Reconstruction from Medial Representations

P.J. Giblin and J.P. Warder*

Abstract

In the analysis of planar shapes there are a number of ways of reducing the shape to a 1-dimensional 'skeleton': the medial axis, the symmetry set and the 'smoothed local symmetry' which we call here the midpoint locus. All depend on circles which are twice-tangent to the boundary curve of the shape. In addition there is a 'pre-symmetry set' which underlies all these constructions. We describe how a shape can be reconstructed from its medial axis or symmetry set, that is the centres of the twice-tangent circles, and a knowledge of their radii. Then we ask the question: can the shape be reconstructed similarly from its midpoint locus—this amounts to giving the midpoints of chords of twice-tangent circles instead of their centres—and the radii? The first answer is 'yes, given an initial condition', but further analysis shows that this is so sensitive to that initial condition as to make the reconstruction difficult. We also suggest other avenues of investigation.

1   Introduction

Given a 'shape' in the plane there are a number of methods for reducing this shape to a 'skeleton'; something simpler, preferably of lower dimension, but which captures the essence of the shape we started with. One of these methods goes back to work of the theoretical biologist Harry Blum in 1967 [2], and is now called the *medial axis* of the shape. Roughly speaking, the medial axis 'goes down the middle of the shape'; it is constructed, for a planar region enclosed by a smooth simple closed curve C, by taking centres of disks lying entirely inside the shape and whose boundary circles are tangent to C in more than one place—we refer to these as 'bitangent circles' for C. The closure of the locus of these centres forms a tree-like structure M, with endpoints and 3-way branches ('Y-junctions'). A typical example is shown in Figure 1(a) (from [13], but see also [14] for an extensive survey and references). It is important that, given M and the radii of the bitangent circles which were used to construct M, we can go backwards and recover the original curve C. For then actual circles are known and the curve C is found as their envelope (Figure 1(b)), i.e.

medial axis + radii of the bitangent circles $\longrightarrow$ original shape.

Since Blum's time the computation and application of medial axes have seen striking developments. Many of these developments appear in the book [14] on Medial Structures. Here are some of the applications which appear in Chapter 11 of that book:

- Sebastian et al. [13] used the medial axis to propose a method for improved recognition of objects of interest in X-ray images.

- Held [7] described ways of using the medial axis to improve tool machining accuracy in applications such as milling, punching and drilling.

- Leymarie and Levine [12] used the medial axis to predict the growth of pseudo-pods of white blood cells, which are used to move these cells around inside the body.

An alternative, but closely related construction was proposed by Michael Brady and Haruo Asada in 1984 (see [1]), under the name of 'smoothed local symmetry'. Here, instead of tracing the centres of
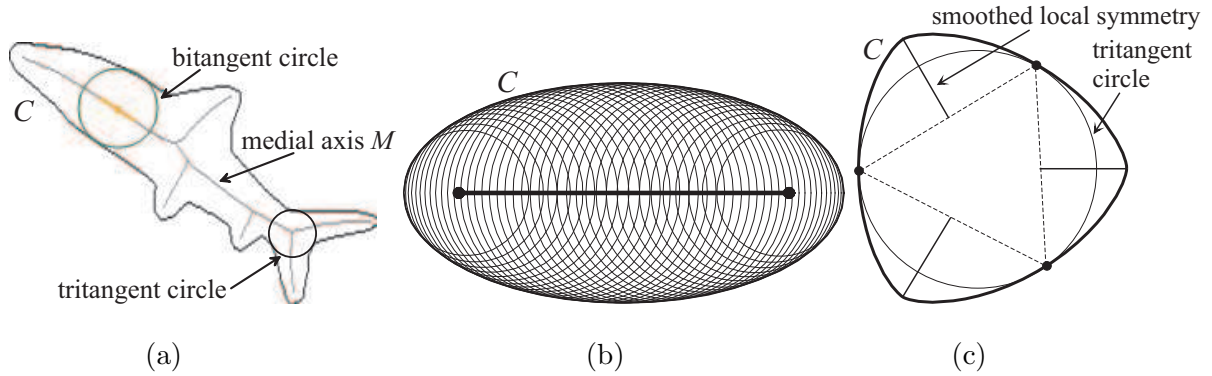
Figure 1: (a) A typical medial axis, which is the trace of the centres of bitangent circles such as the one drawn, lying entirely inside the shape enclosed by C. Note the endpoints and Y-junctions, the latter being centres of 'tritangent circles'. (b) Reconstructing a curve C—here an ellipse—as an envelope from the medial axis (the thick straight segment) and the actual circles, given their radii. (c) For the curve shown, tracing the midpoints of chords of contact of bitangent circles *only inside* the shape ('smoothed local symmetries'; see the text) produces the three straight segments ending on the sides of the triangle, joining the points of contact of the tritangent circle shown.

bitangent circles we trace the *midpoints of the chords* joining the pairs of tangency points between the bitangent circles and C. If we take only circles inside a planar shape then this produces discontinuities where a circle is tangent in three places (see Figure 1, right)[1]. Also, and perhaps more significantly, it is far from obvious that there is a simple way of reconstructing the shape from the trace of midpoints together with some additional information such as the radii of the bitangent circles. Indeed, this difficulty is pointed out in [14, p. 6].

