Additional Material on Corners and Some Other Cases

September 5, 2016

This additional material relates to the book by James Damon, Peter Giblin and Gareth Haslinger: *Local Features in Natural Images via Singularity Theory*, Springer Lecture Notes in Mathematics Vol. 2165 (2016), where it is referred to as [C]. Chapter, section, figure etc. references below are references to this book.

Please note that this material may be added to from time to time. The date of this version is given above.

Contents

1 Additional notes on realizations for a surface with boundary edge, shade and cast shadow of the edge (Case L_3).

These additional notes refer to Chap.6, Figs.6.7 and 6.9 and Chap.13, Sect.13.2, **The Case** L_3, Table 13.1, Figs.13.3 and 13.4.

As in Equation (13.1) we use a quadratic surface $z = x^2 + pxy + \varepsilon_1 y^2$, $\varepsilon_1 = \pm 1$, and a boundary edge given by a linear constraint $y = qx$.

Once the sign of ε_1 is fixed p and q determine the surface. For consistency in what follows we illuminate from the positive y-axis; this makes the cast shadow a physical presence on the surface. We view from the negative x-axis, that is along the direction $(1, 0, 0)$.[1] Reversing the signs of p and q is equivalent to viewing from the opposite direction. We can change the view so that it is along the direction to $(1, 0, \mu)$ for small μ—note that changing to say $(1, \lambda, \mu)$ has the same effect up to first order, that is changes in the y-component produce effects which are negligible compared with changes in the z-component.

In what follows we denote by Σ, K the critical set and kernel line of the projection in the viewing direction at the origin, by E the boundary edge, by S the shade curve and by CS the cast shadow. The view projection itself is denoted by φ. The following table is Table 13.1, with the addition of an abbreviated form for the realization conditions.

Abstract	Description	Realization	Short name
$a - b \neq 0$	Σ not tangent to K	$\mathrm{coeff}(x^2)$ in $f \neq 0$	
$a - 1 \neq 0$	S not tangent to K	$p \neq 0$	
automatic	S is smooth	$\varepsilon_1 \neq 0$	
$a + 1 \neq 0$	CS not tangent to K	$p + \varepsilon_1 q \neq 0$	$A \neq 0$
automatic	E not tangent to K	$q \neq 0$	
automatic	Σ not tangent to E	$2 + pq \neq 0$	$B \neq 0$
$a - 2b + 1 \neq 0$	Σ not tangent to S	$p^2 - 4\varepsilon_1 \neq 0$	$C \neq 0$
$a - 2b - 1 \neq 0$	Σ not tangent to CS	$p^2 - 2\varepsilon_1 + p\varepsilon_1 q \neq 0$	$D \neq 0$
automatic	No two of S, CS, E tangent	$p + 2\varepsilon_1 q \neq 0$	$E \neq 0$
$b - 1 \neq 0$	$\varphi(E), \varphi(S)$ have ordinary contact	$p + p^2 q - 2\varepsilon_1 q \neq 0$	$F \neq 0$
$b + 1 \neq 0$	$\varphi(E), \varphi(CS)$ have ordinary contact	$1 + pq + \varepsilon_1 q^2 \neq 0$	$G \neq 0$
$a^2 - 2ab + 1 \neq 0$	$\varphi(S), \varphi(CS)$ have ordinary contact	$p^3 + p^2 \varepsilon_1 q + 3p\varepsilon_1 + 2q \neq 0$	$H \neq 0$

Remark 1.1 In the table 'Σ not tangent to K' in the surface means the same as 'the apparent contour $\varphi(\Sigma)$ is smooth in the image'. Similarly 'Σ not tangent to S' means the same as '$\varphi(\Sigma)$ has ordinary contact with $\varphi(S)$'. If this latter condition fails, then $\varphi(\Sigma)$ and $\varphi(S)$ have *four*-point contact in general, since they cannot cross in the image. The same applies to Σ and CS or E.

The last four conditions in the table require some additional explanation. If $E = 0$ then *all three* of S, CS, E are tangent on the surface. If $E \neq 0$ then no two of them are tangent. Further, *if* $E = 0$ then, as we would expect, each pair among $\varphi(S), \varphi(CS), \varphi(E)$ has in general 3-point contact in the image. The conditions in the last three rows of the table are separate for each pair among $\varphi(S), \varphi(CS), \varphi(E)$; for example $\varphi(S)$ and $\varphi(CS)$ have 4- point contact if and only if $H = 0$, and similarly for the other pairs. Note that these conditions do *not* require that also $E = 0$.

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[1] According to the convention established in Chap.3, we would then write $\mathbf{L} = (1, 0, 0)$ ('towards the light') and $\mathbf{V} = (-1, 0, 0)$ ('towards the viewer'). But we shall not use $\mathbf{L}, \mathbf{V}$ here.

1.1 Visibility of edge, shade and cast shadow for a surface with boundary edge

First consider $\varepsilon_1 = 1$. Then the regions in the (p,q) plane in Figure 1, left, determine the visibility of the cast shadow and shade curve.

