Bifurcations of affine equidistants

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Dedicated to the seventieth birthday of Vladimir I. Arnold

1 Introduction

This is the continuation of our study in [9, 10, 11] of singularities arising from families of chords intrinsically attached to a hypersurface $M \subset \mathbb{R}^n$ embedded in an affine space.

The origin of this investigation is the paper [14] of Janeczko. He described a generalization of central symmetry in which a single point—the centre of symmetry—is replaced by the bifurcation set of a certain family of ratios. Then [8] Giblin and Holtom and later Giblin and Zakalyukin investigated the singularities of the envelope of a family of chords (called the Centre Symmetry Set or CSS). Here a *chord* is a straight line passing through a pair of points on the given hypersurface $M \subset \mathbb{R}^n$ at which the tangent hyperplanes are parallel.

The singularities of envelopes of families of chords are examples of Lagrangian and Legendrean singularities [2, 15]. In fact, the CSS construction generalizes that of the family of normals of a surface in euclidean space and the family of affine normals of a surface in affine space. The family of affine equidistants arises as the counterpart of parallels or offsets in Euclidean geometry. An *affine equidistant* for us is the set of points of the above chords which divide the chord segments between the base points with a fixed ratio λ, also called the *affine time*. When λ varies the affine equidistant points move along the chords and their singularities sweep out the CSS. Since λ and $1 - \lambda$ give the same equidistant, the value $\lambda = \frac{1}{2}$ has symmetry and hence is special. The values $\lambda = 0, 1$ are also special in that the equidistant lies in the hypersurface.

In [10, 11] we described a general method for analyzing the local structure of the envelope of chords based on the construction of a generating function depending on parameters whose bifurcation set is tangent to all these chords. We listed the generic singularities of CSS for the cases $n = 2$ and $n = 3$. In the present paper we study the generic bifurcations of affine equidistants when the affine time λ varies.

The most interesting case arises when the manifold M is not convex. In particular, arbitrarily close to an ordinary inflexion ($n = 2$) or a parabolic point ($n = 3$), there are pairs of points of M with parallel tangents. The corresponding chords have an envelope with a limit point at the inflexion or parabolic point itself. In [8] the first case of this phenomenon (a simple inflection on a plane curve) was considered; in general the CSS acquires extra components, and singularities resembling the boundary singularities of Arnold.

We describe in this article the structure of the equidistants in this 'local' case. We find, in fact, that all the generic cases have explicit normal forms up to a natural equivalence ('s-equivalence' below) preserving the equidistants up to local diffeomorphism. For the

case $n = 3$, so that M is a smooth surface in $\mathbb{R}^3$, we distinguish ordinary parabolic points (A_2 of the height function), certain other parabolic points which we call A_2^* points, and cusps of Gauss (A_3 points). In each case $\lambda = \frac{1}{2}, 0, 1$ behave differently from the other values of λ, and for A_3 points there are also, in certain cases, other special values of λ which give different normal forms. See Theorems 3.3 and 3.5 for precise statements of these results.

All constructions are local and all objects are assumed to be C^∞-smooth. We base our study on the methods of [10] where we found normal forms for generic types of singularities of the CSS in three dimensions. Some of those are not simple—the diffeomorphism types depend on functional invariants. Nevertheless, as stated above, *all the generic families of equidistants reduce to normal forms without moduli.*

Another interesting case, which we shall consider elsewhere, is that of two distinct surface patches which share a common tangent plane at a_0, b_0 say. Again we are concerned with chords joining pairs of points a, b close to a_0, b_0 respectively, at which the tangent planes are parallel. In [10, 11] we classified the CSS—the envelope of chords—in this situation, but there are extra difficulties in classifying the evolution of the affine equidistants.

Bifurcations of affine equidistants are of interest [1, 6] in some applications, for example in the description of generic singularities of families of equidistants in Finsler geometry. A translation invariant Finsler metric on an affine space A determines a system of Finsler geodesics. Since the respective Hamiltonian is translation invariant, the geodesics are straight lines. Given an initial data hypersurface $I \subset A$ a family of Finsler equidistants I_t is formed.

For completeness we start with necessary singularity theory background and repeat some constructions from [10].

2 Affine generating families

Let M, a_0 and N, b_0 be two germs at points a_0 and b_0 of smooth hypersurfaces in an affine space $\mathbb{R}^n$. Let $\mathbf{r}_i : U_i^{n-1} \to \mathbb{R}^n, i = 1, 2$ be local regular parametrizations of M and N, where U_i are neighbourhoods of the origin in $\mathbb{R}^{n-1}$ with local coordinates x and y respectively, $\mathbf{r}_1(0) = a_0$, $\mathbf{r}_2(0) = b_0$.

A *parallel pair* is a pair of points $a \in M$, $b \in N$, $a \neq b$ such that the hyperplane $T_a M$ which is tangent to M at a is parallel to the tangent hyperplane $T_b N$.

A *chord* is the straight line $l(a, b)$ passing through a parallel pair:

$$l(a, b) = \{q \in \mathbb{R}^n \mid \quad q = \lambda a + \mu b, \ \lambda \in \mathbb{R}, \ \mu \in \mathbb{R}, \ \lambda + \mu = 1\}.$$

An *affine* (λ, μ)–*equidistant* E_λ of the pair (M, N) is the set of all $q \in \mathbb{R}^n$ such that $q = \lambda a + \mu b$ for given $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\lambda + \mu = 1$ and all parallel pairs a, b (close to a_0, b_0). Note that E_0 is contained in the germ of M at a_0 and E_1 is contained in the germ of N at b_0.

The *extended* affine space is the space $\mathbb{R}_e^{n+1} = \mathbb{R} \times \mathbb{R}^n$ with barycentric coooordinate $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\lambda + \mu = 1$ on the first factor (called *affine time*). We denote by $\pi_2 : w = (\lambda, q) \mapsto q$ the projection of $\mathbb{R}_e^{n+1}$ to the second factor.

An *affine extended wave front* $W(M, N)$ of the pair (M, N) is the union of all affine equidistants each embedded into its own slice of the extended affine space: $W(M, N) = \{(\lambda, E_\lambda)\} \subset \mathbb{R}_e^{n+1}$.

The *centre symmetry set* $\mathrm{CSS}(M, N)$ of a family of affine equidistants (or of the family of chords) of the pair M, N is the image under π_2 of the locus of the critical points of the restriction $\pi_r = \pi|_{W(M,N)}$. A point is *critical* if π_r at this point fails to be a regular projection of a smooth submanifold.

Besides being swept out by the momentary equidistants E_λ, the affine extended wave front is swept out by the lifts to $\mathbb{R}_e^{n+1}$ of chords. Each of them (except for limiting chords where the endpoints coincide, as at a parabolic point of a surface or an inflexion point of a curve) has regular projection to the configuration space $\mathbb{R}^n$. Hence the centre symmetry set $\mathrm{CSS}(M, N)$ is the envelope of the family of chords. Its singularities were studied in [10, 11].

In this paper we are interested in the equidistants and we consider instead the projection π_1 of $\mathbb{R}^{n+1}$ to the first factor.

Definition 2.1 We say that two germs of families of affine equidistants have *equivalent bifurcations* if there is a diffeomorphism germ $\theta : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ mapping one extended affine equidistant to the other and respecting the fibers of π_1. In other words there is a diffeomorphism germ $\widetilde{\theta} : \mathbb{R} \to \mathbb{R}$ of an affine time axis such that $\pi_1 \circ \theta = \widetilde{\theta} \circ \pi_1$.

So families have equivalent bifurcations if via some appropriate reparametrization of time each affine equidistant of one family is diffeomorphic to the respective equidistant of the other family. From the theory of Legendre and Lagrange singularities, singularities of families of equidistants are closely related to singularities of families of functions depending on two groups of parameters (time-space unfoldings), which will be used below and which differ from the space-time unfoldings considered in [10]. Compare e.g. [2, 15, 12].

A germ of a family $F(u, v)$ of functions in variables $u \in \mathbb{R}^k$ with parameters $v = (\lambda, q) \in \mathbb{R}_e^{n+1}$ where $\lambda \in \mathbb{R}$ and $q \in \mathbb{R}^n$ determines the following collection of varieties:

- The *fibrewise critical set* is the set $\mathcal{C}_F \subset \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n$ of solutions (u, v) of the so-called *Legendre equations*:

$$F(u, v) = 0, \qquad \frac{\partial F}{\partial u}(u, v) = 0.$$

- The *big wave front* (or *discriminant*) is $W(F) = \{v = (\lambda, q) \mid \exists u : (u, v) \in \mathcal{C}_F\}.$

- The intersection of the (big) wave front with $\lambda = \mathrm{const}$ is called the *momentary wave front* $W_\lambda(F)$.