This article is organized as follows: In §2 we introduce the medial axis, a closely associated but larger set called the 'symmetry set' and the 'midpoint locus' which includes the smoothed local symmetry mentioned above. We also give some examples to help the reader's intuition. In §3 we introduce 'pre-symmetry sets', which underlie all of these constructions. In §4 we fill in the details of the reconstruction of a curve C as an envelope of circles, centred on the medial axis and of specified radii. In §5 we take up the question mentioned above:

> *Given the midpoints of the chords of contact of the bitangent circles and their radii, can we still reconstruct C?*

The situation here is curiously delicate. We show that in principle we need just one 'initial condition', at an endpoint of the trace of midpoints, but getting this slightly wrong may result in a grossly incorrect reconstruction. We also consider the possibility of using a different initial condition, at a local maximum or minimum of radius of the bitangent circles—a point of locally greatest or least 'width' of the shape. This is more promising, having essentially two solutions, but it raises other difficulties. We illustrate all these ideas with explicit examples, which involve some pleasant exercises in the solution of differential equations. Finally, in §6 we draw our work together.

2 Symmetry set, medial axis and midpoint locus

We shall give definitions for the planar case, assuming that all shapes under consideration are connected regions bounded by a single curve C.

Definition 2.1 Given a simple closed smooth curve C in $\mathbb{R}^2$ the *symmetry set* of C, or of the region bounded by C, is the closure of the set of centres of all circles bitangent to C, i.e. tangent in more than one place. The *medial axis* is the subset of the symmetry set formed by the centres of bitangent circles which are *maximal* [2].

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[1]For this reason we relax the 'inside only' condition below.

[2]A circle is maximal if its radius equals the absolute minimum distance from its centre to C: such a circle cannot be expanded about its centre without crossing C.

We shall occasionally give examples where C is not closed (notably C a parabola) in order to illustrate ideas with computable formulas. In this case there is no region to consider but otherwise the definitions remain the same.

Let C be parametrized by the circle S^1, say $\gamma : S^1 \to \mathbb{R}^2$ (or by the real line $\mathbb{R}$ if C is not closed), and suppose that $\gamma(p)$ and $\gamma(q)$ are two points of tangency with a circle, radius r. Then the centre of the circle is

$$c = \gamma(p) \pm r\, N(p) = \gamma(q) \pm r\, N(q), \tag{1}$$

where the $\pm$ signs are independent. Here N is the unit normal to C, obtained by rotating $90°$ anticlockwise from the unit tangent, oriented by γ'. Figure 6, right, explains why the $\pm$ signs are necessary. Note that the criterion of maximality does not require that the circle, and the disk which it bounds, lie entirely inside C. It does however mean that the disk lies either wholly inside or wholly outside[3] of the curve C. For the medial axis, with the circles entirely inside C, we can consistently choose the $+$ signs in (1).

Definition 2.2 Given a curve C as above the *midpoint locus* of C is the closure of the set of midpoints of chords joining contact points of all circles bitangent to C. Thus, if $\gamma(p)$ and $\gamma(q)$ are two points of tangency then the corresponding point of the midpoint locus is $m = \frac{1}{2}(\gamma(p) + \gamma(q))$.

Note that we take *all* circles in our definition. This is to avoid the discontinuities which would otherwise occur when a circle is 'tritangent' (i.e. tangent in three places) as shown in Figure 1(c).

The most basic example of a curve C in this context is an ellipse, and in Figure 2 we show the three constructions for this curve. The vertical straight segments of the symmetry set and midpoint locus arise from bitangent circles which are tangent externally to the ellipse; one of these circles is partially drawn in Figure 2(b).
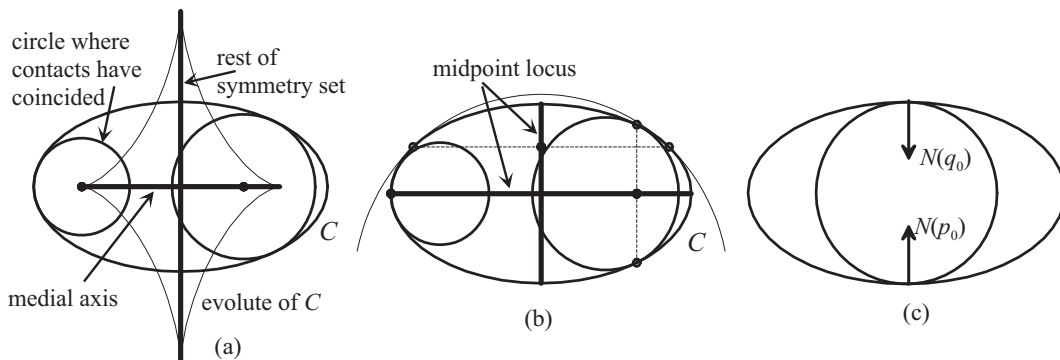

Figure 2: C an ellipse. (a) The symmetry set is two straight segments, ending in the cusps of the evolute. The medial axis is one of these segments, joining the centres of curvature at the maxima of curvature. (b) The midpoint locus is two segments joining the maxima, resp. the minima, of curvature. (c) This bitangent circle has a local maximum of radius, and the normals are in the same straight line. See §3.3.

It can happen that the two points where a circle is tangent to C come together into one place P; then the circle has $2 + 2 = 4$ 'coincident intersections' with C which means that P is a *vertex* of C, that is an extremum of curvature. This results in an endpoint for the medial axis and the symmetry set, lying at the centre of curvature of C at P; for the midpoint locus the endpoint lies at P itself. See Figure 2(b), where P is the left-hand end of the horizontal axis of the ellipse for the circle drawn. Several basic properties of these constructions are proved in [5] and there are detailed discussions in [3, 14].