Now consider $\varepsilon_1 = -1$, which implies that the surface is hyperbolic since $p^2 - 4\varepsilon_1 = p^2 + 4 > 0$. In Figure 1, right, the curve in two pieces is $p^2 - pq + 2 = 0$, along which the contour generator is tangent to the cast shadow on the surface, and the 'half- curve' is $p^2 q + p + 2q = 0$, $p \leq 0$, along which the edge and the shade curve have higher contact in the image. In this case, the q axis is not a barrier between different behaviours.
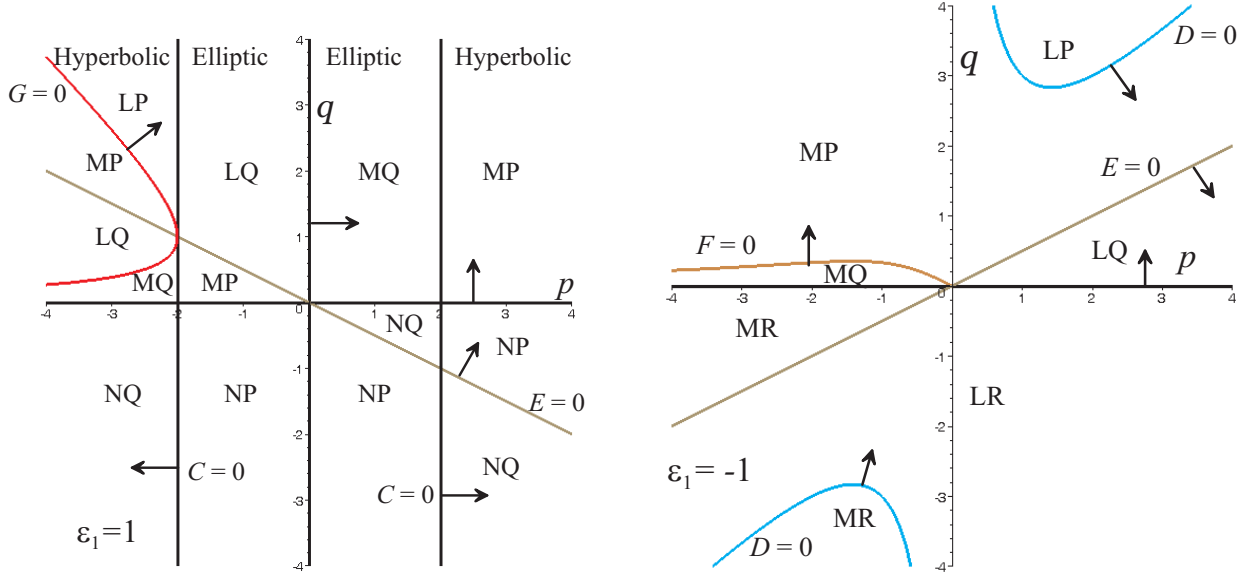

Figure 1: The possibilities for visibility of the cast shadow (CS) and the shade curve (S), from direction $(1, 0, \mu)$ when (left) $\varepsilon_1 = 1$ and when (right) $\varepsilon_1 = -1$. L, M, N refer to CS and P, Q, R to S, as follows. L: CS is visible for $\mu = 0$ and a small piece disappears for small μ with the sign of ε_1; M: a small piece of the CS appears for small μ with the sign of $-\varepsilon_1$; N: CS is never visible; P: S is visible for $\mu = 0$ and a small piece disappears for small μ with the sign of $-\varepsilon_1$; Q: a small piece of S appears for small μ with the sign of ε_1. R: S is never visible. The equations such as $D = 0$ refer to the above table, and the arrows to the directions in which D, etc., are > 0. Note that LP, LQ, MP, MQ occur only for $q > 0$ and LR, MR, NP, NQ only for $q < 0$. From Proposition 1.4 this means that for the first group the whole local edge is visible and for the second group only half the local edge is visible, at the transition moment.

The following subsections indicate how the above results are obtained.

1.2 Visibility of the cast shadow

Let us write the cast shadow as the set of points $(x, \sigma(x), f(x, \sigma(x)))$ on the surface. The function σ is determined by the condition that light in direction $(0, 1, 0)$ strikes the edge and strikes the surface again at the cast shadow point. So, for some t, $(x, qx + t, f(x, qx + t))$ lies on the surface, giving $f(x, qx) = f(x, qx + t)$, and t will be a function of x. Neglecting the trivial solution $t \equiv 0$ we find that $t = -\varepsilon_1 (p + 2\varepsilon_1 q)x + $ h.o.t. and

$$\sigma(x) = -\varepsilon_1 (p + \varepsilon_1 q)x + \text{ h.o.t.} \tag{1}$$

Note that for the point $(x, \sigma(x), f(x, \sigma(x)))$ to lie on the surface which is physically present, we need $\sigma(x) \leq qx$ which says

$$\varepsilon_1 (p + 2\varepsilon_1 q)x \geq 0, \text{ so } x \text{ has the sign of } (p + 2\varepsilon_1 q)\varepsilon_1 = \varepsilon_1 E \text{ [see the table].} \tag{2}$$

In order a point of the cast shadow to be visible, we require that

(i) the cast shadow point is on the 'same side' of the surface as the point where the visual ray strikes the surface. This regards the surface as a thin sheet in $\mathbb{R}^3$ with two sides facing opposite ways.