- The Legendre subvariety $\Lambda(F)$ is a subvariety of the projectivised cotangent bundle $PT^*(\mathbb{R}^{n+1})$:

$$\Lambda(F) = \left\{ (v, \bar{v}) \,\middle|\, \exists u, (u, v) \in \mathcal{C}_F, \bar{v} = \left[\frac{\partial F}{\partial v}\right] \right\}$$

Here [ ] stands for the projectivisation of a vector.

The family F is called a *generating* family for $\Lambda(F)$. The germ of $\Lambda(F)$ is smooth provided that the Legendre equations are locally regular, i.e. that the standard Morse conditions are fulfilled (see e.g. [2]).

Definition 2.2 Two germs of families $F_i,\quad i = 1, 2$ are *contact*-equivalent (c-equivalent for short) if there exist a non-zero function $\phi(u, v)$ and a diffeomorphism $\theta : \mathbb{R}^k \times \mathbb{R}^{n+1} \to \mathbb{R}^k \times \mathbb{R}^{n+1}$, of the form $\theta : (u, v) \mapsto (X(u, v), V(v))$ such that $\phi F_1 = F_2 \circ \theta$.

In particular, they are *time-space-contact*-equivalent (s-equivalent for short) if there exist a non-zero function $\phi(u, \lambda, q)$ and a diffeomorphism $\theta : \mathbb{R}^k \times \mathbb{R}^{n+1} \to \mathbb{R}^k \times \mathbb{R}^{n+1}$, of the form

$$\theta : (u, \lambda, q) \mapsto (U(u, \lambda, q), \Lambda(\lambda), Q(\lambda, q))$$

such that $\phi F_1 = F_2 \circ \theta$.

The sum of the family $F(u, \lambda, q)$ with a non-degenerate quadratic form in extra variables $y_1, \ldots, y_m$ is called a *stabilization* of F. Two germs of families are *stably* equivalent if they are equivalent (we mean either c, or s equivalence), to stabilizations of one and the same family in fewer variables.

We now recall some standard facts.

- The discriminants of stably c-equivalent families are diffeomorphic. The families of momentary wave fronts of stably s-equivalent families of functions are equivalent.

- Legendre submanifolds L_F of c-stable equivalent families are Legendre equivalent: the germ of θ determines a contactomorphism of the projectivised cotangent bundle $PT^*\mathbb{R}^{n+1}$ which preserves the fibres and maps one Legendre submanifold onto the other.

Standard arguments of singularity theory (see e.g. [2]) imply that versality and infinitesimal versality conditions for c- or s-groups yield stability of wavefronts or of bifurcation of momentary wave fronts: any small perturbation of a versal family produces a wavefront or bifurcation diffeomorphic to the unperturbed one.

We now specialise to the case at hand, of an *affine generating family* ([10]) which describes the affine equidistants and to which we can apply the above results. Let, as above, $\lambda, \mu = 1 - \lambda$ be barycentric cooordinates on $\mathbb{R}$; let $\langle , \rangle$ be the standard pairing of vectors from $\mathbb{R}^n$ and covectors p from the dual space $(\mathbb{R}^n)^\wedge$. Let $\mathbf{r}_1, \mathbf{r}_2$ be local parametrizations of M, N close to a_0, b_0 respectively, defined on neighbourhoods U, V of the origin in $\mathbb{R}^{n-1}$, with coordinates u, v and satisfying $\mathbf{r}_1(0) = a_0, \ \mathbf{r}_2(0) = b_0$.

Definition 2.3 An *affine generating family* $\mathcal{F}$ of a pair M, N is a family of functions in variables $x, y, p \in U \times V \times ((\mathbb{R}^n)^\wedge \backslash \{0\},\ 0)$ with parameters $\lambda, q \in \mathbb{R} \times \mathbb{R}^n$ of the form

$$\mathcal{F}(x, y, p, \lambda, q) = \lambda \langle \mathbf{r}_1(x) - q, p \rangle + \mu \langle \mathbf{r}_2(y) - q, p \rangle.$$

We have shown in [10] the following crucial fact.

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Proposition 2.4 *The wave front $W(\mathcal{F})$ coincides with the affine extended wavefront $W(M, N)$, and the affine equidistants of M, N coincide with the sections of the wave front $W(\mathcal{F})$ by the hyperplanes $\lambda = \text{const}$.*

In fact, $W(\mathcal{F})$ is reducible and consists of three components:
the germ of $N : \lambda = 0, q = \mathbf{r}_2(y)$
the germ of $M : \mu = 0, q = \mathbf{r}_1(x)$
and the germ of the set of (λ, q) such that $\lambda \neq 0, \mu \neq 0$, and q is a point on a chord joining the parallel pair $\mathbf{r}_1(x)$, $\mathbf{r}_2(y)$.

The germ of $W(\mathcal{F})$ at any point (λ_0, q_0) where $q_0 = \lambda_0 a_0 + (1 - \lambda_0)b_0$, corresponding to $x = 0, y = 0$, $[p] = [d\mathbf{r}_1|_{a_0}] = [d\mathbf{r}_2|_{b_0}]$ coincides with the germ of the extended wavefront $W(M, N)$ at this point.

The simplest case is that of two distinct hypersurfaces with parallel but distinct tangent planes. In such a case choose affine coordinates $q = (h, s_1 \ldots, s_{n-1})$ in $\mathbb{R}^n$ such that $a_0 = (c, 0, \ldots, 0)$, $b_0 = (-c, 0, \ldots, 0)$ for some constant c and the hyperplanes tangent to M, a_0 and N, b_0 are parallel to the $h = 0$ coordinate hyperplane.

Take local parametrizations of M in Monge form:

$$\mathbf{r}_1(x) = (c + f(x), x_1, \ldots, x_{n-1}), \qquad \mathbf{r}_2(y) = (-c + g(y), y_1, \ldots, y_{n-1}).$$

Here $x = (x_1, \ldots, x_{n-1}) \in U$; $y = (y_1, \ldots, y_{n-1}) \in V$, where U, V are neighbourhoods of the origin in $\mathbb{R}^{n-1}$ and the smooth functions f, g have zero 1-jet at the origin: $f \in \mathcal{M}_x^2$, $g \in \mathcal{M}_y^2$.

Lemma 2.5 *The germ of the family $\mathcal{F}$ at the point $x = 0$, $y = 0$, $p_0 = (1, 0, \ldots, 0), \lambda = \lambda_0, q_0 = (h_0, 0, \ldots, 0)$, $h_0 = c(\lambda_0 - \mu_0)$ (which corresponds to the point $q_0 = \lambda_0 a_0 + \mu_0 b_0$ on the chord $l(a_0, b_0)$) is stably s-equivalent to the germ of the following family G of functions in $z \in \mathbb{R}^{n-1}$ with parameters $q = (h, s), \lambda$ at the point $z = 0, \lambda = \lambda_0, q = q_0 = (h_0, 0)$:*

$$G = -h + \lambda \left(c + f(s + \mu z) \right) + \mu \left(-c + g(s - \lambda z) \right).$$

Remarks 2.6 1. The lemma is essentially the stabilization lemma from [10], but for the case of $s-$ equivalence. Due to its importance we repeat the proof.

2. The proof of the lemma actually provides not only stable s-equivalence, but also an underlying identity diffeomorphism of the extended parameters λ, h, s: these parameters remain unchanged.