Example 2.3 The symmetry set shown in Figure 3 has six end points, corresponding to vertices on C, and also six singular points (cusps). These in fact lie at the centres of circles which are tangent at one point of C and *osculate* C at the other—that is, such a circle is the circle of curvature at the second point (see for example [5]). There are also two triple crossings; these are the centres of

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[3]We shall not be concerned with the 'external' medial axis here, but it is often included in the definition.

tritangent circles to C. Notice that the structure of the symmetry set is very complicated, even for a relatively simple oval shape.
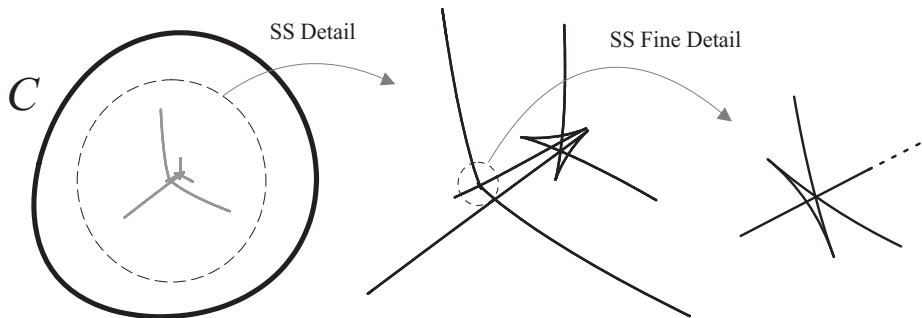

Figure 3: Details of the symmetry set (SS) of an oval C, with six endpoints, six cusps and two triple crossings.

The medial axis in this example is much simpler than the symmetry set: it consists of just a Y-junction, as in Figure 4, left. The midpoint locus has the structure of three smooth branches joining pairs of vertices on C, as in Figure 4, right. In fact the midpoint locus consists only of smooth branches joining vertices, for a generic curve C: in general there are no cusps at all. This is proved in [5].
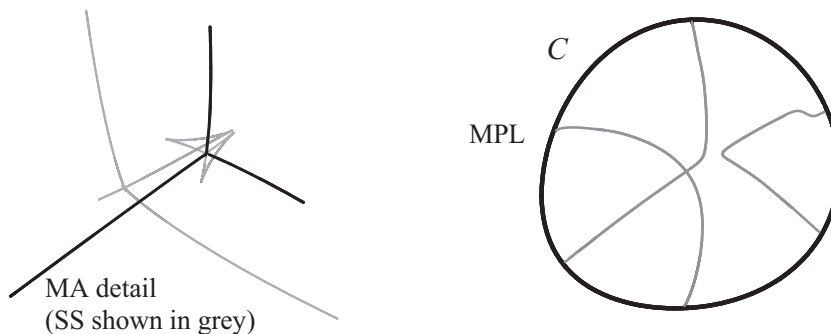

Figure 4: Left: The medial axis (MA, dark line) of the curve C shown in Figure 3. Right: the midpoint locus (MPL). Despite being constructed from *all* bitangent circles, the structure here is much simpler than the symmetry set in Figure 3: it consists of just three smooth arcs.

Notice that in Figure 4, right, the branches of the midpoint locus do not necessarily strike the curve at right-angles. In fact the angle between the limiting tangent to the midpoint locus and the curve C varies considerably and depends on the first *five* derivatives of the parametrizing function γ at the point of contact. Also, at degenerate vertices of C (points where the first and second derivatives of curvature are zero), the midpoint locus approaches C directly along the *tangent* to C at the point of contact (see [15, §2.4]).

Example 2.4 Let us consider the simplest case of a curve having an ordinary vertex at the origin, namely the parabola $C : \ y = x^2$ (Figure 5). This curve is not closed but nevertheless it provides a useful and computable example. There is a circle tangent to the curve at the symmetric points $(\pm p, p^2)$, $p > 0$, and a short calculation shows that the centre of this circle is $(0, p^2 + \frac{1}{2})$. Thus the centres trace the y-axis for $y > \frac{1}{2}$, the limit point $(0, \frac{1}{2})$ being the centre of the circle of curvature of C at the origin: here, the two tangency points have coincided. Thus the symmetry set of the parabola is the set of points $(0, y)$ where $y \geq \frac{1}{2}$. The radius of the circle is $r(p) = \sqrt{p^2 + \frac{1}{4}}$, so that we can express r as a smooth function of arclength $s = y - \frac{1}{2} \geq 0$ on the symmetry set, namely $r(s) = \sqrt{s + \frac{1}{4}}$. (Curiously, this function is smooth for $s > -\frac{1}{4}$ and not just for $s > 0$.) The value of $\frac{dr}{ds}$ at $s = 0$ (strictly as $s \to 0$ from above) is 1. In a similar way the midpoint locus of the parabola lies along
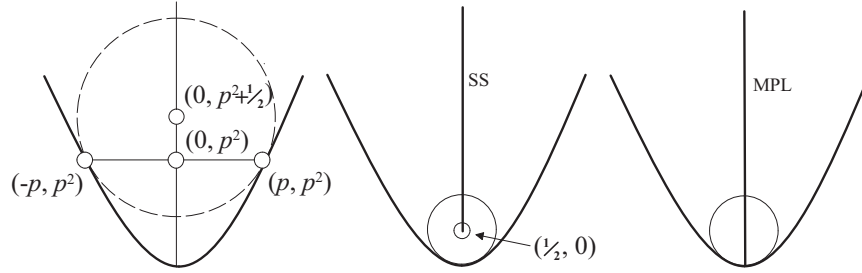
Figure 5: A parabola with its symmetry set (SS) and its midpoint locus (MPL).

