. We would like to know when this component of the AESS of the composite curve $E \cup U$ (which, following §6.2, comprises the two parametrized curves $\gamma(s), \delta(t)$) is *non-empty* (that is, has real points). This is equivalent to finding the condition for

$$[\gamma(s) - \delta(t), \gamma'(s) + \delta'(t)] = 0, \tag{6.13}$$

to have *any* real solutions for s, t (this comes from the AESS Condition of Proposition 2.2.2). If we can show that there exist real (s, t) for which (6.13) holds, then the non-trivial AESS corresponding to these two curves is non-empty. Substituting $\gamma(s)$ and $\delta(t)$ into expression (6.13) we get

$$\begin{vmatrix} a\cos s - c - \cos t & -a(ab)^{-1/3}\sin s - \sin t \\ b\sin s - d - \sin t & b(ab)^{-1/3}\cos s + \cos t \end{vmatrix} = 0.$$

Expanding this expression and rearranging leads to the following:

Proposition 6.4.1. *The* necessary and sufficient *condition for the non-trivial component of the AESS of the curve $E \cup U$ (i.e. $\gamma \cup \delta$) to be non-empty is that*

$$(ab)^{2/3} - 1 + \left(a - \left(\frac{b^2}{a}\right)^{1/3}\right)\cos s\cos t + \left(b - \left(\frac{a^2}{b}\right)^{1/3}\right)\sin s\sin t$$
$$- \left(\frac{b^2}{a}\right)^{1/3} c\cos s - \left(\frac{a^2}{b}\right)^{1/3} d\sin s - c\cos t - d\sin t = 0, \tag{6.14}$$

has real *solutions for s, t.*

From now on we will disregard the two trivial components of the AESS of $E \cup U$, and refer simply to the non-trivial component of as 'the AESS' of $E \cup U$. The aim is then to derive conditions on parameters a, b, c, d for there to exist real solutions s, t to (6.14). In §6.4.2, we will find a condition on a and b under which there *always* exists solutions for s and t, regardless of the values of c and d. This condition will necessitate the introduction of the MPTL. We then suppose that this condition is not satisfied, and deduce a further condition on c and d for solutions s and t to exist. This leads to what we will call the *'Singular-Point Curve'* for the AESS, which we will define (see Definition 6.4.4) to be the set of points in (c, d)-space which, as centres of the unit circle U, lead to an *isolated point* or a *crossing* on the AESS, that is, either to the *'birth'* or *'death'* of a component of the AESS, or the *'marriage'* or *'divorce'* of two components (that is, their *'joining up'* or *'splitting apart'*). We study the Singular-Point Curve in §6.4.3.

As far as this analysis is concerned, we will consider the terms 'birth' and 'death' to be interchangeable, both relating to the same phenomenon on the AESS, namely an isolated point. Similarly, we will consider the terms 'marriage' and 'divorce' to be interchangeable since they both refer to crossings on the AESS. Thus we will reduce our terminology by simply referring to births and marriages. Of course, births and deaths, and marriages and divorces, become distinguished in families, which have a direction.

6.4.2 The MPTL for $E \cup U$

As in Chapter 2, we will find it more useful to study the union of the AESS and the MPTL (see Definition 2.4.8), since the structure of the MPTL will enable us to explain certain phenomena that occur on the AESS.

The MPTL of $E \cup U$ is *always* non-empty, since we can always find parallel tangent pairs between E and U, and the midpoint of the chord joining the points of contact of these parallel tangents can always be located. Moreover, the correspondence between a pair of points on E and U, given by the curves having parallel tangents at these points, is fixed for a given a and b: as (c, d) is moved, this correspondence does not change. We also note that moving the centre, (c, d), of U does not change the *Euclidean* curvature of U at any point. We have the following:

Proposition 6.4.2. *Under the assumption that* $0 < b < a$, *the MPTL of* $E \cup U$ *is* non-smooth *(that is, exhibits cusps) if and only if*

- $ab \leq 1$ *and* $a^2 \geq b$; *or*
- $ab > 1$ *and* $a \geq b^2$.