Proof of Lemma 2.5 The family $\mathcal{F}$ differs only by a non-zero factor from its restriction $\mathcal{F}_r$ to the subspace $p = (1, p_1, \ldots, p_{n-1})$ which is

$$\mathcal{F}_r = -h + c(\lambda - \mu) + \lambda f(x) + \mu g(y) + \sum_{i=1}^{n-1}(\lambda x_i + \mu y_i - s_i)p_i.$$

Let $w_i = \lambda x_i + \mu y_i - s_i$ and $z_i = x_i - y_i$ for $i = 1, \ldots, n - 1$. The determinant of the transformation $x, y \mapsto z, w$ equals 1. In the new variables z, w the family takes the form $\mathcal{F}_* = G_*(z, w, \lambda, q) + \sum w_i p_i$ where

$$G_* = -h + c(\lambda - \mu) + \lambda f(x(z, w)) + \mu g(y(z, w)).$$

By D'Adamard's lemma $\mathcal{F}_* = G_*(z, 0, \lambda, q) + \sum w_i \left(p_i + \phi_i(z, w, \lambda, q) \right)$, where ϕ_i are smooth functions which vanish at $w = z = 0$, $\lambda = \lambda_0$, $q = q_0$. Hence $\mathcal{F}_*$ is a stabilization of $G_*(z, 0, \lambda, q)$. Since the restriction to $w = 0$ of the inverse mapping $z, w \mapsto x, y$ yields $x = s + \mu z$, $y = s - \lambda z$ we obtain

$$G(z, \lambda, q) = G_*(z, 0, \lambda, q) = -h + c(\lambda - \mu) + \lambda f(s + \mu z) + \mu g(s - \lambda z).$$

3 Normal forms of families of equidistants at inflections and parabolic points

For systems of chords through parallel pairs of points close to a parabolic point on a single hypersurface $M \in \mathbb{R}^n$ [10] the lemma 2.5 (setting $c = 0$, $f = g$) proves that the generating family $\mathcal{F}$ is stably s-equivalent to the form:

$$G = -h + \lambda f(s + \mu z) + \mu f(s - \lambda z).$$

The germ of the function f (defining the hypersurface) will be taken to have vanishing 1-jet at the origin and degenerate second differential at the origin.

According to the results of the previous section, studying s-equivalence of these families will give us a classification of equivalence classes of families of affine equidistants.

This formula implies the following properties of the family G:

1. The function $G|_{z=0} = -h + f(s)$ vanishes exactly at the hypersurface M.
2. The family G contains no terms linear in z : $\frac{\partial G}{\partial z}|_{z=0} = 0$.
3. The family G is invariant under the symmetry $\lambda \mapsto \mu$, $z \mapsto -z$ which has a fixed point at $\lambda = \mu = 1/2$, $z = 0$.
4. The extended wavefront contains the axis $q = 0$, $\lambda \in \mathbb{R}$, which projects to the origin in q space. So the mapping $\pi|_{W(F)}$ is not a proper map but is a kind of blowing-down map. In contrast to this, the projection π_1 to the affine time axis is generically a proper map. We shall see below the generic affine equidistant bifurcations are stable and simple with respect to s-equivalence.

3.1 Planar inflection case

Starting with the planar case $z \in \mathbb{R}$, $s \in \mathbb{R}$ observe that up to an appropriate affine transformation a germ of a generic plane curve C at an ordinary inflection point is the graph $h = f(s)$ of a function $f(z) = z^3 + c_4 z^4 + c_5 z^5 + \ldots$ with $c_4 \neq 0$.

Hence

$$\begin{aligned} G = -h + f(s) + \lambda \mu \big[(\mu - \lambda) z^3 + (\mu^2 + \lambda^2 - \mu\lambda) c_4 z^4 + \cdots + \\ s z^2 (3 + 6 c_4 s + \ldots) + (\mu - \lambda) s z^3 (4 c_4 + \ldots) \big]. \end{aligned}$$

We shall write $\lambda = \lambda_0 + \varepsilon$ and $\tilde{h} = h - f(s)$. In all cases below the equidistant contains a redundant component which is the original curve C given by $\tilde{h} = 0$. All the reductions to normal form in the following theorem leave the 'time' component ε unchanged.

Theorem 3.1 *(i) For $\lambda_0 \neq 0, \frac{1}{2}, 1$ the germ of $G(z, \lambda, s)$ at $(0, \lambda_0, 0)$ is s-equivalent to the germ at the origin of*

$$H = -\tilde{h} + sz^2 + z^3,$$

which corresponds to a trivial family of affine equidistants, independent of ε. The germs of equidistants are smooth when passing through the origin. Each of them has 3-point contact with C.

(ii) For $\lambda_0 = 0$ or $\lambda_0 = 1$ the normal form is

$$H = -\tilde{h} + \varepsilon(sz^2 + z^3).$$

The equidistant coincides with C for $\varepsilon = 0$, and for $\varepsilon \neq 0$ is family smooth curves having 3-point contact with C at the origin.

(iii) For $\lambda_0 = \frac{1}{2}$ the family is s-equivalent to the normal form

$$H = -\tilde{h} + sz^2 + \varepsilon z^3 + z^4.$$

For $\varepsilon = 0$ this is a 'half-parabola' $(-2z^2, -z^4)$ and for $\varepsilon \neq 0$ it is a curve having 3-point contact with C and a cusp at $z = -\frac{3}{8}\varepsilon$. The cusps trace out the curve $(2z^2, \frac{1}{3}z^4)$ which is the caustic, also called the centre symmetry set (CSS) of C. See Figure 1 for a real example.
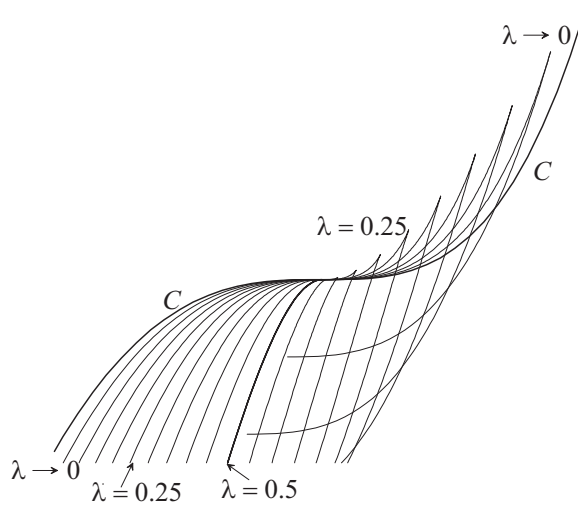

Figure 1: The equidistants for a curve with C an inflexion and various values of λ from 0 to $\frac{1}{2}$ (note that λ and $1 - \lambda$ give identical equidistants). As $\lambda \to 0$ (or 1) the equidistant wraps itself along the curve C. For $\lambda = \frac{1}{2}$ it is a 'half parabola'; for other values of λ it is smooth at the point of contact with C but has a cusp elsewhere.

Remarks 3.2 (1) In case (iii) the caustic and the initial curve $\tilde{h} = 0$ form the bifurcation diagram of Arnold's boundary singularity B_2.

(2) In the case (iii), in [10, Prop. 5.2] we were able to prove only weak stability (stability of the caustic) for this normal form. Here, in contrast, we prove s-stability of the normal

form, thereby obtaining an accurate representation of the family of equidistants up to diffeomorphism. As we shall see, the same s-stability holds also in all the generic cases of surfaces in $\mathbb{R}^3$.

Proof. (i). An appropriate rescaling of z, s and a diffeomorphism $s \mapsto 3s + 6c_4 s^2 + \ldots$ reduce the family to the following one:

$$G = -\tilde{h} + sz^2(1 + A(\varepsilon)) + z^3(1 + B(z, \varepsilon, s))$$

with smooth functions A, B such that $A(0) = 0$, and $B(0) = 0$.

Applying the standard Moser homotopy method it is easy to prove that all such germs are s-equivalent. We give the details of this case below.

Consider a homotopy

$$G_\tau = -\tilde{h} + sz^2(1 + \tau A(\varepsilon)) + z^3(1 + \tau B(z, \varepsilon, s)), \quad \tau \in [0, 1],$$

joining $G_1 = G$ and G_0 - which is the normal form. Try to find a family of s-equivalences reducing G_τ to G_0. In other words we need a family of non-zero functions $P_\tau(z, s, h, \varepsilon)$, and a family of diffeomorphisms of the form

$$\Theta_\tau : (z, s, \tilde{h}, \varepsilon) \mapsto \left(Z_\tau(z, s, \tilde{h}, \varepsilon), S_\tau(s, \tilde{h}, \varepsilon), H_\tau(s, \tilde{h}, \varepsilon), E_\tau(\varepsilon) \right)$$

such that $P_\tau(G_\tau \circ \Theta_\tau) = G_0$ for any $\tau \in [0, 1]$.

Differentiating with respect to τ and dividing by P we get so-called homological equation

$$-\frac{\partial G_\tau}{\partial \tau} = \frac{1}{P} \frac{\partial P}{\partial \tau} G_\tau + \frac{\partial G_\tau}{\partial z} \frac{\partial Z_\tau}{\partial \tau} + \frac{\partial G_\tau}{\partial \tilde{h}} \frac{\partial H_\tau}{\partial \tau} + \frac{\partial G_\tau}{\partial s} \frac{\partial S_\tau}{\partial \tau} + \frac{\partial G_\tau}{\partial \varepsilon} \frac{\partial E_\tau}{\partial \tau}.$$

Notice that all the partial derivatives of G_τ are taken at the point $\Theta_t(z, s, h, \varepsilon)$.