the y-axis, consisting of all the points on the positive y-axis, the midpoint of the chord for the circle tangent at $(\pm p, p^2)$ being of course $(0, p^2)$. So as a function of arclength $s_m = y \geq 0$ on the midpoint locus, the radius is $r_m(s_m) = \sqrt{s_m + \frac{1}{4}}$, also a smooth function. In this simple example the midpoint locus and symmetry set points maintain a constant distance apart of $\frac{1}{2}$ on the y-axis. See Figure 5. In general the midpoint locus and symmetry set will not be related in this simple way, of course. Indeed they will not even have parallel tangents at their endpoints for an un-symmetric ordinary vertex such as $y = x^2 + x^5$.

3 Pre-symmetry sets

The pre-symmetry set underlies all of the symmetry constructions introduced above; it has also found direct application recently—see [9, 10]. It first appears, though without a special name, in [5].

Definition 3.1 Given a smooth simple *closed* curve C in $\mathbb{R}^2$ the *pre-symmetry set* of C is the closure of the set of all pairs of distinct points on C such that there is a circle tangent to C at both points.

Thus, given a parametrization γ of C by the circle S^1, we can regard the pre-symmetry set as lying in the space $S^1 \times S^1$ consisting of parameter values (p, q) where there is a circle tangent to C at $\gamma(p)$ and $\gamma(q)$, together with the limit points of these pairs. These limit points are diagonal points (p, p) where $\gamma(p)$ is a vertex of C: recall from above that the pair of contact points comes into coincidence at a vertex. The pre-symmetry set will consist of loops lying on the surface of a torus $S^1 \times S^1$, which we represent as a square with opposite sides identified. Those loops which are non-contractible on the torus are termed *essential* loops and the others are *inessential*. An example is shown in Figure 6, left. There are in fact curves with *no* essential loops in their pre-symmetry sets, and it is conjectured that
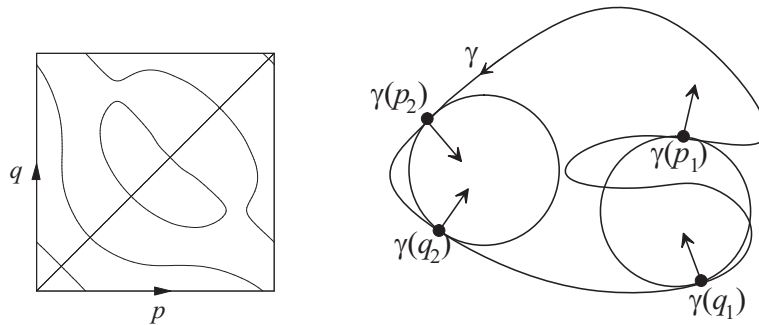

Figure 6: Left: Pre-symmetry set for the oval in Figure 3. Note: there are two essential loops and one inessential loop on the torus obtained by identifying left and right edges, and top and bottom edges, of the square (see the text). Right: $\gamma(p_1) - \gamma(q_1)$ is perpendicular to the *difference* of the two oriented unit normals at these points, while $\gamma(p_2) - \gamma(q_2)$ is perpendicular to the *sum* of the unit normals. This is why both signs are needed in (2).

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there are always *zero* or *two* essential loops for any C. See [9, 10].

To find the entire pre-symmetry set of C, parametrized by γ, we can use (1) to define two functions:

$$f^{\pm}(p,q) = (\gamma(p) - \gamma(q)) \cdot (N(p) \pm N(q)), \qquad (2)$$

where $\cdot$ is scalar product and N is the oriented unit normal to C. See Figure 6, right. The pre-symmetry set is the union of the zero sets of f^+ and f^-, once certain unwanted components have been removed, namely those for which (i) $p = q$ or (ii) $N(p) = \mp N(q)$. Here are two cases of particular interest.

3.2 Vertex of C

Suppose that $\gamma(0)$, say, is an ordinary vertex of C; consider pairs (p, q) close to $(0, 0)$ which satisfy (2) with the $+$ sign. In that case $N(p) + N(q)$ will not be zero and the only spurious solutions of (2) are (p, p), $p \neq 0$. In fact we find[4] that f^+ has the form, up to a multiplicative nonzero constant, $(p - q)^3(p + q +$ higher order terms). The last factor represents the true pre-symmetry set[5], which is naturally perpendicular to the diagonal because it has to be symmetric about the diagonal. There are six such perpendicular crossings of the diagonal in the example of Figure 6, left, one of them just visible in the top right corner!