Proof. We know (by Proposition 2.4.9) that the condition for the MPTL of $E \cup U$ to exhibit a cusp is that E and U have the same Euclidean curvature at points which have parallel tangents. Now the Euclidean curvature of U is fixed at 1, and we know that the Euclidean curvature κ_E of E is bounded as follows:

$$\min(\kappa_E) = \frac{b}{a^2} \leq \text{ Euclidean curvature of } E \leq \frac{a}{b^2} = \max(\kappa_E),$$

since the curvature of E is continuous and attains its maximum and minimum values at its (Euclidean) vertices. Thus, the necessary and sufficient condition for there to exist a point of E where the curvature is 1 is that

$$\frac{b}{a^2} \leq 1 \leq \frac{a}{b^2}. \tag{6.15}$$

If this holds, then there exists a point on E and a point on U having parallel tangents and equal Euclidean curvatures, and thus, by Proposition 2.4.9, the MPTL of $E \cup U$ is non-smooth. Now (6.15) is equivalent to

$$a \geq b^2 \text{ and } a^2 \geq b,$$

holding simultaneously. Since we are assuming that $a > b > 0$, the necessary and sufficient condition for the MPTL to be non-smooth is

$$ab \leq 1 \text{ and } a^2 \geq b, \text{ or } ab > 1 \text{ and } a \geq b^2.$$

□

Figures 6.13, 6.14 and 6.15 contain [LSMP] plots illustrating the result of Proposition 6.4.2.

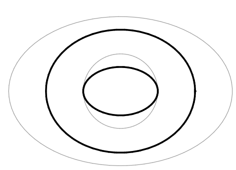


Figure 6.13: *[LSMP] plot for $a = 3, b = 2, c = d = 0$ (so $ab > 1, a < b^2$): the MPTL is the dark curve, E and U the grey curves. The MPTL is smooth.*

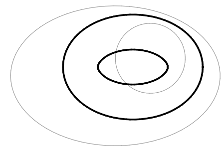


Figure 6.14: *[LSMP] plot for* $a = 3, b = 2, c = 1, d = 0.5$ *(so* $ab > 1, a < b^2$*): the MPTL remains a smooth curve, regardless of position of* (c, d)*.*

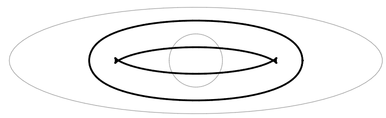


Figure 6.15: *[LSMP] plot for* $a = 7, b = 2, c = d = 0$ *(so* $ab > 1, a > b^2$*): this time, the MPTL is non-smooth, as predicted by Proposition 6.4.2.*

Convention: *From now on, the term* **Singular MPTL Condition** *will be used to denote the two conditions under which the MPTL is singular, i.e.*

$$ab \leq 1 \text{ and } a^2 \geq b, \text{ or } ab > 1 \text{ and } a \geq b^2.$$

From Proposition 2.4.9 we know that the necessary and sufficient condition for a singularity to appear on the MPTL is that there exists a conic having 3-point contact with the curve at each of the two points of contact of the parallel tangents. Thus the Singular MPTL Condition, which implies the existence of a singularity on the MPTL of $E \cup U$, in turn implies the existence of *at least one* 3+3 conic. Hence the Singular MPTL Condition is a *sufficient* condition for the existence of real AESS points.

Proposition 6.4.3. *For* $a > b > 0$*, suppose the Singular MPTL Condition holds. Then the AESS of* $E \cup U$ *is* non-empty.

From Proposition 6.4.1, a short calculation shows that the Singular MPTL Condition is a necessary *and* sufficient condition for the AESS of $E \cup U$ to be non-empty in the special case where $(c, d) = (0, 0)$. Figure 6.16 shows an [LSMP] plots of the AESS $\cup$ MPTL for $ab > 1$ and $a > b^2$. We notice that the AESS segments fit neatly in between the two pairs of cusps on the MPTL. We also observe that each of the two segments of the AESS exhibit six cusps, four of which are due to the existence of four *4+3* conics, and correspond to the four *vertical* or *horizontal* tangents on the relevent branch of the pre-AESS (see Figure 6.16). The two others come from the fact that we have parallel tangents at the points of contact of a $3+3$ conic: these cusp-points are also cusps of the MPTL, and the (dual)-beaks singularity (as predicted in §2.5.7) can clearly be seen.