The key idea of the method consists of the following. For a given left hand side function $-\frac{\partial G_\tau}{\partial \tau}$ of the variables $z, s, h, \varepsilon, \tau$ we have to find a decomposition from the right hand side with some smooth functions $\frac{\partial Z_\tau}{\partial \tau}, \frac{\partial H_\tau}{\partial \tau}, \frac{\partial S_\tau}{\partial \tau}, \frac{\partial E_\tau}{\partial \tau}$.

The phase flow of the vector field in z, s, h, ε-space with these components provides the required family of diffeomorphisms Θ_τ. Then knowing $\frac{1}{P} \frac{\partial P}{\partial \tau}$ we can reconstruct family of functions P_τ and therefore, establish the s-equivalence of the families G_τ.

In other words, we have to show that the tangent space

$$T_{G_\tau} \mathcal{A} = \left\{ -\frac{\partial G_\tau}{\partial \tau} \right\}$$

at G_τ to the space $\mathcal{A}$ of all families (which is a $\mathcal{O}_{z, \varepsilon, q}$-module) belongs to the tangent space

$$TO_s(G_\tau) = \left\{ \widetilde{P} G_\tau + \frac{\partial G_\tau}{\partial z} \dot{Z} + \frac{\partial G_\tau}{\partial \tilde{h}} \dot{H} + \frac{\partial G_\tau}{\partial s} \dot{S} + \frac{\partial G_\tau}{\partial \varepsilon} \dot{E} \right\}$$

of the orbit of s-equivalencies through G_τ.

Here $\widetilde{P}(z,s,h,\varepsilon,\tau)$, $\dot{Z}(z,s,h,\varepsilon,\tau)$, $\dot{H}(s,h,\varepsilon,\tau)$, $\dot{S}(s,h,\varepsilon,\tau)$, $\dot{E}(\varepsilon,\tau)$ are arbitrarily germs in respective variables. This condition is called the s-infinitesimal stability (or versality).

Notice that $TO_s(G_\tau)$ contains a subspace

$$T_* = \mathcal{O}_{z,\varepsilon,q}\left\{\frac{\partial G_\tau}{\partial z}z\right\} + \mathcal{O}_{\varepsilon,q}\frac{\partial G}{\partial s} \subset TO_s(G_\tau),$$

Now we will show that $T_* = T_{G_\tau}\mathcal{A}$ and hence s-infinitesimal versality of G_τ holds. Moreover in this case we don't need to modify affine time ε : $E = \mathrm{id}$ since the component $\dot{E}$ can always be chosen equal to zero.

For completeness we state now the Malgrange preparation theorem (see e.g. [3]) which is the main tool to proof similar inclusions. This theorem will be intensively used in each theorem below.

Malgrange Preparation Theorem Let $f : x \mapsto y$ be a germ (at the origin) of a C^∞ mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. Assume that M is a finitely generated module over the algebra $\mathcal{O}_x$ of germs at the origin of smooth functions in x and that the factor module $M/I_f M$ where the ideal I_f is generated by the components of f is isomorphic to $\mathbb{R}$-module with the generators $g_1(x),\ldots,g_k(x)$. Then M is isomorphic to the module over the algebra of composed functions $\mathcal{O}_f = \{h(y_1(x),\ldots,y_m(x))\}$ with the same set of generators $g_1,\ldots,g_k$.

Let now M be the $\mathcal{O}_{z,s,h,\varepsilon,\tau}$ module of germs at $z = s = h = \varepsilon = 0, \tau = \tau_0$ (for any $\tau_o \in [0,1]$) of smooth functions in z,s,h,ε,τ which are divisible by z^2. Consider the mapping $f : (z,s,h,\varepsilon,\tau) \mapsto (s,h,\varepsilon,\tau - \tau_0, \frac{\partial G_\tau}{\partial z}z$. Apparently, the factor module $M/I_f M$ is isomorphic to the factor module of the space $z^2\mathcal{O}_z$ of functions in z only divisible by z^2 over the ideal generated by the z^5. In fact, we can restrict each function to subset $\varepsilon = h = s = 0, \tau = \tau_0$ since the respective coordinate functions belong to the ideal. Clearly the restriction of $\frac{\partial G_\tau}{\partial z}z$ is $z^3 q(z)$ with $q(0) \neq 0$, and the restriction of $\frac{\partial G_\tau}{\partial s}$ equals $z^2 p(z)$ with $p(0) \neq 0$. Hence $M/I_f M$ is generated over $\mathbb{R}$ by the classes of $\frac{\partial G_\tau}{\partial s}$, $\frac{\partial G_\tau}{\partial z}z$, $\frac{\partial G_\tau}{\partial z}z^2$. Now Malgrange's preparation theorem implies that any function $\varphi \in M$ can be represented as the following linear combination with smooth functions C_1, C_2, C_3 :

$$\varphi = \frac{\partial G_\tau}{\partial s}C_1\left(s,h,\varepsilon,\tau,\frac{\partial G_\tau}{\partial z}z\right) + \frac{\partial G_\tau}{\partial z}zC_2\left(s,h,\varepsilon,\tau,\frac{\partial G_\tau}{\partial z}z\right) + \frac{\partial G_\tau}{\partial z}z^2 C_3\left(s,h,\varepsilon,\tau,\frac{\partial G_\tau}{\partial z}z\right).$$

Decomposing a given function germ φ at any τ_0 and then using the compactness of the unit segment $[0,1]$ and appropriate distribution of the unit, we get the decomposition with the coefficients being smooth functions with respect to τ on this segment. The $s-$infinitesimal stability and hence s-stability in this easy case is proven. As we have seen neither modification of ε, nor of h, nor any non-trivial factor P are needed.

The proof of (ii) is similar. From now on we will omit repeating complete details describing only the essential part of the solvability of the homological equation.

Now the space of deformations

$$\mathcal{A} = \left\{\varepsilon z^2 s\hat{A}(s,\varepsilon) + \varepsilon z^3 \hat{B}(z,\varepsilon,s)\right\}$$

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is exhausted by sum of the subspaces of the right orbit generated (over the corresponding algebras of germs of functions) by:

$$z\frac{\partial G}{\partial z} = 2\varepsilon s A(\varepsilon, s)z^2 + 3\varepsilon z^3 B(z, \varepsilon, s);$$

$$G = -h + \varepsilon s A(\varepsilon, s)z^2 + \varepsilon z^3 B(z, \varepsilon, s);$$

$$s\frac{\partial G}{\partial s} = \varepsilon s \tilde{A}(\varepsilon, s)z^2 + \varepsilon z^3 \tilde{B}(z, \varepsilon, s)$$

with smooth functions $A \neq 0$, $\hat{A} \neq 0$, $B \neq 0$ and $\hat{B}$.

To prove (iii) observe that family G is σ-invariant, where the involution $\sigma : (q, \lambda, \mu, z) \mapsto (q, \mu, \lambda, -z)$ transposes λ and μ (and so takes ε to $-\varepsilon$), and reverses the sign of z (in fact it transposes th basic points along a chord). Denote by $\mathcal{O}^\sigma_{z,\varepsilon,q}$ the space of germs at the origin of smooth σ-invariant functions in z, ε, q. According to Malgrange preparation theorem $\mathcal{O}^\sigma_{z,\varepsilon,q}$ consists of composed functions germs, being functions in basic σ-invariants $\varepsilon^2, z\varepsilon, z^2, q$.

Germs at the origin $\varepsilon = \frac{1}{2}(\lambda - \mu) = 0, q = 0, z = 0$ of affine generating families G after some obvious s-transformation G take the form

$$G = h + sz^2 A + \varepsilon z^3 B + z^4 C$$

with non-vanishing at the origin σ-invariant functions $A, B \in \mathcal{O}^\sigma_{\varepsilon,q}$ in parameters ε, q only, and non-vanishing at the origin σ-invariant functions $C \in \mathcal{O}^\sigma_{z,\varepsilon,q}$. Denote by $\mathcal{O}_G$ the space of all such families G.

The tangent space $T\mathcal{O}_G$ to $\mathcal{O}_G$ at any G consist of germs

$$G' = sz^2 A' + \varepsilon z^3 B' + z^4 C'$$

with $A', B' \in \mathcal{O}^\sigma_{\varepsilon,q}$, and $C' \in \mathcal{O}^\sigma_{z,\varepsilon,q}$ which are divisible by z^2.