(ii) either the second point in which the visual ray through a cast shadow point strikes the surface is beyond the cast shadow, or it is a 'virtual' point in that it lies on the half surface which has been removed, that is (2) is violated for such a point.

To test for (i) we can use a normal to the surface, given say by $N(x,y) = (f_x, f_y, -1)$ evaluated at the cast shadow point. Then the 'same side' condition is $(N \cdot (0,-1,0))(N \cdot (1,0,\mu)) > 0$ for small values of x and μ, that is $f_y(\mu - f_x) > 0$. Using the above parametrization of the cast shadow, and taking just the quadratic terms of the resulting condition, this comes to

$$\varepsilon_1(p^2 - 2\varepsilon_1 + p\varepsilon_1 q)(p + 2\varepsilon_1 q)x^2 + (p + 2\varepsilon_1 q)x\mu < 0. \tag{3}$$

Note that x has the sign given in (2), so (3) can be rewritten as

$$(p^2 - 2\varepsilon_1 + p\varepsilon_1 q)x + \varepsilon_1\mu < 0, \text{ i.e. } Dx + \varepsilon_1\mu < 0 \text{ [see the table]}. \tag{4}$$

To test for the visual ray striking the surface again *after* striking the cast shadow, we consider points of the form $(x, \sigma(x), f(x, \sigma(x)) + t(1, 0, \mu)$: this belongs to the surface if $f(x, \sigma(x)) + t\mu = f(x + t, \sigma(x))$, where we require $t > 0$ for the solution other than $t \equiv 0$. Expanding t as a function of x and μ this gives

$$(p^2 - 2\varepsilon_1 + p\varepsilon_1 q)x + \varepsilon_1\mu > 0,$$

which is of course incompatible with (4). Therefore the only possibility for a visible cast shadow is that (4) holds, together with the second strike of the visual ray through a cast shadow point being 'virtual', that is being on the missing half surface $y \geq qx$. This comes to

$$\varepsilon_1 p(1 + pq + \varepsilon_1 q^2)x + q\mu < 0, \text{ i.e. } \varepsilon_1 pGx + q\mu < 0 \text{ [see the above table]}. \tag{5}$$

For $\mu = 0$, that is with initial viewing direction $(1,0,0)$, the condition for all of the cast shadow close to the origin to be visible is therefore that

$$(p^2 - 2\varepsilon_1 + p\varepsilon_1 q)(p + 2\varepsilon_1 q)\varepsilon_1 \quad < \quad 0, \text{ i.e. } \varepsilon_1 DB < 0, \tag{6}$$
$$p(1 + pq + \varepsilon_1 q^2)(p + 2\varepsilon_1 q) \quad < \quad 0, \text{ i.e. } pGB < 0, \tag{7}$$

and when $\mu \neq 0$ the points in the (x, μ)-plane which give some visibility of the cast shadow can be recovered. For example, suppose that $\varepsilon_1 > 0, q > 0, B = p + 2\varepsilon_1 q > 0$ (so that $x > 0$ on the cast shadow by (2)), and (6), (7) both fail. Then the situation is depicted in Figure 2.

Proposition 1.2 *The condition for the cast shadow to be visible, for small x with sign given by (2), from viewing direction $(1, 0, \mu)$ for small μ, is that (4) and (5) hold.*

1.3 Visibility of the shade curve

This is similar to describing the visibility of the cast shadow. We need to find the condition for points of the shade curve to be on the 'same side' of the surface as the point where the visual ray strikes, and we need to examine the second point where the visual ray through a shade point strikes the surface.

As in the case of the cast shadow (2),

$$x \text{ for the shade curve has the sign of } (p + 2\varepsilon_1 q)\varepsilon_1 = \varepsilon_1 E.$$
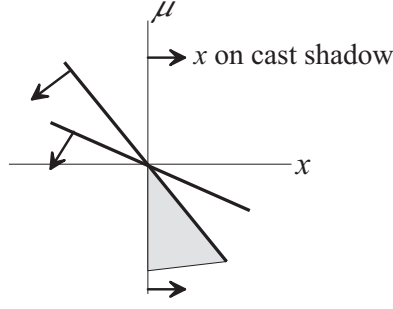
Figure 2: The situation with $\varepsilon_1 > 0, q > 0, E = p + 2\varepsilon_1 q > 0$ and (6), (7) both failing. The cast shadow has $x \geq 0$ and the shaded region represents the points (x, μ), $x \geq 0$ which satisfy both (4) and (5). Then for $\mu > 0$ there is no cast shadow visible but for $\mu < 0$ a small piece of cast shadow becomes visible.
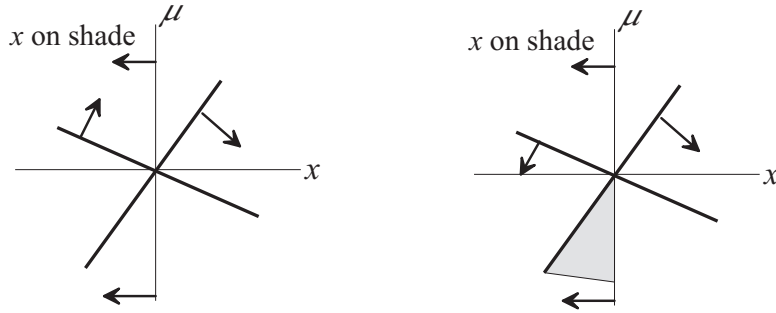

Figure 3: The sides of the lines indicate solutions of (9) and (10) in two cases where $\varepsilon_1 < 0$. Left: $E > 0, F > 0, q < 0$, where there are no solutions for any values of x, μ so the shade curve is always invisible. Right: $E > 0, F < 0, q > 0$, where the solutions intersect in the shaded region, so that for small $\mu < 0$ there will be a small part of the shade curve visible.