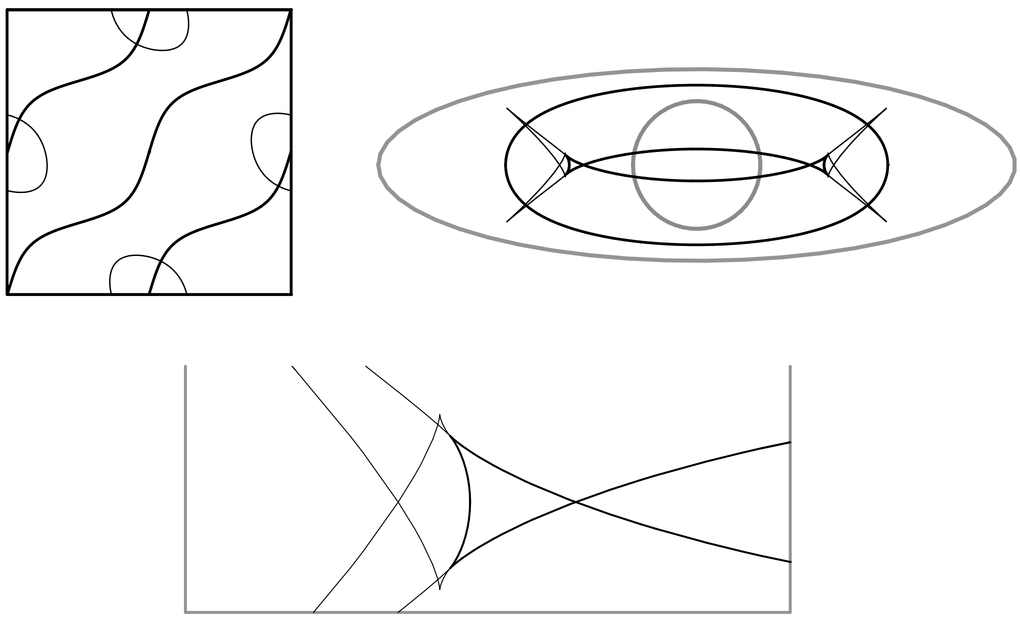


Figure 6.16: *(above) On the right is an [LSMP] plot of the AESS* $\cup$ *MPTL for* $a = 5, b = 1.5$ *(so* $ab > 1$ *and* $a > b^2$*), and* $c = d = 0$. *The AESS is the thinner dark curve, the MPTL the thicker dark curve, and ellipse* E *and circle* U *are in grey. Note that the two AESS segments fit between the two cusp pairs on the MPTL. On the left, we have the pre-sets for the AESS and MPTL, the thicker curve being the pre-MPTL. We can see that the pre-AESS comprises two closed curves each with four horizontal or vertical tangents, and each crossing the pre-MPTL twice. This accounts for the six cusps on each branch of the AESS. (below) This plot shows the magnification of the plot above, and clearly illustrates that the AESS (thin) and MPTL (thick) has two beaks singularities on each branch of the AESS, corresponding to crossings on the pre-sets.*

6.4.3 The *Singular-Point Curve*

Proposition 6.4.3 tells us nothing about the existence of real AESS points in the case when the Singular MPTL Condition does *not* hold, that is, when the MPTL is *smooth*. In this case, the AESS will be empty or non-empty depending on the position of (c, d). In this section, we will consider the following two problems:

- **Isolated Points on the AESS:** Suppose we fix ellipse E (that is, we fix a and b), and suppose also that the Singular MPTL Condition does not hold. Then we know that, for $(c, d) = (0, 0)$, the AESS is empty. Let us allow the centre of U, point (c, d), to vary. Then we would like to identify the values of (c, d) which correspond to the *birth* of an AESS segment. In effect, we would like to plot the locus of points in (c, d)-space for which the AESS has *isolated points*.

- **Crossings on the AESS:** Suppose we fix ellipse E, and suppose now that the Singular MPTL Condition *does* hold. Then we know that the AESS has real points for any position of (c, d). We ask the question: *Can any other components of AESS appear?* In other words, can we still plot a (non-empty) locus of points in (c, d)-space for which a component of AESS is 'born'? Computer experiments using [LSMP] suggest that *no other components of AESS are created as (c, d) varies.* However, another interesting phenomenon is observed: when the Singular MPTL Condition holds, the AESS of $E \cup U$ is in two disjoint components for $(c, d) = (0, 0)$. Now, as (c, d) is varied, we notice that these two disjoint components can join together (or split apart), and this leads us to a similar problem of plotting the curve in (c, d)-space which, with (c, d) as centres of U, correspond to a *marriage* of two AESS segments: that is, we require the locus of points (c, d) for which the AESS has a *crossing*.