Up to reversing signs of s, h, z and G itself we may assume the germ G belongs to the connected component of the normalized germ $G_0 = h + sz^2 + \varepsilon z^3 + z^4$.

Following again Moser's method connect G_0 and G by a homotopy G_τ, $\tau \in [0,1]$ and prove that all G_τ are s-equivalent to G_0. Now it is sufficient to show that $T\mathcal{O}_G$ for any G_τ is contained in the tangent space $T\mathcal{O}_s G_\tau$ of the orbit of the action of s-equivalences, which are σ-equivariant. This space has the following form (for shortness we omit suscript τ.

$$TS_G = \mathcal{O}^\sigma_{z,\varepsilon,q}\left\{G, z\frac{\partial G}{\partial z}\right\} + \mathcal{O}^\sigma_{\varepsilon,q}\left\{\frac{\partial G}{\partial h}, \frac{\partial G}{\partial s},\right\} + \mathcal{O}^\sigma_\varepsilon \frac{\partial G}{\partial \varepsilon}.$$

The Malgrange preparation theorem implies that for any $g \in T\mathcal{O}_G$ there is a decomposition

$$g = z\frac{\partial G}{\partial z}a(z, \varepsilon, s) + \varepsilon z^3 b(\varepsilon^2, s) + z^2 c(\varepsilon^2, s)$$

with some smooth germs $a \in O^\sigma_{z,\varepsilon,q}$ $b, c \in O^\sigma_{\varepsilon,q}$

This decomposition for $G - h$ provides

$$G = z\frac{\partial G}{\partial z}a_0(z, \varepsilon, s) + \varepsilon z^3 b_0(\varepsilon^2, s) + z^2 c_0(\varepsilon^2, s) + h$$

with $a_0 \in \mathcal{O}^\sigma_{z,\varepsilon,q}$ $b_0, c_0 \in \mathcal{O}^\sigma_{\varepsilon,q}$ and $b_0(0) \neq 0$, but $c_0(0) = 0$.
Decomposing $\frac{\partial G}{\partial s}$ we get

$$\frac{\partial G}{\partial s} = z\frac{\partial G}{\partial z}a_1(z, \varepsilon, s) + \varepsilon z^3 b_1(\varepsilon^2, s) + z^2 c_1(\varepsilon^2, s)$$

with $c_1(0) \neq 0$.

Hence, classes of 1, G and $\frac{\partial G}{\partial s}$ generates over $O^\sigma_{\varepsilon,q}$ the factor module $T\mathcal{O}_G/\mathcal{O}^\sigma_{z,\varepsilon,q}\{z\frac{\partial G}{\partial z}\}$, and the required inclusion holds. The infinitesimal s-versality condition holds for the family G in the subspace of σ-invariant functions G'.

Moreover, the decompositions do not involve variations of ε, so the resulting s-equivalence is preserving values of affine time ε.

3.2 Space case

Let a germ of a generic surface M be the graph of a function $h = f(s,t)$ $\quad h,\ s,t\ \in \mathbb{R}$. Let

$$f = f_2 + f_3 + \dots, \quad f_k = \sum_{i+j=k} a_{i,j}s^i t^j$$

be Taylor decomposition of f into homogeneous forms.

On a generic surface M parabolic points form a smooth curve. At any of them quadratic form f_2 has rank 1.

At a *generic* parabolic point on the parabolic curve the dual surface has A_2 singularity (cuspidal edge). By an appropriate affine transformation (of s, t plane) the 3-jet of f at such a point can be reduced to the form:

$$A_2 : f(s,t) = s^2 + t^3 + a_{2,1}ts^2 + a_{3,0}s^3 + \dots$$

After this normalization of 3-jet at some isolated points the 4-th order form f_4 can vanish on the line $s = 0$. These special points will be called A_2^* points. The notation A_2 remains for generic points with non-vanishing $f_4\,|_{s=0}$.

In these cases (A_2, A_2^*) the organizing centre of the affine generating family takes the form:

$$G_0 = \lambda\mu \left[x^2 + (\mu - \lambda)\left(y^3 + a_{2,1}x^2 y + a_{3,0}x^3\right) + (\mu^2 - \lambda\mu + \lambda^2)f_4 + \dots\right].$$

Theorem 3.3 *As in the curve case, the redundant component M is part of every equidistant.*

(i) In the cases A_2, A_2^* if $\lambda_0 \neq 0, \frac{1}{2}, 1$ the affine generating family is s-equivalent to the normal form

$$H = -\tilde{h} + ty^2 + y^3$$

which gives a trivial family of smooth germs of affine equidistants at the origin, independent of time ε.

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If $\lambda_0 = 0,\ 1$ *the affine generating family is s-equivalent to the normal form*

$$H = -\tilde{h} + \varepsilon(ty^2 + y^3).$$

(ii) In the case A_2 (that is, $a_{0,4} \neq 0$) at $\lambda_0 = \frac{1}{2}$ the germ of the generating family is s-equivalent to the normal form :

$$H = -\tilde{h} + ty^2 + \varepsilon y^3 + y^4.$$

(iii) In the case A_2^* (that is, $a_{0,4} = 0$) at $\lambda_0 = \frac{1}{2}$ the generating family is s-equivalent to the normal form

$$H = -\tilde{h} + ty^2 + \varepsilon y^3 + sy^4 + y^6,$$

We now describe these cases geometrically.

In all the cases (i) and (ii) the family of equidistants is diffeomorphic to the product of a line with the family which arises in the plane case, as in Theorem 3.1. Thus in case (ii) the cuspidal edge of the equidistant (Figure 2) approaches the parabolic curve on M as affine time tends to $\frac{1}{2}$ and coincides with it completely at this instant. At $\lambda = \frac{1}{2}$ the affine equidistant becomes a smooth surface with a boundary, which coincides with the parabolic curve. The surface is tangent to $M : \tilde{h} = 0$ along the boundary.
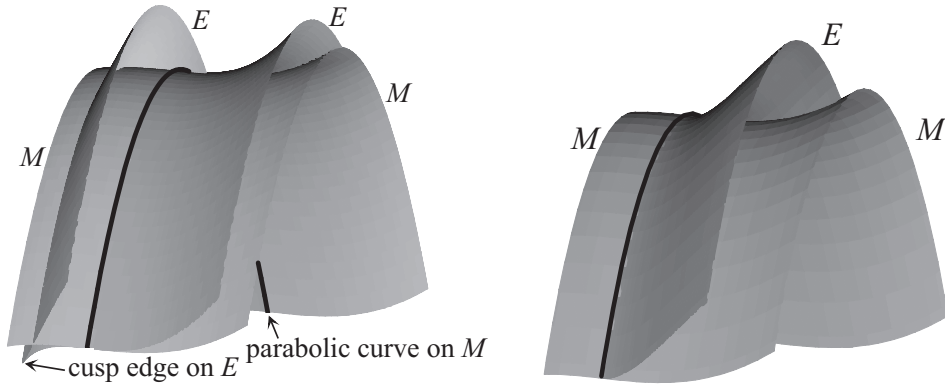

Figure 2: Left: the equidistant $E = E_\lambda$ for $\lambda_0 = \frac{1}{2}$ and $\lambda = \lambda_0 + \varepsilon, \varepsilon \neq 0$. As $\varepsilon \to 0$ the cusp edge comes into coincidence with the parabolic curve on the original surface M and the two sheets of E_λ coincide. Right: $\varepsilon = 0$.