Consider the light ray striking the surface tangentially at a point of the shade curve. We will call the *outward normal* N to the surface the normal such that moving the ray in this direction it ceases to intersect the surface locally. We find N is parallel to $\varepsilon_1(f_x, f_y, -1)$, where the derivatives are evaluated at the shade point. The 'same side' condition then becomes $N \cdot (1, 0, \mu) < 0$, that is $\varepsilon_1(\mu - f_x) > 0$. Putting in a parametrization of the shade curve, $y = -px/2\varepsilon_1 + $ h.o.t., we find the 'same side' condition for the shade curve to be

$$(p^2 - 4\varepsilon_1)x + 2\varepsilon_1\mu > 0, \text{ i.e. } Cx + 2\varepsilon_1\mu > 0 \text{ [see the table]}. \tag{8}$$

The condition for the visual ray to strike the surface *after* the shade curve comes to

$$\varepsilon_1(p^2 - 4\varepsilon_1)x + 2\mu > 0, \text{ i.e. } \varepsilon_1 Cx + 2\mu > 0. \tag{9}$$

When $\varepsilon_1 > 0$ the conditions (8) and (9) are identical, but when $\varepsilon_1 < 0$ they are contradictory and then the only possibility for the shade curve to be visible is that (8) holds and the second strike of the visual ray on the surface is 'virtual', that is on the missing half-sheet $y \geq qx$. The condition for this comes to

$$\frac{p + p^2 q - 2\varepsilon_1 q \varepsilon_1}{x} + 2q\mu < 0, \text{ i.e. } \varepsilon_1 Fx + 2q\mu < 0, \tag{10}$$

when x has the sign of $\varepsilon_1 E$ on the shade curve.

Thus when $\varepsilon_1 > 0$ the signs of C and E are important, while for $\varepsilon_1 < 0$ we have $C = p^2 - \varepsilon_1 > 0$ automatically and we need the signs of E, F and q to determine visibility of the shade curve.

As an example, suppose that $\varepsilon_1 = -1, E > 0$ (so that $x \le 0$ for the shade curve), $F > 0, q < 0$. Then the situation is depicted in Figure 3, left, and no shade curve is visible at all. On the other hand if $\varepsilon_1 = -1, E > 0, F < 0, q > 0$ then from Figure 3, right, a small piece of shade curve becomes visible for $\mu < 0$.

Proposition 1.3 *The condition for the shade curve to be visible, for small x with sign given by (2), from viewing direction $(1, 0, \mu)$ for small μ, is that (i) $\varepsilon_1 > 0$, (8) or (ii) $\varepsilon_1 < 0$, (8), (10) hold.*

1.4 Visibility of the edge

This is simpler than the corresponding cases of shade and cast shadow. The edge is the set of points $(x, qx, f(x, qx))$ for small x. For this to be visible from the direction $(1, 0, \mu)$ one of two conditions must hold:
(i) the visual ray striking the edge does so before striking the surface again, or
(ii) the strike of the visual ray besides that on the edge is on the 'missing' part of the surface, $y \ge qx$.
    A general point on the visual ray is $(x, qx, f(x, qx)) + t(1, 0, \mu)$ and this belongs to the surface if and only if $f(x, qx) + t\mu = f(x + t, qx)$. Writing $t = t_{10}x + t_{01}\mu +$ h.o.t. and substituting we find that

$$t = -(2 + pq)x + \mu + \text{ h.o.t..}$$

The edge point given by x is therefore visible for $\mu = 0$ if and only if $(2 + pq)x \le 0$ or $q(2 + pq)x \ge 0$. Thus, if $q > 0$ the whole edge is visible (for all small x and μ) and if $q < 0$ then just those x for which $(2 + pq)x \le 0$ are visible. When $q < 0$ and $\mu \ne 0$ we require $\mu > (2 + pq)x$ for visibility. Hence:

Proposition 1.4 *The edge for all small x is visible when viewing in the direction $(1, 0, \mu)$ for small μ if and only if $q > 0$. When $q < 0$ the edge is visible on one side of the origin and the amount visible increases when $\mu > 0$ and decreases when $\mu < 0$.*

1.5 Properties of the image: directions for contour, cast shadow and shade

One of the features of the image which clearly distinguishes the abstract cases are the directions in which the apparent contour and the images of the shade and cast shadow point along the image of the edge. These can be distinguished in the realization as follows. The images are as follows, where the letters A etc. refer to the table and '...' referring to terms of degree ≥ 2:

$$\text{Shade image: } \left(-\frac{p\varepsilon_1}{2}x + ..., \; ...\right), \qquad x \text{ having the sign of } \varepsilon_1 E \tag{11}$$

$$\text{Shadow image: } \left(-\frac{A\varepsilon_1}{2}x + ..., \; ...\right), \qquad x \text{ having the sign of } \varepsilon_1 E \tag{12}$$

$$\text{Contour image: } \left(-\frac{2}{p}x + ..., \; ...\right), \qquad x \text{ having the sign of } pB \tag{13}$$

From this we deduce the following.