3.3 Cusps on the symmetry set and extrema of r

Let us consider a point (p_0, q_0) at which the pre-symmetry set has a horizontal (or vertical) tangent. At these points, one of the parameters 'turns back'—for a vertical tangent it will be p—while the other continues in the same direction. It can be shown (see [5]) that this happens when the corresponding circle, tangent to C at $\gamma(p_0)$ and $\gamma(q_0)$, is in fact *osculating* at $\gamma(q_0)$: its centre is the centre of curvature of C at $\gamma(q_0)$. Furthermore the symmetry set has a cusp (the symmetry set 'turns back')—compare Example 2.3, Figures 3 and 6, left. For more details on the structure of the pre-symmetry set see [5, 9, 10]. It can be shown that the radius function r (as in (1)) has an extremum at the cusp point, with the extreme value being the radius of curvature of C at $\gamma(q_0)$.

There is also another, more obvious situation in which the radius function has an extremum, illustrated in Figure 2(c), where the drawn circle has a maximum of radius. Orienting the normals towards the centre of the circle, $N(p_0) = -N(q_0)$. If instead the curve has a narrow 'waist' then the radius will have a local minimum. These both correspond to the case where the centre of a bitangent circle coincides with the midpoint of the chord joining the contact points, a situation we return to in §5, in particular equation (5) below.

4 Reconstruction of C using its symmetry set or medial axis, and the radius function

In this section, we make precise the idea that the curve C can be reconstructed as an envelope of circles centred on the symmetry set, as in the ellipse example of Figure 1(b). This reconstruction will be important, too, in the next section.

Suppose we are given the symmetry set (or medial axis) of a smooth curve C locally as a unit speed (in particular smooth) parametrised curve $c(s) = (x(s), y(s))$ and, for each s, the radius $r(s)$ of the bitangent circle centred at $c(s)$. We can reconstruct local parametrizations γ_1, γ_2 of the two corresponding arcs of C as the envelope of bitangent circles as follows. Let $w \in \mathbb{R}^2$; then the equation of the circle of radius $r(s)$ centred at $c(s)$, is $F = 0$ where $F(s, w) = (c(s) - w) \cdot (c(s) - w) - r(s)^2$. The envelope of this family is given by $\mathcal{D}_F = \{ w \in \mathbb{R}^2 : \exists s \in \mathbb{R} \text{ with } F = \partial F / \partial s = 0 \}$. If $T(s)$ and

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[4]Make a substitution $p = x + y$, $q = x - y$; then it can be shown that $F(x, y) = f^+(x + y, x - y)$ has the property $F(x, 0) \equiv 0$, $\frac{\partial F}{\partial y}(x, 0) \equiv 0$, $\frac{\partial^2 F}{\partial y^2}(x, 0) \equiv 0$ and $\frac{\partial^3 F}{\partial y^3}(x, 0) \not\equiv 0$. Thus $F(x, y) \equiv y^3 F_1(x, y)$ for a smooth F_1, by Hadamard's Lemma (see for example [3, Ch. 4]).

[5]In fact even when the vertex is degenerate (the second and maybe higher derivatives of the curvature vanishing there) it is still true that there is only one real branch of the solution to (2) besides the diagonal, so that the true pre-symmetry set, and with it the symmetry set itself, has a single branch with an endpoint at the centre of curvature at the vertex. A careful geometrical analysis of this situation, proving more than is stated here, appears in [11].

$N(s)$ are the unit tangent and normal to $c(s)$ then they are linearly independent and so can be used as a basis for $\mathbb{R}^2$. So writing $c(s) - w = \lambda T(s) + \mu N(s)$ (where $\lambda, \mu \in \mathbb{R}$) and using $F = \partial F / \partial s = 0$ it follows that for each circle the two points of tangency are, writing $'$ for d/ds,

$$\gamma_i(s) = c(s) - (r(s)r'(s))\, T(s) \pm \left(r(s)\sqrt{1 - (r'(s))^2} \right) N(s) \quad (i = 1, 2). \tag{3}$$

Note that, when $(r')^2 = 1$, the two points $\gamma_1(s)$ and $\gamma_2(s)$ coincide: this occurs at an endpoint of the symmetry set, when the two contact points of a bitangent circle have coincided at a vertex of C. Indeed it is clear that $(r')^2 \leq 1$ is a necessary condition for the envelope to be real: if the radius changes too quickly with respect to distance along the symmetry set then the circles do not form an envelope. We can also express this condition as

$$\textit{The envelope of circles reconstructing } C \textit{ is real if and only if } r'^{\,2} \leq x'^{\,2} + y'^{\,2}, \tag{4}$$

where, here, the prime $'$ can be interpreted as differentiation with respect to any regular parameter. When reconstructing C globally, the above method provides a unique reconstruction over the *smooth* arcs of the symmetry set or medial axis, and, in view of the fact that we *start* with a smooth curve, the reconstructed pieces must 'fit together' smoothly[6].

5 Reconstruction using the midpoint locus and radius function

5.1 In Theory

We turn now to the question raised in the Introduction: suppose that we are given the midpoint locus of C, together with the radius function. Is this sufficient to recover C, and if not, what additional information is needed? If we can reconstruct the symmetry set of C from the given information, then of course we can use the method of §4 to recover the curve C. But it is far from clear that the centres of the bitangent circles are determined by knowledge of the midpoints of chords of contact and the radii. It follows from (3), by adding γ_1 and γ_2, that the line joining the centre c of a bitangent circle to the midpoint m of the chord of contact is *tangent* to the symmetry set at c. (Of course this is problematic when c and m coincide, but then we understand the result in a limiting sense, as also will be the case at singular points of the symmetry set.) The symmetry set is the envelope of the lines through the midpoints m, perpendicular to the chords of contact joining $\gamma_1(s)$ and $\gamma_2(s)$. This follows from equation (3) since we have

$$m = c - \left(r\, \frac{dr}{ds} \right) T \tag{5}$$

where s is arclength on c, so that T is parallel to $m - c$. Note that this equation requires the symmetry set to be *smooth*, for otherwise we cannot use an arclength parameter. Thus $dr/ds = 0$ for the circle shown in Figure 2(c) whose radius is a local maximum. As described in §3.3 the radius can also have a local extremum when the symmetry set has a cusp. In this case dr/ds is not defined, so (5) does not apply, but r is a smooth function of another parameter (on C) with zero derivative.