The case (iii) of A_2^* for $\lambda = \frac{1}{2} + \varepsilon$ is quite different. With $\varepsilon = 0$ the equidistant has a cusp edge ending on the boundary of the equidistant, which lies along the parabolic curve of M. For $\varepsilon \neq 0$ the cusp edge splits, one piece ending in a swallowtail point which moves away from the origin carrying a third cusp edge with it. See Figure 4. Using the normal form, the swallowtail point p_0 corresponds to $y = y_0 = (\varepsilon/16)^{1/3}$, which gives $p_0 = (s, t, \tilde{h}) = (-9y_0^4, -9y_0^2, -y_0^6)$. We can check that the equidistant at this point is locally diffeomorphic to a standard swallowtail as follows. Regard H for a fixed ε as an unfolding of a function of y by three unfolding parameters $s, t, \tilde{h}$ (that is, regard H as defining the equidistant as an envelope of planes in $\mathbb{R}$ parametrized by y). Then it is easy to check that the family H is a versal unfolding at p_0 of an A_3 singularity at y_0, for all sufficiently small ε (compare [5, §6.18]).
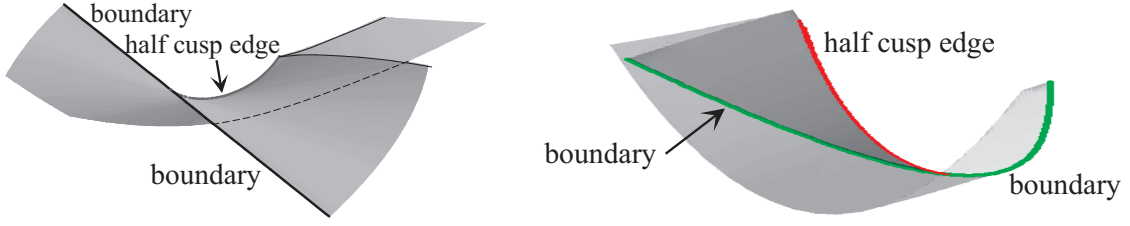
Figure 3: The equidistant E_λ, for $\lambda = \lambda_0 = \frac{1}{2}$, ($\varepsilon = 0$) in the case A_2^* (Theorem 3.3, (iii)). Left: using the normal form; right: a real example, the surface M being $z = x^2 + xy^2 + y^3 + xy^3$; the boundary lies along the parabolic curve on M.
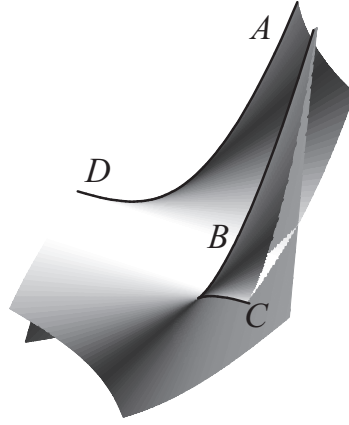

Figure 4: The equidistant E_λ for $\lambda = \frac{1}{2} + \varepsilon$, $\varepsilon \neq 0$ (ε small) in the case A_2^* (Theorem 3.3, (iii)). Note the swallowtail point; the cusp edges A and B become identical as $\varepsilon \to 0$ and the cusp edges C and D join to form the boundary of the equidistant for $\varepsilon = 0$.

Remarks 3.4 (1) Similarly to the plane case, the germs of affine generating families at $q = 0, z = 0, \lambda = \mu = \frac{1}{2}$ are σ-symmetric, and the affine equidistant for this value of affine time is covered twice via the respective Legendre mapping.

(2) Notice that in the case of A_2^* the generic caustic is diffeomorphic to the image of a half plane under the simple mapping $\tilde{A}_4$ (from the classification by D.Mond of mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$):

$$(t, \tau) \mapsto \left(t, \ \tau^2, \ \tau^3 + t^5\tau\right).$$

The criminant is $\tilde{h} = 0$ [10].

Proof. Recall that the organizing centre and terms linear in parameters $\tilde{h}, s, t$ from

the affine generating family are given by the formula:

$$G_1 = -h + \lambda\mu \left[f_2(x,y) + \sum_{i>2} f_i(x,y)(\mu^{i-1} + (-1)^i \lambda^{i-1}) \right.$$

$$\left. + \sum_{i>2} (\frac{\partial f_i(x,y)}{\partial x} s + \frac{\partial f_i(x,y)}{\partial y} t)(\mu^{i-2} + (-1)^{i-1}\lambda^{i-2}) \right].$$

When $\lambda_0\mu_0 \neq 0$ (also when $\lambda_0 = 0, 1$ but dealing with non-redundant components) the variable x can be eliminated by stabilization:

$$\frac{1}{\lambda\mu}\frac{\partial G_1}{\partial x} = 2x + 2a_{2,1}sy + 6a_{3,0}xs + 2a_{2,1}tx$$

$$+(\mu - \lambda)\left[2a_{2,1}xy + 3a_{3,0}x^2 + \dots\right]$$

$$+(\mu^2 + \lambda^2 - \mu\lambda)\left[\sum_{i+j=3} a_{i,j}ix^{i-1}y^j + \dots\right] + \dots.$$

Solving the latter for x and substituting the result into the expression of the family G provides the family in y, q, λ only with the following low degree terms

$$\hat{G} = -\tilde{h} + \mu\lambda\left[(\mu - \lambda)y^3 + (\mu^2 + \lambda^2 - \mu\lambda)a_{0,4}y^4 + 3ty^2 + \dots\right].$$

Clearly, the reduced family remains symmetric under the involution $\sigma : (s, t, \mu, \lambda, y) \mapsto (s, t, \lambda, \mu, -y)$.

If $a_{0,4} \neq 0$ or ($a_{0,4} = 0$ but $\lambda_0 \neq \mu_0$) the reduced family has the form already considered in the Theorem 3.1. Hence the result in **I**, and **II** cases follows.

In the remaining A_2^* case $\mu = \frac{1}{2} - \varepsilon$, $\lambda = \frac{1}{2} + \varepsilon$ and $a_{0,4} = 0$ the family which is σ invariant after an appropriate s-equivalence takes the following form:

$$\hat{G} = -\tilde{h} + \left(ty^2 + \varepsilon A(\varepsilon^2, s, t)y^3 + sB(\varepsilon^2, s, t)y^4 + \varepsilon C(\varepsilon^2, s, t)y^5 + D(\varepsilon^2, s, t)y^6 + \dots\right).$$

Here $\dots$ stands for terms divisible by y^7 and the functions A, B, C, D do not vanish at the origin generically. For example, $D = a_{0,6} + \frac{1}{4}(\mu^2 + \lambda^2 - \lambda\mu)^2 a_{1,3}^2$.

Similarly to the proof of the proposition 3.1 it is sufficient to show the inclusion of tangent spaces $T\mathcal{O}_{\hat{G}} \subset TS_{\hat{G}}$. The first one has the following representation:

$$T\mathcal{O}_{\hat{G}} \subset \mathcal{O}_{y,\varepsilon,q}^\sigma y^6 + \mathcal{O}_{\varepsilon,q}^\sigma \left\{y^2, y^4, \varepsilon y^3, \varepsilon y^5\right\}$$

while the second takes the form

$$TS_{\hat{G}} = \mathcal{O}_{y,\varepsilon,q}^\sigma \left\{\hat{G}, y\frac{\partial\hat{G}}{\partial y}\right\} + \mathcal{O}_{\varepsilon,q}\left\{\frac{\partial\hat{G}}{\partial h}, \frac{\partial\hat{G}}{\partial s}, \frac{\partial\hat{G}}{\partial t},\right\} + \mathcal{O}_\varepsilon \frac{\partial\hat{G}}{\partial\varepsilon}.$$

Again according to Malgrange preparation theorem for any $g \in T\mathcal{O}_{\hat{G}}$ there is a decomposition

$$g = y\frac{\partial \hat{G}}{\partial y}a(y, \varepsilon, s, t) + \varepsilon y^5 b(e^2, s, t) + y^4 c(\varepsilon^2, s, t) + \varepsilon y^3 d(\varepsilon^2, s, t) + y^2 e(\varepsilon^2, s, t)$$

with some smooth germs $a \in O^\sigma_{y,\varepsilon,q}$ $b, c, d, e \in O^\sigma_{\varepsilon,q}$.

This decomposition for $\hat{G} - h$ provides

$$\hat{G} = y\frac{\partial \hat{G}}{\partial y}a_0(y, \varepsilon, s) + \varepsilon y^5 b_0(\varepsilon^2, s, t) + y^4 c_0(\varepsilon^2, s, t) + \varepsilon y^3 d_0(\varepsilon^2, s, t) + y^2 e_0(\varepsilon^2, s, t) + h$$

with $a_0 \in \mathcal{O}^\sigma_{y,\varepsilon,q}$ $b_0, c_0 \in \mathcal{O}^\sigma_{\varepsilon,q}$ and $d(0) \neq 0$.

Take now the decomposition for $y^2\hat{G} \in T\mathcal{O}_{\hat{G}}$:

$$y^2\hat{G} = y\frac{\partial \hat{G}}{\partial y}a_*(y, \varepsilon, s) + \varepsilon y^5 b_*(\varepsilon^2, s, t) + y^4 c_*(\varepsilon^2, s, t) + \varepsilon y^3 d_*(\varepsilon^2, s, t) + y^2(e_*(\varepsilon^2, s, t) + h)$$

which is obtained from the previous one via multiplication by y^2 and forgoing decomposing of terms $\varepsilon y^7 b_0$ and $y^6 c_0$ of higher degrees in y appearing thereafter.