Proposition 1.5 (i) *All three images $\varphi(S), \varphi(CS), \varphi(C)$ point in the same direction along the image of the edge if and only if p, A and BE have the same sign.*
(ii) *$\varphi(S), \varphi(CS)$ point one way and $\varphi(C)$ the other if and only if $p > 0, A > 0, BE < 0$ or $p < 0, A < 0, BE > 0$.*
(iii) *$\varphi(S), \varphi(C)$ point one way and $\varphi(CS)$ the other if and only if $p > 0, A < 0, BE > 0$ or $p < 0, AA > 0, BE < 0$.*
(iv) *$\varphi(CS), \varphi(C)$ point one way and $\varphi(S)$ the other if and only if $p > 0, A < 0, BE < 0$ or $p < 0, A > 0, BE > 0$.*

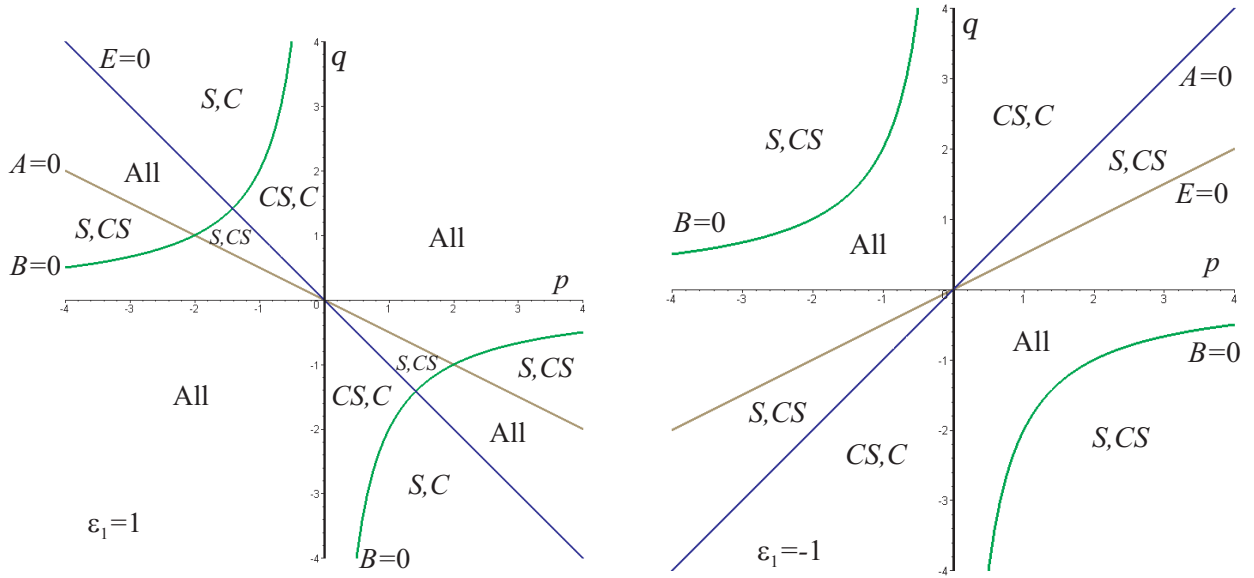These regions can be illustrated on a diagram; see Figure 4.

Figure 4: The various possibilities for the relative directions of the *images* of the shade curve (S), the cast shadow (CS) and the contour (C) for viewing in direction $(1, 0, 0)$. For A, B, E see the table. 'All' means that all the above point in the same direction in the image, while for example CS, C means that the images $\varphi(CS), \varphi(C)$ point in the same direction while the remaining one, $\varphi(S)$, has the opposite direction.

1.6 Properties of the image: which side of the edge the images of shade, cast shadow and contour lie

The images have equations, in the y, z-plane which take the form

$$\text{Edge:} \quad z \;=\; \frac{G}{q^2} y^2 + \dots \tag{14}$$

$$\text{Shade:} \quad z \;=\; -\frac{C\varepsilon_1}{p^2} y^2 + \dots \tag{15}$$

$$\text{Shadow:} \quad z \;=\; \frac{G}{A^2} y^2 + \dots \tag{16}$$

$$\text{Contour:} \quad z \;=\; -\frac{C}{4} y^2 + \dots \tag{17}$$

From these we deduce the following.