Combining these two problems, we will consider the problem of plotting the locus of points (c, d) which, as centres of U, lead to an *isolated point or a crossing* on the AESS of $E \cup U$. We make the following:

Definition 6.4.4. *The* **Singular-Point Curve** *for the AESS is the set of points (c, d) which, as centres of the unit circle U, lead to isolated points or* *crossings on the AESS.*

Returning to our analysis of §6.4.1, we found that the condition for the existence of real AESS points was identical to the condition that there exist *any* s, t for which (6.14) of Proposition 6.4.1 holds. We will modify this to suit out current analysis. If we suppose that a and b are fixed, then we can use the above to deduce a condition on c and d for such an s and t to exist. Let us set

$$\begin{array}{llll} A = (ab)^{2/3} - 1 & B = -c\left(\frac{b^2}{a}\right)^{1/3} & C = -d\left(\frac{a^2}{b}\right)^{1/3} & D = -c \\ E = -d & F = a - \left(\frac{b^2}{a}\right)^{1/3} & G = b - \left(\frac{a^2}{b}\right)^{1/3} & \end{array}$$

Then the problem is finding real s, t for which

$$A + B\cos s + C\sin s + D\cos t + E\sin t + F\cos s\cos t + G\sin s\sin t = 0. \quad (6.16)$$

This problem can be restated as follows[2]: for fixed real s, the condition that there exists real t such that

$$A + B\cos s + C\sin s = -(D + F\cos s)\cos t - (E + G\sin s)\sin t,$$

is equivalent to the inequality

$$\|A + B\cos s + C\sin s\| \leq \sqrt{(D + F\cos s)^2 + (E + G\sin s)^2},$$

which follows from the fact that numbers of the form $p\cos t + q\sin t$ fill the interval $[-\sqrt{p^2+q^2}, \sqrt{p^2+q^2}]$. Thus, we have reduced the problem to solving the equation

$$(A + B\cos s + D\sin s)^2 - (D + F\cos s)^2 + (E + G\sin s)^2 \leq 0. \quad (6.17)$$

We can re-express the LHS of (6.17) as

$$f(u) \equiv \alpha_4 u^4 + \alpha_3 u^3 + \alpha_2 u^2 + \alpha_1 u + \alpha_0,$$

[2]This line of attack was originally suggested by Dr.Mariusz Zajac (Warsaw Technical University).

where

$$u \equiv \tan\left(\frac{s}{2}\right), \quad \cos s = \frac{1-u^2}{1+u^2} \text{ and } \sin s = \frac{2u}{1+u^2},$$

and where

$$\begin{aligned}
\alpha_4 &\equiv \left((a+c)^2-1\right)\left(\left(\frac{b^2}{a}\right)^{2/3}-1\right)-d^2 \\
\alpha_3 &\equiv 4d\left(b-(ab)^{1/3}(a+c)\right) \\
\alpha_2 &\equiv 2\left((ab)^{2/3}-1\right)^2-2c^2+4d^2\left(\frac{a^2}{b}\right)^{2/3} \\
&\quad -4\left(b-\left(\frac{b^2}{a}\right)^{1/3}\right)^2-2c^2\left(\frac{b^2}{a}\right)^{2/3} \\
&\quad +2\left(a-\left(\frac{b^2}{a}\right)^{1/3}\right)^2-2d^2 \\
\alpha_1 &\equiv 4d\left(b-(ab)^{1/3}(a-c)\right) \\
\alpha_0 &\equiv \left((a-c)^2-1\right)\left(\left(\frac{b^2}{a}\right)^{2/3}-1\right)-d^2.
\end{aligned}$$

Thus Proposition 6.4.1 can be rephrased as:

Proposition 6.4.5. *The AESS of $E \cup U$ is non-empty if and only if there exists real u such that $f(u) \leq 0$.*

Remark 6.4.6. *Note that f is symmetric under the maps:*

$$c \mapsto -c, u \mapsto 1/u,$$

$$\textit{and} \quad d \mapsto -d, u \mapsto -u.$$

We expect, by the reflexional symmetry of the ellipse, that we should have the same situation if we substitute c for $-c$ or d for $-d$, and this implies that the graph of $y = f(u)$ is symmetric about the y-axis.