So $b_*(0) \neq 0$, and $d_*(0) = 0$.

Decomposing also $\frac{\partial \hat{G}}{\partial s}$ and $\frac{\partial \hat{G}}{\partial t}$ we get the set of generators 1, $\frac{\partial \hat{G}}{\partial s}$, $\frac{\partial \hat{G}}{\partial t}$, $\hat{G}$, $y^2\hat{G}$ of the $\mathcal{O}^\sigma_{\varepsilon,q}$-module $T\mathcal{O}_{\hat{G}}/\mathcal{O}^\sigma_{y,\varepsilon,q}\{y\frac{\partial \hat{G}}{\partial y}\}$. Hence the required inclusion holds. The infinitesimal s-versality condition holds for the family $\hat{G}$ in the subspace of admissible σ-invariant deformations of $\hat{G}$.

Again, the decompositions do not involve variations of ε, so the resulting s-equivalence is preserving values of affine time ε in this case as well.

On the parabolic set of M there are also special isolated points of A_3 type: cusps of Gauss or godrons. We may take the asymptotic direction at the cusp of Gauss to be the line $s = 0$, so that the surface takes the form

$$h = s^2 + a_{3,0}s^3 + a_{2,1}s^2 t + a_{1,2}st^2 + a_{4,0}s^4 + \ldots + a_{0,4}t^4.$$

For exactly type A_3 (nondegenerate cusp of Gauss) we need $a_{1,2}^2 \neq 4a_{0,4}$ and for a smooth parabolic set we need $a_{1,2} \neq 0$. Then the dual surface has swallowtail (A_3 discriminant singularity) at the corresponding dual point.

The 4-jet of h can then by an appropriate affine transformation be reduced to the form:
$$A_3: \quad f = s^2 + st^2 + a_{3,0}s^3 + a_{0,4}t^4 + s\varphi(s, t), \quad a_{0,4} \neq \tfrac{1}{4},$$

for some cubic polynomial $\varphi(s, t)$.

Theorem 3.5 (i) *In the case A_3 the germ of G at point of λ different from $\lambda_0 = 0$, 1, $\frac{1}{2}$ and the values in* (iii) *below is s-equivalent to the family*

$$H = -\tilde{h} + sy^2 + ty^3 + y^4,$$

which is independent of ε.

For $\lambda_0 = 0$ or 1, the family becomes

$$H = -\tilde{h} + \varepsilon(sy^2 + ty^3 + y^4).$$

(ii) *For $\lambda_0 = \frac{1}{2}$ the family is s-equivalent to*

$$H = -\tilde{h} + sy^2 + \varepsilon ty^3 + y^4,$$

provided $a_{0,4} \neq 0$.

(iii) *Provided $0 < a_{0,4} < \frac{1}{3}$, there are also two values of λ_0 symmetric with respect to $\lambda_0 = \frac{1}{2}$ the germ of G at which is s-equivalent to the germ at the origin of the family*

$$H = -\tilde{h} + sy^2 + ty^3 + (\varepsilon + t)y^4 + y^5.$$

(This also requires the two further open conditions $a_{1,3} - 4a_{0,5} + 8a_{0,4}a_{0,5} \neq 0$ and $a_{1,3} + 6a_{1,3}a_{0,4} - 5a_{0,5} \neq 0$.)

We now give some more detail and describe these cases geometrically.

(i) The equidistant is a folded Whitney umbrella. See Figure 5, left. As $\lambda_0 \to 0$ or 1, the umbrella flattens to become the surface M itself, on one side of the parabolic curve.

(ii) The equidistant becomes a smooth surface with boundary along the parabolic curve of M.

(iii) The two values of λ_0 are

$$\lambda^* = \frac{1}{2} \pm \frac{\sqrt{a_{0,4}}}{\sqrt{(1 - 3a_{0,4})}}.$$

For $\varepsilon = 0$ there is an open swallowtail at the A_3 point; see Figure 5, right. Two cusp edges converge on the cusp of Gauss, and meet there in a cusp point, but there is no self-intersection. As ε changes, an ordinary swallowtail forms (see Figure 6). Using the normal form, the swallowtail point p_0 corresponds to $y = y_0 = (\varepsilon/16)^{1/3}$, which gives $p_0 = (s, t, \tilde{h}) = (-9y_0^4, -9y_0^2, -y_0^6(1 - 256y_0^6))$. We can check that the equidistant at this point is locally diffeomorphic to a standard swallowtail as follows. Regard H for a fixed ε as an unfolding of a function of y by three unfolding parameters $s, t, \tilde{h}$ (that is, regard H as defining the equidistant as an envelope of planes in $\mathbb{R}$ parametrized by y). Then it is easy to check that the family H is a versal unfolding at p_0 of an A_3 singularity at y_0,

Proof. Generically, at A_3 point the coefficients a, b, c, k in the Taylor series of $f(s, t) = s^2 + st^2 + as^3 + bt^4 + cst^3 + kt^5 + \dots$ where $\dots$ stand for terms of weighted degree greater than $\frac{5}{4}$ (other than s^3) with weights $\frac{1}{2}$ of s and $\frac{1}{4}$ of t.

Hence the affine generating family near the origin has a low degree terms

$$\begin{aligned}
G &= -h + \lambda\mu\left\{x^2 + (\mu - \lambda)(xy^2 + ax^3) + (\mu^2 + \lambda^2 - \mu\lambda)(by^4 + cxy^3)\right. \\
&\quad \left. + s(y^2 + 3ax^2) + 2xyt + (\mu - \lambda)(scy^3 + 4tby^3 + 3tcxy^2) + \dots\right\}
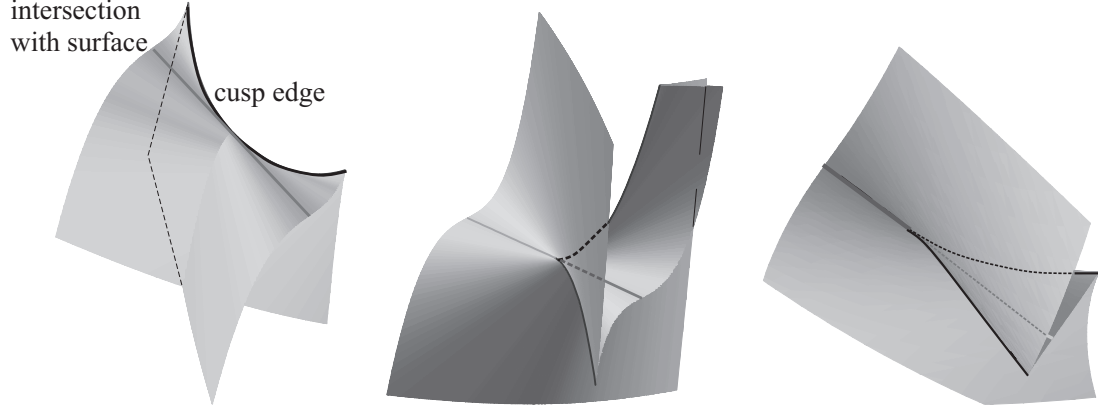\end{aligned}$$

Figure 5: Left: using the normal form of Theorem 3.5, the equidistant for $\lambda \neq \frac{1}{2}, 0, 1$ at an A_3 point (cusp of Gauss on the original surface M) is a folded Whitney umbrella which intersects M with inflexional contact along the parabolic curve. For $\lambda = \frac{1}{2}$ the two sheets of the umbrella collapse together and the equidistant is a smooth surface with boundary on the parabolic curve. Centre: For λ one of the special values λ^*, the equidistant is an open swallowtail, with two cusp edges (dark lines) meeting at the cusp of Gauss on M. The equidistant intersects M as before along the parabolic curve (grey line) but there is no self-intersection. Right: The equidistant for $\lambda^* = \frac{1}{4}$ for the surface $z = x^2 + xy^2 + \frac{1}{7}y^4 + xy^3$. Again the parabolic curve on M, where the equidistant meets the surface with inflexional contact, is shown in grey.

where $\dots$ mean the terms either of order greater than 1 with respect to s, t variables either of high enough weighted order mentioned.