Proposition 1.6 (i) *The images $\varphi(S), \varphi(CS), \varphi(C)$ are all on the same side of the image of the edge if and only if E, F, pG all have the same sign.*
(ii) *The images $\varphi(S), \varphi(CS)$ are on one side and $\varphi(C)$ on the other side of the image of the edge if and only if $E > 0, F < 0, pG < 0$, or $E < 0, F > 0, pG > 0$.*
(iii) *The images $\varphi(S), \varphi(C)$ are on one side and $\varphi(CS)$ on the other side of the image of the edge if and only if $E > 0, F > 0, pG < 0$, or $E < 0, F < 0, pG > 0$.*
(iv) *The images $\varphi(C), \varphi(CS)$ are on one side and $\varphi(S)$ on the other side of the image of the edge if and only if $E > 0, F < 0, pG > 0$, or $E < 0, F > 0, pG < 0$.*

These regions can be illustrated on a diagram; see Figure 5.

1.7 Visibility

When considering realizations, for the two cases $\varepsilon_1 = 1, \varepsilon_1 = -1$, the curves $A = 0, \dots, H = 0$, in the table, and the axes $q = 0$ and $p = 0$, divide up the (p, q)-plane into a total of 72 regions,
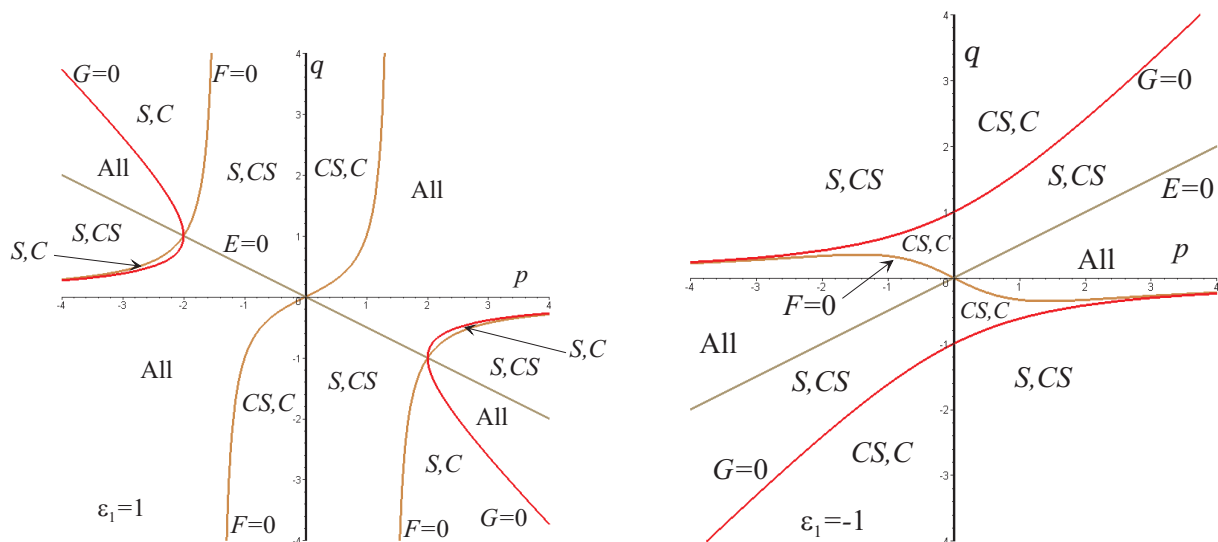
Figure 5: The various possibilities for the relative sides of the edge where lie the *images* of the shade curve (S), the cast shadow (CS) and the contour (C) for viewing in direction $(1, 0, 0)$. For E, F, G see the table. Note that the regions are symmetric with respect to the origin in the (p, q)-plane. 'All' means that all the above lie on the same side of the image of the edge, while for example CS, C means that the images $\varphi(CS), \varphi(C)$ lie on one side in the edge while the remaining one, $\varphi(S)$, lies on the opposite side.

and within each region the visibility such as LP, LQ, etc., the abstract type $1, \ldots, 12, 1' \ldots, 12'$, and the various attributes from Figures 4, 5 are constant. However, when we examine the images, including their unfoldings by moving viewpoint, we find a much smaller number of visually distinct cases, in fact 20 in all. This assumes that in the image we can distinguish between the shade curve S, the cast shadow CS, the apparent contour C and the edge E, but we cannot tell for example which way the invisible continuation of an edge will bend. Each of the 20 cases can be realized in various ways which correspond with several different abstract types: it is the visibility which makes them the same for a given one of the cases.

We proceed to give diagrams of each of the 20 visually distinct cases, together with information about which abstract types are associated with each one. Concrete examples of surfaces exhibiting these cases follow in §1.8.

In Figures 6 and 7 the 20 cases of visibility are displayed in the same way as the abstract cases of Figures 13.3 and 13.4 of Chap.13. Green lines are apparent contours (C), red lines are cast shadow curves (CS), thin black lines are shade curves (S) and the thick vertical black line is the edge (E). Each row represents an unfolding by moving the viewpoint. At the left of each row is the visibility type, as in Figure 1, and at the right is a bracket containing the possible abstract types which give the displayed visual image, taking into account all occlusions. The notation $(a; b, c, \ldots)$ means that the displayed diagram uses type a, while types $b, c, \ldots$ can also give the same visual image.