We shall now rewrite (5) in terms of a general (regular) parameter t, in order to derive differential equations connecting m and c. Let us write $c(t) = (x(t), y(t))$, $m(t) = (u(t), v(t))$. From (5) we have, writing $'$ for d/dt and omitting the variable t from the notation,

$$(c - m) \cdot (x', y') = rr'; \quad (c - m) \cdot (-y', x') = 0.$$

Rearranging these equations, and assuming $c - m \neq \mathbf{0}$, we get

$$x' = \frac{rr'(x - u)}{(x - u)^2 + (y - v)^2}, \quad y' = \frac{rr'(y - v)}{(x - u)^2 + (y - v)^2}. \tag{6}$$

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[6]It is quite a different matter to write down necessary and sufficient conditions for a pre-chosen collection of arcs purporting to be a symmetry set (or medial axis), and a radius function defined on them, to yield a global smooth curve C by means of envelope constructions on the smooth arcs. Even at triple crossings (or Y-junctions for the medial axis) there are stringent conditions. We shall not go into this here; details can be found in [14, Ch. 2] and in [6] while higher dimensional cases are studied in [14, Ch. 3] and [4].

Substituting $X = x - u$, $Y = y - v$ the equations take the form

$$X' = \frac{rr'X}{X^2 + Y^2} - u' \quad \text{and} \quad Y' = \frac{rr'Y}{X^2 + Y^2} - v'. \tag{7}$$

Recall that $u(t), v(t), r(t)$ are known, $'$ can be taken as the derivative with respect to a regular parameter t on the midpoint locus $m(t) = (u(t), v(t))$; note that the midpoint locus of a generic curve is smooth. We seek $X(t), Y(t)$ from which the symmetry set is parametrized $(x(t), y(t)) = (X(t) + u(t), Y(t) + v(t))$. Existence and uniqueness of solution are now assured, given an initial condition, using standard results on ODEs (see for example Ince [8, §3.3]). At points where $X = Y = 0$ (equivalently $m = c$), and also at points where the symmetry set is singular, so that (5) is not valid, we must understand the equations (7) in a limiting sense and uniqueness may break down.

How, then, do we provide an initial condition? This can be done by using an *endpoint* of the midpoint locus, which occurs at a vertex V of C, as in Figure 2(b). This gives us a point of C but not a point of the symmetry set since for that we need the centre of the osculating circle at V and we only know the radius r of this circle. So the endpoint of the symmetry set is restricted to lie on a circle, centre V, radius r. There is thus a 1-parameter family of solutions, depending on the choice of point on this circle.

5.2 In Practice

To see how this reconstruction method might work in practice we will return to Example 2.4. Thus we take C to be a parabola; this is not a closed curve, but we shall shortly find that perturbing the initial condition unexpectedly reconstructs closed curves!

We start with the midpoint locus m of the parabola, together with the radius function r and an initial condition which is a point on the circle, centre $(0,0)$, radius $\frac{1}{2}$, and from this we show how to find a symmetry set and a curve C which fit all this data. The 'standard' solution is the original parabola, when the initial condition is the point $(0, \frac{1}{2})$. As we shall see, the other solutions can all be computed explicitly in this case.

The midpoint locus can here be parametrized by arclength s_m, which for simplicity of notation we will write here as t. The general point of the midpoint locus is $(0, t)$, $t \geq 0$, with radius function r given by $r^2 = t + \frac{1}{4}$, so that $rr' = \frac{1}{2}$. We take the differential equations in the form (6):

$$x' = \frac{x}{2(x^2 + (y - t)^2)}, \quad y' = \frac{y - t}{2(x^2 + (y - t)^2)}.$$

where $'$ means d/dt. The 'standard' solution for the symmetry set, with initial condition $x(0) = 0$, $y(0) = \frac{1}{2}$, is $x = 0$, $y = t + \frac{1}{2}$, which gives rise to the original parabola as the envelope of circles centred at $(x(t), y(t))$ of radius $r(t)$. To find other solutions, where the initial condition is say $(x(0), y(0)) = (\frac{1}{2}\cos\theta, \frac{1}{2}\sin\theta)$, we change variables from x, y to x, z where $y - t = zx$. This gives $y' - 1 = zx' + z'x$, and substituting in the differential equations for x', y' we get the greatly simplified equations

$$x' = \frac{1}{2x(1 + z^2)}, \quad 1 + z'x = 0.$$

It is easy to eliminate x (using $x = -1/z'$) giving

$$2(1 + z^2)z'' + z'^3 = 0, \tag{8}$$

which can be solved explicitly for t as a function of z. For writing $w = z'$, $z'' = w\,(dw/dz)$ reduces the equation to