Eliminating x variable by stabilization solve for x the equation

$$0 = \frac{\partial G}{\partial x} = 2x + (\mu - \lambda)(y^2 + 3ax^2) + (\lambda^2 + \mu^2 - \mu\lambda)cy^3 + 6sax + 2ty + 3t(\mu - \lambda)cy^2 + \dots$$

and get

$$x = -\frac{1}{2}(\mu-\lambda)y^2 - frac12(\lambda^2+\mu^2-\lambda\mu)cy^3 + \frac{3}{4}(\mu-\lambda)y^2sa + frac34(\lambda^2+\mu^2-\lambda\mu)scay^3 + \dots$$

After the substitution of this expression into the formula for G we get

$$\hat{G}(y, \lambda, \mu, q) = -h + \lambda\mu \left\{ [(b(\mu^2 + \lambda^2 - \mu\lambda) - \frac{1}{4}(\mu - \lambda)^2]y^4 + (\mu - \lambda)[(\lambda^2 + \mu^2) \right.$$

$$\left. -\frac{1}{8}(\mu^2 + \lambda^2 - \lambda\mu)(6 + 4c)]y^5 + sy^2 + (\mu - \lambda)y^3[(4b - 1)t + sc] + \frac{1}{4}(3as(\mu - \lambda)^2 - 5ct)y^4 + \dots \right\}$$

where dots stand for either terms of degree greater than 1 in s, t or of degree greater than 5 in y.

Now it is clearly visible that if $\lambda \neq 0; \frac{1}{2}; 1$, and $4b-1 \neq 0$, $4b(\mu^2+\lambda^2-\lambda\mu)-(\mu-\lambda) \neq 0$ then the family is s-equivalent to a versal family $g_0 = -h + y^2s + y^3t + y^4$.
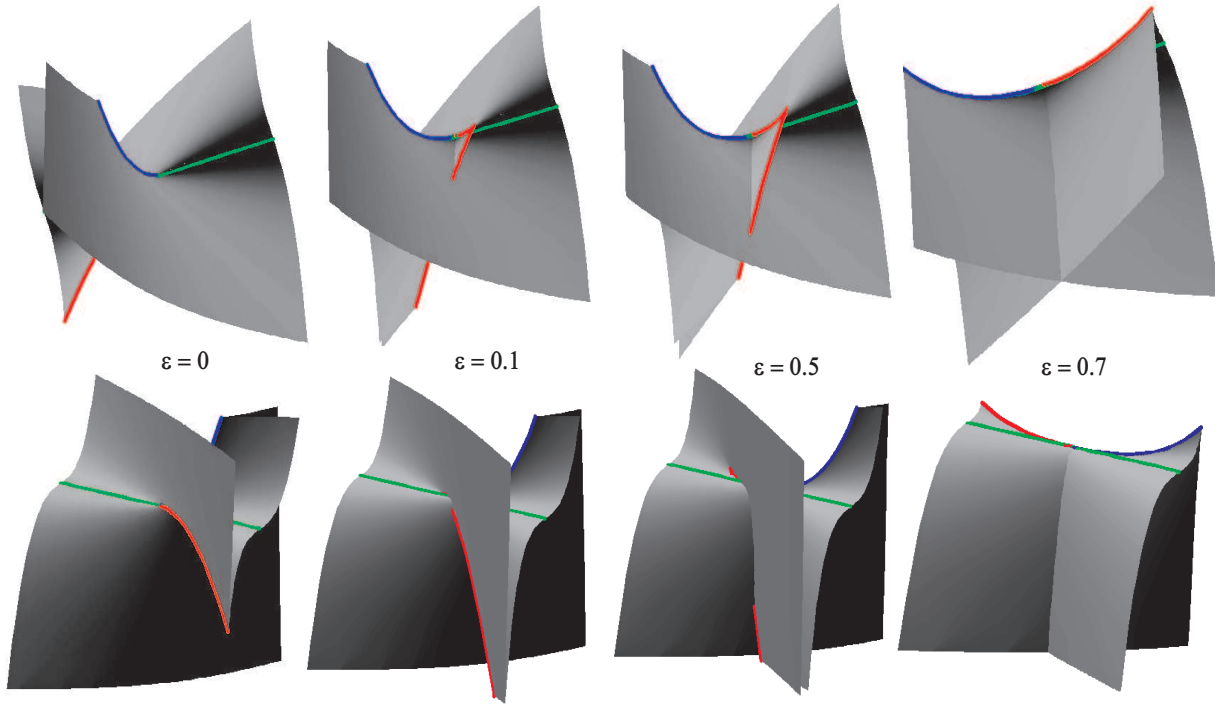
Figure 6: Equidistants for an A_3 point (cusp of Gauss on the original surface M), for $\lambda = \lambda^* + \varepsilon$. Each equidistant is shown from two views. For $\varepsilon = 0$ the equidistant has an open swallowtail singularity, as in Figure 5, right, and as ε increases an ordinary swallowtail appears (see the text) and moves away from the cusp of Gauss. One of the cusp edges persists while the other pierces the equidistant and creates a self-intersection.

If $\lambda = \frac{1}{2}$ the family is σ-symmetric and is s-equivalent (with respect to equivariant mappings) to the normal form $g_1 = -h + y^2 s + \varepsilon t y^3 + y^4$, where as usual $\varepsilon = \frac{1}{2}(\lambda - \mu)$ is the affine time measured from the middle point $\lambda = \mu = \frac{1}{2}$ on a chord. The proof of equivariant stability is analogous to that of the theorem 3.1.

Finally, if the coefficient at y^4 vanishes for some affine time $4b(\mu^2 + \lambda^2 - \mu\lambda) - (\mu - \lambda) = 0$ then generically the derivative of this coefficient with respect to λ does not vanish and the family is s-equivalent to the normal form $g_2 = -h + sy^2 + ty^3 + \varepsilon y^4 + y^5$. In fact, the proof follows from the fact that the classes of y^2, y^3, g_2, $g_2 y^2$ and g_y^3 generates over $\mathcal{O}_{\varepsilon,s,t,h}$ the factor module $y^2 \mathcal{O}_{y,\varepsilon,s,t,h} / \mathcal{O}_{y,\varepsilon,s,t,h} \left\{ \frac{\partial g_2}{\partial y} y \right\}$.

References

[1] A.A.Agrachev, *Methods of control theory in non-holonomic geometry,* Proc. Int. Congr. Math., Zürich, 1994, (Birkhäuser, Basel,1995),12-19.

[2] V.I.Arnold, *Singularities of caustics and wave fronts,* Math. and its Appl., Vol. 62, Kluwer, 2001.

[3] Th.Bröcker and L.Lander, *Differentiable Germs and Catastrophes*, London Mathematical Society Lecture Notes 17, Cambridge University Press 1975.

[4] J.W.Bruce, *A classification of 1-parameter families of map germs $\mathbb{R}, 0 \to \mathbb{R}, 0$, with applications to condensation problems,* J. London Math. Soc. **33** (1986), 375–384.

[5] J.W.Bruce and P.J.Giblin, *Curves and Singularities*, Second Edition, Cambridge University Press 1992.

[6] Buchin Su, *Affine differential geometry,* Science Press, Beijing; Gordon and Breach, New-York, 1983.

[7] A.A.Davydov, V.M.Zakalyukin, *Classification of relative minima singularities*, Geometry and Topology of Caustics, Banach Center Publications, Vol **50**, ed. S.Janeczko and V.M.Zakalyukin, Warsaw, 1999, pp.74–90.

[8] P.J.Giblin and P.A.Holtom, *The centre symmetry set,* Geometry and Topology of Caustics, Banach Center Publications, Vol **50**, ed. S.Janeczko and V.M.Zakalyukin, Warsaw, 1999, pp.91–105.

[9] P.J.Giblin, V.M.Zakalyukin, FA 2004

[10] P.J.Giblin, V.M.Zakalyukin,*Singularities of centre symmetry sets*, Proc. London Math.Soc. 90(2005), n. 3,132-166.

[11] P.J.Giblin, V.M.Zakalyukin, Recognition ...

[12] V.V.Goryunov, *Projections of generic surfaces with boundary* in: Theory of Singularities and its Applications, V.I.Arnold (ed.), Adv.Soviet Math 1, Amer. Math.Soc., Providence, 1990, 157-200.

[13] P.A. Holtom, *Local Central Symmetry for Euclidean Plane Curves,* M.Sc. Dissertation, University of Liverpool, Sept. 1997.

[14] S.Janeczko, *Bifurcations of the center of symmetry, Geom. Dedicata* **60** (1996), 9-16.

[15] V.M.Zakalyukin *Envelopes of families of wave fronts and control theory,* Proc. Steklov Math. Inst. **209** (1995), 133-142.