1.8   Examples

There now follow examples of these various behaviours, in Figures 8, 9 and 10, which show in all cases the transitional moment, drawn schematically in the centre of each row in Figures 6, 7. Note that, in most cases, there are other geometrical surfaces giving the same qualitative image, including the unfoldings, as noted above.
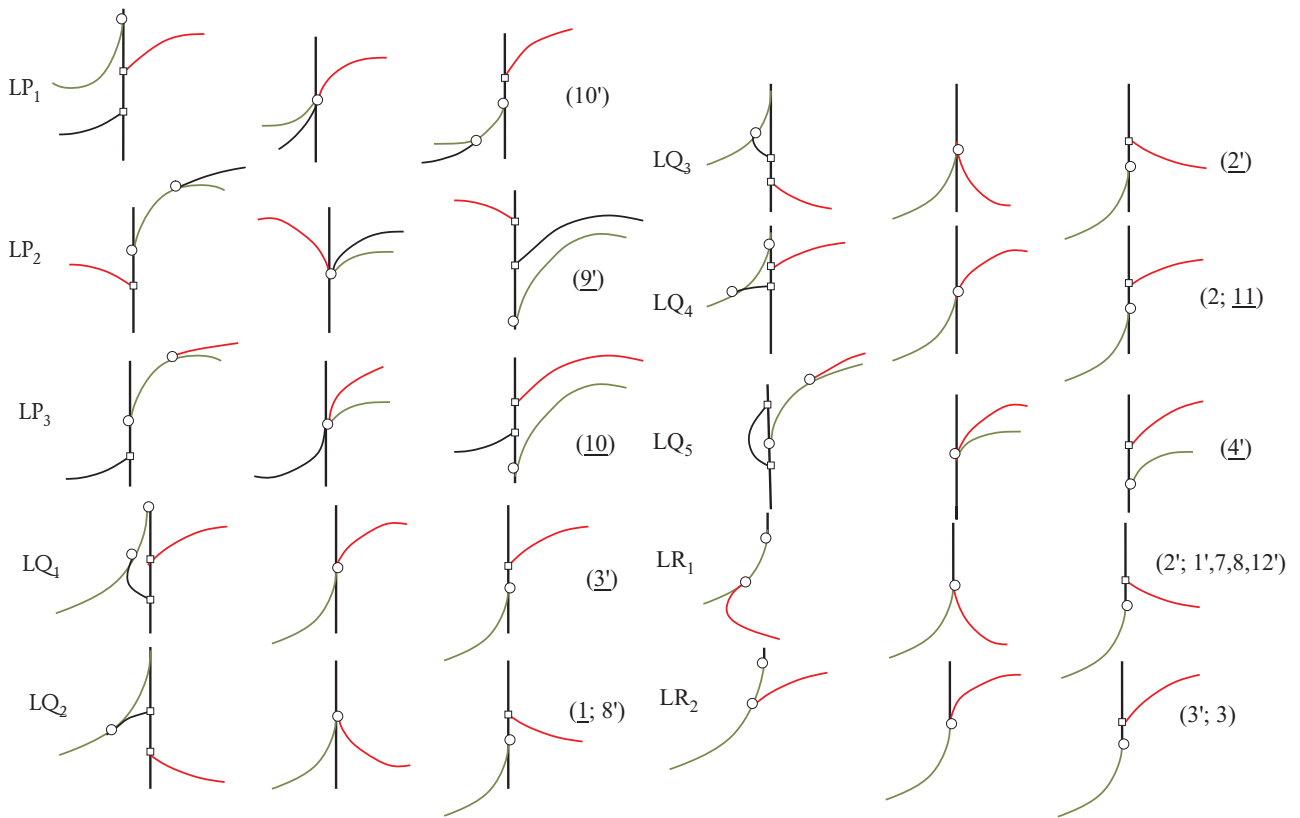
Figure 6: The first ten cases, drawn schematically but taking into account all occlusions. See the text. The underlining of a number indicates that this was used to illustrate the corresponding case in Figure 13.3 or 13.4 in Chap.13.
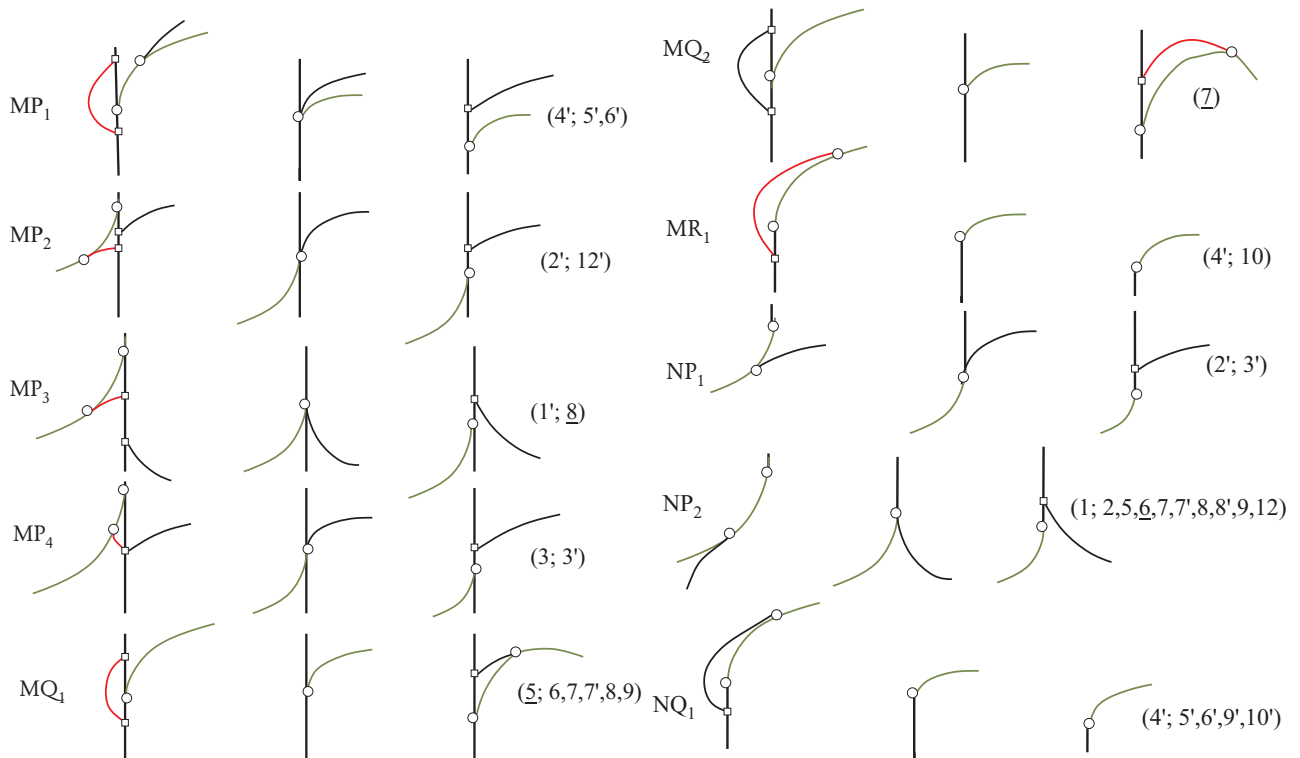