Now we are no longer interested simply in the existence (or otherwise) of real points on the AESS: we would *also* like to know when two components of the AESS join or split apart. However, the function $f(u)$ gives us all the information we require. Suppose we fix a and b. Then $f(u)$ is a degree four polynomial in u with parameters c and d. The segments of the graph of $f(u) = 0$ below the u-axis correspond to real branches of the AESS. We are thus interested in the values of (c, d) which result in critical points of $f(u)$ for which $f(u) = 0$, that is, critical points of f on its zero-level. These are precisely the situations in which segments of AESS are born (see Figure 6.17(a)) or marry (see Figure 6.17(b)). Hence we require the *discriminant* of f, which is a polynomial of degree twelve in u, and factorises. Some calculation using [MAPLE] shows that it comprises the three following curves:

Proposition 6.4.7. *The* **Singular-Point Curve** *consists of three curves:*

- *the two Euclidean* parallels *at distance* 1 *to the ellipse,*
- *the (squared)* conic

$$\frac{b^{2/3}c^2}{(a^{2/3} - b^{1/3})(a^{2/3} + b^{1/3})} - \frac{a^{2/3}d^2}{(a^{1/3} - b^{2/3})(a^{1/3} + b^{2/3})} = a^{2/3}b^{2/3} - 1,$$

where we have assumed that $b \neq a^2$ and $a \neq b^2$.

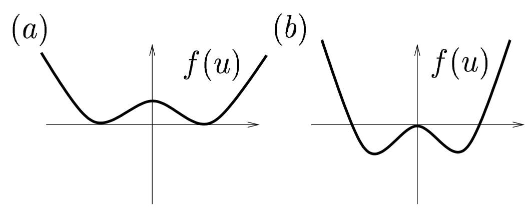


Figure 6.17: *(a) The birth of a segment of AESS, as a minimum of the graph crosses the u-axis. Strictly speaking, a birth corresponds to the graph crossing the axis moving downwards, and a death to the graph crossing the axis and moving upwards. (b) The marriage of two segments of AESS, as a maximum of the graph of f crosses the u-axis. Strictly speaking, a marriage corresponds to the graph crossing the axis moving downwards, and a divorce to the graph crossing the axis moving upwards.*

Case (i): $\alpha_4 < 0$

In this case, f has two maxima each side of a minimum, and must have at least two real roots. Thus there will *always* exist some real range for u for

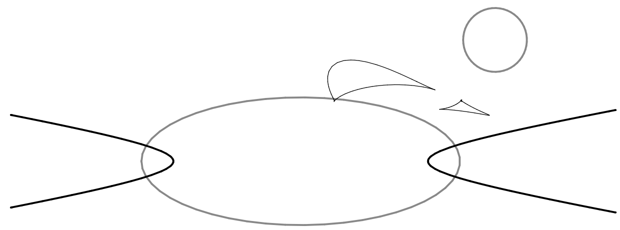


Figure 6.18: *The thicker dark curve is a component of the Singular-Point Curve for the ellipse and circle as shown ($a = 5, b = 2, c = 6.1, d = 3.8$). The thinner dark curve is the AESS for $E \cup U$. The other components of the Singular-Point Curve, the Euclidean parallels at distance 1 from the ellipse, are not shown.*

which $f(u) < 0$, and therefore $\alpha_4 < 0$ is a *sufficient* condition for there to exist real AESS points. Now, the zero-level of α_4 in (c, d)-space is

$$(a+c)^2 + \frac{d^2}{\left(1 - \left(\frac{b^2}{a}\right)^{2/3}\right)} = 1,$$

which is an *ellipse* with centre at $(-a, 0)$ if $a > b^2$, and an *hyperbola* with centre at $(-a, 0)$ if $a < b^2$. We can interpret this in the following way,