$$-\frac{dw}{w^2} = \frac{dz}{2(1 + z^2)},$$

which integrates twice to give

$$t = \tfrac{1}{2} z \tan^{-1} z - \tfrac{1}{4}\ln(1 + z^2) + az + b, \tag{9}$$

where a, b are constants[7]. Clearly $z = $ constant is also a solution, but we wish avoid $z' = 0$. Thus

$$\frac{dt}{dz} = \frac{1}{2}\tan^{-1} z + a$$

which will have a single zero, when $z = -\tan(2a)$. For z less than, or greater than, this value, the equation (9) defines (implicitly) z as a smooth function of t. In any case, substituting for t in $x = -1/z'$, $y = t + zx$ gives the symmetry set parametrized as a smooth curve by z:

$$(x(z), y(z)) = \left(-\tfrac{1}{2}\tan^{-1} z - a, -\tfrac{1}{4}\ln(1 + z^2) + b\right). \tag{10}$$

Furthermore the end-point, given by $t = 0$ corresponding to $z = z_0$ say, lies on the circle, centre $(0,0)$ with radius $\frac{1}{2}$. Using (9) and $(x(z_0), y(z_0)) = (\frac{1}{2}\cos\theta, \frac{1}{2}\sin\theta)$ we find that

$$z_0 = \tan\theta, \quad a = -\tfrac{1}{2}(\theta + \cos\theta), \quad b = \tfrac{1}{2}(\sin\theta - \ln\cos\theta).$$

Thus θ determines the constants a, b in the solution (9) and hence completely determines the reconstructed symmetry set, at least so long as it remains nonsingular[8]. We can now explicitly compute the envelope of circles centred on the symmetry set $(x(z), y(z))$, of radius $r(z)$ obtained by substituting for t from (9) into $r^2 = t + \frac{1}{4}$. Hence we can determine the range of values of z for which the envelope is real and the condition for the envelope to have cusps[9]. We find that, for $-\frac{1}{2}\pi < \theta < 0$, the envelope is never real, but for $0 < \theta < \frac{1}{2}\pi$: (i) the envelope is real over a finite range of values of z, of the form $z_1 \leq z \leq \tan\theta$ so that the envelope C is (somewhat surprisingly!) a *closed* curve, and (ii) within the range of values of z for which the envelope C is real there are two values which give a *cusp* on C.

Figures 7 and 8 show some symmetry sets and the resulting reconstructed curves C for various values of $\theta < \frac{1}{2}\pi$. These show exclusively closed curves with cusps. Notice that the 'standard' solution, with symmetry set along the y-axis and C a parabola, does not arise by this method since it has $\theta = \frac{1}{2}\pi$, which means z_0 and b are undefined. The standard solution is thus only a limit of these general solutions as $\theta \to \frac{1}{2}\pi$.
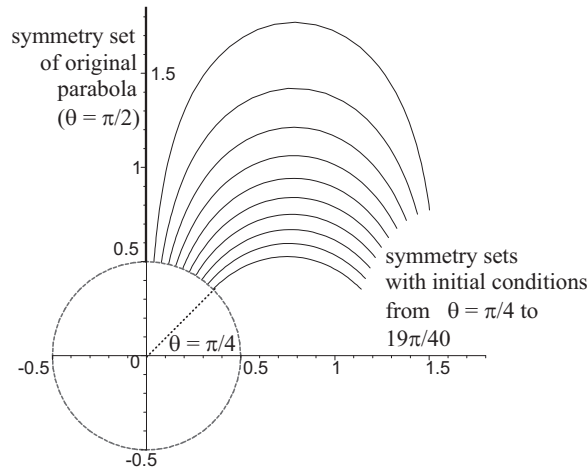

Figure 7: Starting from the midpoint locus of the parabola $y = x^2$ and the corresponding radius function, these are the symmetry sets which result from different choices of initial condition.

The obvious question now arises: how rapidly do these double cusped closed curves tend to the parabola $y = x^2$ as $\theta \to \pi/2$ from below? That is, if we make a small error in the initial condition, how

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[7]We used Maple to find this explicit solution for (8) but are grateful to Victor Goryunov for pointing out how to get the solution ourselves!

[8]Without loss of generality we can take $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$; note that, at $z = z_0$, $dt/dz = -\cos\theta < 0$ so that as t increases from 0, z will *decrease* from z_0.