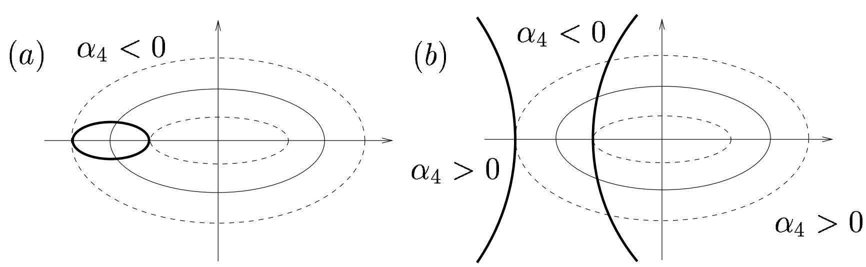


Figure 6.19: *The zero-levels of α_4 are (a) an ellipse and (b) an hyperbola shown as the thickest curves: α_4 is positive inside the ellipse in (a), and has the sign shown in the other regions.*

referring back to the earlier results concerning the sufficient condition for real AESS points in terms of the Singular MPTL Condition:

- If we have $ab > 1$ and $a > b^2$, then $\alpha_4 < 0$ for (c, d) in the area of Figure 6.19(a) as shown. Also illustrated are the ellipse E and the two Euclidean parallels at distance 1. We see that for any unit circle that is disjoint from the ellipse, its centre (c, d) must lie inside the inner parallel or outside the outer parallel, and hence in the area with $\alpha_4 < 0$. Thus for any (c, d) for which the curves are disjoint, there is some real AESS, since then α_4 is guaranteed to be negative. (Of course, if the unit circle intersects the ellipse, then real AESS points exist, for example at the intersection points.)

- If we have $ab > 1$ and $a < b^2$, then $\alpha_4 < 0$ for (c, d) in the area shown in Figure 6.19(b). Again, we also illustrate the ellipse E. However, this time we can position the unit circle such that it is *not* centred in the area having $\alpha_4 > 0$, and then $\alpha_4 > 0$. We may thus deduce nothing about the existence of real AESS points.

Thus we have confirmed the result of Proposition 6.4.3.

Case (ii): $\boldsymbol{\alpha_4 > 0}$

In this case, f has two minima each side of a maximum. By Remark 6.4.6, the minima are on the same level, and the maximum occurs on the y-axis. This implies that, at a birth of the AESS (that is, when these minima sit on the zero-level), two segments are born simultaneously, as the two minima cross the u-axis. This fact is confirmed by [LSMP] plots.

6.5 Further research

This study of the ADSS and AESS of a curve comprising two ellipses is the first step towards the consideration of the ADSS and AESS for *piecewise-conic* curves. One route towards a fuller understanding of the affine symmetry sets for genuinely piecewise-conic curves might be:

(i) Mirror the analysis of this chapter for a curve comprising a union of hyperbolas, or a union of an ellipse (taken to be a circle) and an hyperbola.

(ii) Consider any exceptional structures which may appear on the affine symmetry set of a union of conics when one of the conics is a parabola, or a line-pair.

(iii) Consider how two conic segemnts may be splined together. What does it mean for the resulting curve to be smooth? It may be enough to insist that the corresponding *affine tangent vectors* agree at a join, or alternatively that the *affine normal lines* to the curve agree at a join.

(iv) What structures appear on the affine symmetry sets of such a composite curve? Are any of the generic structures that may occur on the affine symmetry sets of a generic plane curve disallowed from occuring on the affine symmetry sets of piecewise-conic curves?

(v) Consider the transitions on the affine symmetry sets of piecewise-conic curves.

Appendix A

Partial derivatives of $v_1(t_1, t_2)$ and $v_2(t_1, t_2)$

During the analysis of §§2.5.4-2.5.7, we need expressions for the partial derivatives of v_1 and v_2 with respect to t_1, t_2, up to the third order derivative, and each of these evaluated at $t_1 = t_2 = 0$. We denote derivatives by subscripts: for example, the third partial derivative of v_2 with respect to t_1, t_1 and t_2 will be denoted $v_{1_{t_1 t_1 t_2}}$. We will omit the parameters t_1, t_2 for brevity.

The list is as follows:

$$v_1 = (t_1 - c - t_2)(f' + g') - 2(f - d - g)$$

$$v_1^0 = -cg' + 2d$$

$$v_{1_{t_1}} = -f' + g' + (t_1 - c - t_2)f''$$

$$v_{1_{t_1}}^0 = g' - cf''$$

$$v_{1_{t_2}} = -f' + g' + (t_1 - c - t_2)g''$$

$$v_{1_{t_2}}^0 = g' - cg''$$

$$v_{1_{t_1 t_1}} = (t_1 - c - t_2)f'''$$

$$v_{1_{t_1 t_1}}^0 = -cf'''$$

$$v_{1_{t_1 t_2}} = g'' - f''$$
$$v^0_{1_{t_1 t_2}} = g'' - f''$$

$$v_{1_{t_2 t_2}} = (t_1 - c - t_2)g'''$$

$$v^0_{1_{t_2 t_2}} = -cg'''$$

$$v_{1_{t_1 t_1 t_1}} = f''' + (t_1 - c - t_2)f''''$$

$$v^0_{1_{t_1 t_1 t_1}} = f''' - cf''''$$

$$v_{1_{t_1 t_1 t_2}} = -f'''$$

$$v^0_{1_{t_1 t_1 t_2}} = -f'''$$

$$v_{1_{t_1 t_2 t_2}} = g'''$$

$$v^0_{1_{t_1 t_2 t_2}} = g'''$$

$$v_{1_{t_2 t_2 t_3}} = -g''' + (t_1 - c - t_2)g''''$$

$$v^0_{1_{t_2 t_2 t_2}} = -g''' - cg''''$$

$$v_2 = 2(t_1 - c - t_2)f'g' - (f - d - g)(f' + g')$$

$$v^0_2 = dg'$$

$$v_{2_{t_1}} = f'g' + 2(t_1 - c - t_2)f''g' - f'^2 - (f - d - g)f''$$

$$v^0_{2_{t_1}} = -2cf''g' + df''$$

$$v_{2_{t_2}} = -f'g' + 2(t_1 - c - t_2)f'g'' + g'^2 - (f - d - g)g''$$

$$v^0_{2_{t_2}} = g'^2 + dg''$$

$$v_{2_{t_1t_1}} = 3f''g' + 2(t_1 - c - t_2)f'''g' - 3f''f' - (f - d - g)f'''$$

$$v^0_{2_{t_1t_1}} = 3f''g' - 2cf'''g' + df'''$$

$$v_{2_{t_1t_2}} = f'g'' - f''g' + 2(t_1 - c - t_2)f''g''$$

$$v^0_{2_{t_1t_2}} = -f''g' - 2cf''g''$$

$$v_{2_{t_2t_2}} = -3f'g'' + 2(t_1 - c - t_2)f'g''' + 3g'g'' - (f - d - g)g'''$$

$$v^0_{2_{t_2t_2}} = 3g'g'' + dg''' v_{2_{t_1t_1t_1}} = 5f'''g' + 2(t_1 - c - t_2)f''''g' - 3f''^2 - 4f'''f' - (f - d - g)f''''$$

$$v^0_{2_{t_1t_1t_1}} = 5f'''g' - 2cf''''g' - 3f''^2 + df''''$$

$$v_{2_{t_1t_1t_2}} = 3f''g'' - f'''g' + 2(t_1 - c - t_2)f'''g''$$

$$v^0_{2_{t_1t_1t_2}} = 3f''g'' - f'''g' - 2cf'''g''$$

$$v_{2_{t_1t_2t_2}} = f'g''' - 3f''g'' + 2(t_1 - c - t_2)f''g''$$

$$v^0_{2_{t_1t_2t_2}} = -3f''g'' - 2cf''g'''$$

$$v_{2_{t_2t_2t_2}} = -5f'g''' + 2(t_1 - c - t_2)f''''g' + 3g''^2 + 4g'g''' - (f - d - g)g''''$$

$$v^0_{2_{t_2t_2t_2}} = -2cf''''g' + 3g''^2 + 4g'g''' + dg''''$$

$$
\begin{aligned}
F_{t_1t_1}(t_1,t_2) = {} & (v_{1_{t_1}} v_{2_{t_1t_1}} + v_1 v_{2_{t_1t_1t_1}} - v_{2_{t_1}} v_{1_{t_1t_1}} - v_2 v_{1_{t_1t_1t_1}})(v_2 - g' v_1) \\
& +2(v_1 v_{2_{t_1t_1}} - v_2 v_{1_{t_1t_1}})(v_{2_{t_1}} - g' v_{1_{t_1}}) \\
& +(v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}})(v_{2_{t_1t_1}} - g' v_{1_{t_1t_1}}) \\
& -2(v_{1_{t_1}} v_{2_{t_2}} + v_1 v_{2_{t_1t_2}} - v_{2_{t_1}} v_{1_{t_2}} - v_2 v_{1_{t_1t_2}})(v_{2_{t_1}} - f'' v_1 - f' v_{1_{t_1}}) \\
& -(v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}})(v_{2_{t_1t_1}} - f''' v_1 - 2f'' v_{1_{t_1}} - f' v_{1_{t_1t_1}}), \\
& -(v_{1_{t_1t_1}} v_{2_{t_2}} + 2v_{1_{t_1}} v_{2_{t_1t_2}} + v_1 v_{2_{t_1t_1t_2}} - v_{2_{t_1t_1}} v_{1_{t_2}} - 2v_{2_{t_1}} v_{1_{t_1t_2}} - v_2 v_{1_{t_1t_1t_2}})(v_2 - f' v_1)
\end{aligned}
$$