[9]With the circles given by $F(z, \overline{x}, \overline{y}) = 0$ in current coordinates $(\overline{x}, \overline{y})$ the latter condition is $F = \partial F/\partial z = \partial^2 F/\partial z^2 = 0$; see for example 'points of regression', [3, §5.26]).
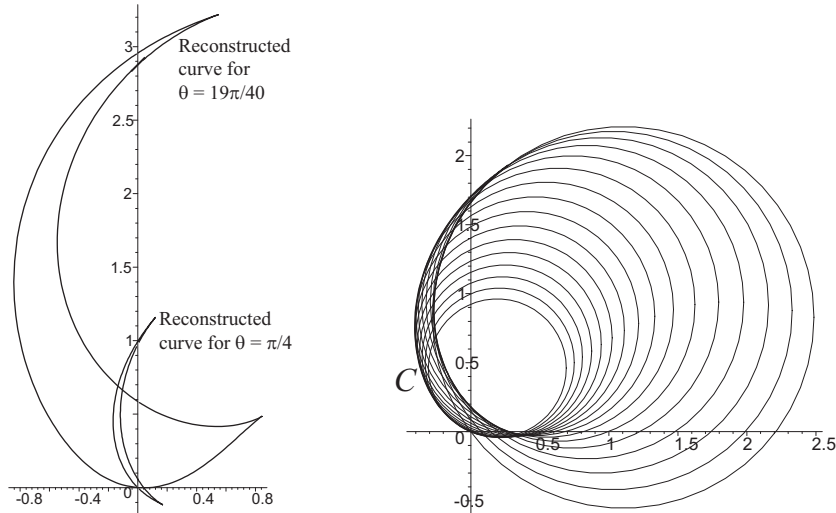
Figure 8: Left: Taking the two extreme values of θ in Figure 7, apart from $\theta = \frac{1}{2}\pi$ itself, these are the reconstructed curves C. Note that these curves are *closed*, and also that they have cusps! Note also that for θ close to $\frac{1}{2}\pi$, the part of C near the origin does resemble the parabola $y = x^2$ which is the reconstructed curve for $\theta = \frac{1}{2}\pi$. Right: The circles centred on the reconstructed symmetry set for the intermediate value $\theta = \frac{3}{8}\pi$. The 'first' and 'last' circles are tangent to the envelope curve C respectively at the origin, and at the third intersection with the vertical axis. Before and after this, the radius is changing too fast for the circles to form an envelope, and in between the circles are bitangent to C. Both the 'first' and 'last' points on C must be vertices, since the contacts with a bitangent circle have coincided there.

badly will this affect the reconstruction? Figure 8, left, shows that with $\theta = 19\pi/40$ the reconstructed curve looks like the original parabola close to the origin, but is still very much a double cusped closed curve away from it. The interval of values of z which give a real envelope (mentioned in (i) above) is of the form $I = (z_1, \tan\theta)$ where $z_1 < 0$ for $\theta \in (0, \pi/2)$. As $\theta \to \pi/2$ from below then $\tan\theta \to \infty$ quite rapidly. However z_1 is much more reluctant to tend to $-\infty$. In fact the condition for a real envelope in this example is $\mathcal{R}(z) > 0$ where

$$\mathcal{R}(z) = -(1+z^2)(\tan^{-1} z + 2a)^2 + 2z(\tan^{-1} z + 2a) + 4b + 1 - \ln(1+z^2) \qquad (11)$$

with a, b are as above, and $z_1 < \tan\theta$ are the zeros of $\mathcal{R}(z)$. Indeed as $\theta \to \pi/2$ from below we do find that $z_1 \to -\infty$ (note that $a \to -\pi/4$ and $b \to \infty$). However calculations suggest that, to make $z_1 = -\tan(\frac{1}{2}\pi - \alpha)$, we need $\theta = \frac{1}{2}\pi - \tau$ where $\tau \approx \alpha \exp(-\pi^2/2\alpha^2)$. For $\alpha = 0.03$, that is $z_1 \approx -33$ we get $\tau = 0.16 \times 10^{-2382}$, a value approximately confirmed by numerical calculation from (11). Thus an infinitesimal error in θ means that the reconstruction is very far indeed from producing the whole parabola with which we started.

6 Conclusions and Further Investigation

In this paper we have investigated the issue of reconstructing a smooth plane curve C given its midpoint locus m and a smooth function r describing the radius of the circle generating each point of m. We chose first to start the reconstruction process from an end point on m since this will always coincide with a vertex on C, and hence give us a point of C from which to start. (For a simple closed curve there will always be at least four vertices to choose.) We showed that the symmetry set c of C (and hence also the medial axis) is retrievable given an initial condition in the form of the endpoint of c corresponding to the vertex on C: there is a one-parameter family of choices for this initial condition and hence a one-parameter family of symmetry sets c consistent with the given midpoint locus and radius function.

Once we have c and r we can reconstruct C as the envelope of circles centred on c, of radius r. However, for the parabola example $C : y = x^2$ it turned out that all the reconstructed symmetry sets

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and curves C were singular with the single exception of the 'original' symmetry set and curve from which we started. It turns out that we need to be *very precise* in selecting the initial condition if we are to recover the correct original symmetry set and curve. Thus what appears to be a clear-cut reconstruction of C, given a single initial condition, is very sensitive to the accuracy of that initial condition.

There is at least one other possibility for a choice of initial condition which is worth considering. At a local widest or narrowest point of C the radius function has a local maximum or minimum, and the midpoint m of the chord and the circle centre c coincide, since $dr/ds = 0$ in (5). What happens if we start integrating the equations (7) from there? This has the advantage that we know the starting point c of the symmetry set since it is the same as m. We shall not give the details here but it appears that, in general, there are only two solutions to the differential equations (7) for deriving the symmetry set and hence two solutions for C. However this method has its own associated issues! One of these is the following: given just the midpoint locus and the radius function can we identify for sure those points m which correspond to local widest or narrowest points of C? Unfortunately there are other places where r has a maximum or minimum, as noted in §3.3, namely points where the symmetry set has a cusp—but the midpoint locus is smooth so we will not notice that. At such points we do not have $m = c$ in general. Is there some way of selecting from among the points of the midpoint locus at which r has an extremum just those which correspond with 'widest-narrowest' points of C? If so, this might give a more effective method of reconstruction not so extremely sensitive to errors in the initial conditions.

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Department of Mathematical Sciences
The University of Liverpool
Liverpool L69 7ZL, England
pjgiblin@liv.ac.uk, paul.warder@gmail.com