$$
\begin{aligned}
F_{t_2t_2}(t_1,t_2) = {} & 2(v_{1_{t_2}} v_{2_{t_1}} + v_1 v_{2_{t_1t_2}} - v_{2_{t_2}} v_{1_{t_1}} - v_2 v_{1_{t_1t_2}})(v_{2_{t_2}} - g'' v_1 - g' v_{1_{t_2}}) \\
& +(v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}})(v_{2_{t_2t_2}} - g''' v_1 - 2g'' v_{1_{t_2}} - g' v_{1_{t_2t_2}}) \\
& -(v_{1_{t_2}} v_{2_{t_2t_2}} + v_1 v_{2_{t_2t_2t_2}} - v_{2_{t_2}} v_{1_{t_2t_2}} - v_2 v_{1_{t_2t_2t_2}})(v_2 - f' v_1) \\
& -2(v_1 v_{2_{t_2t_2}} - v_2 v_{1_{t_2t_2}})(v_{2_{t_2}} - f' v_{1_{t_2}}) \\
& -(v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}})(v_{2_{t_2t_2}} - f' v_{1_{t_2t_2}}) \\
& +(v_{1_{t_2t_2}} v_{2_{t_1}} + 2v_{1_{t_2}} v_{2_{t_1t_2}} + v_1 v_{2_{t_1t_2t_2}} - v_{2_{t_2t_2}} v_{1_{t_1}} - 2v_{2_{t_2}} v_{1_{t_1t_2}} - v_2 v_{1_{t_1t_2t_2}})(v_2 - g' v_1)
\end{aligned}
$$

$$
\begin{aligned}
F_{t_1t_2}(t_1,t_2) = {} & (v_{1_{t_2}} v_{2_{t_1t_1}} + v_1 v_{2_{t_1t_1t_2}} - v_{2_{t_2}} v_{1_{t_1t_1}} - v_2 v_{1_{t_1t_1t_2}})(v_2 - g' v_1) \\
& +(v_1 v_{2_{t_1t_1}} - v_2 v_{1_{t_1t_1}})(v_{2_{t_2}} - g'' v_1 - g' v_{1_{t_2}}) \\
& +(v_{1_{t_2}} v_{2_{t_1}} + v_1 v_{2_{t_1t_2}} - v_{2_{t_2}} v_{1_{t_1}} - v_2 v_{1_{t_1t_2}})(v_{2_{t_1}} - g' v_{1_{t_1}}) \\
& +(v_1 v_{2_{t_1}} - v_2 v_{1_{t_1}})(v_{2_{t_1t_2}} - g'' v_{1_{t_1}} - g' v_{1_{t_1t_2}}) \\
& -(v_{1_{t_1}} v_{2_{t_2t_2}} + v_1 v_{2_{t_1t_2t_2}} - v_{2_{t_1}} v_{1_{t_2t_2}} - v_2 v_{1_{t_1t_2t_2}})(v_2 - f' v_1) \\
& -(v_{1_{t_1}} v_{2_{t_2}} + v_1 v_{2_{t_1t_2}} - v_{2_{t_1}} v_{1_{t_2}} - v_2 v_{1_{t_1t_2}})(v_{2_{t_2}} - f' v_{1_{t_2}}) \\
& -(v_1 v_{2_{t_2t_2}} - v_2 v_{1_{t_2t_2}})(v_{2_{t_1}} - f'' v_1 - f' v_{1_{t_1}}) \\
& -(v_1 v_{2_{t_2}} - v_2 v_{1_{t_2}})(v_{2_{t_1t_2}} - f'' v_{1_{t_2}} - f' v_{1_{t_1t_2}}).
\end{aligned}
$$

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