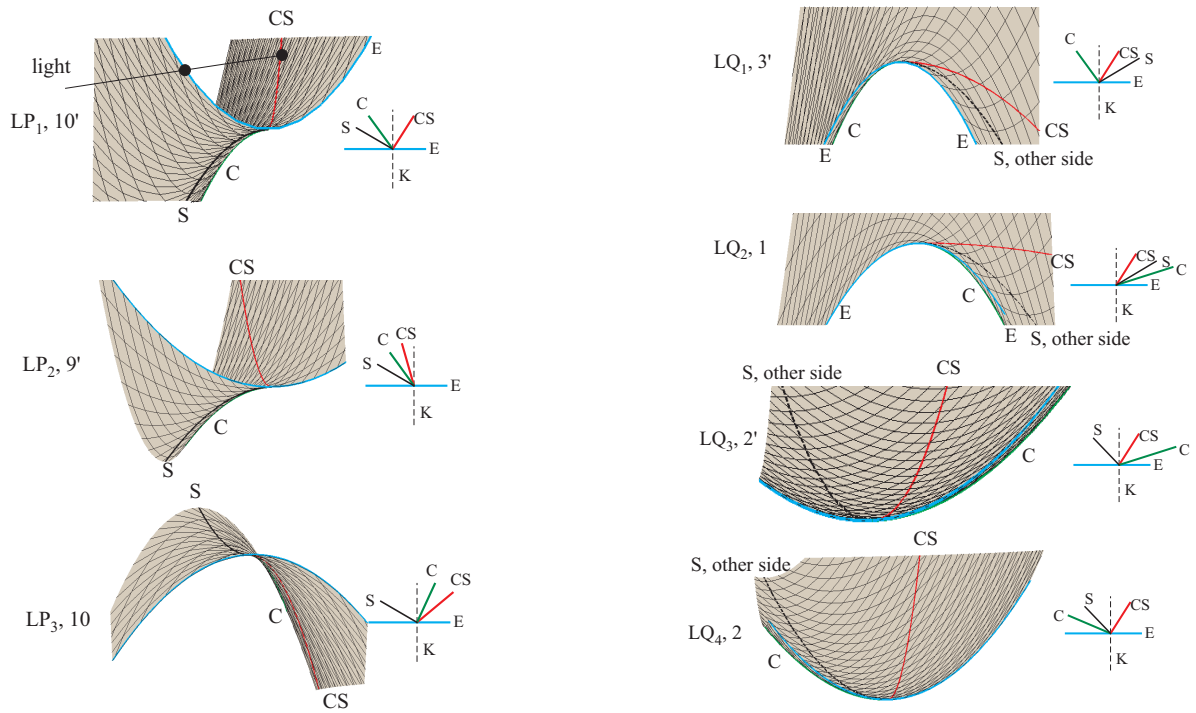Figure 7: The second group of ten cases, as in the previous figure.

Figure 8: Examples of cases in Figure 6, with the abstract type (Fig.6.9 of Chap.6) of this example also noted. At the right of each image is a schematic drawing of the tangent plane to the surface showing, at the transition moment, the kernel of the view projection. 'Other side' means that, regarding the surface as a thin sheet, the curve in question is on the side of that sheet invisible to the viewer.
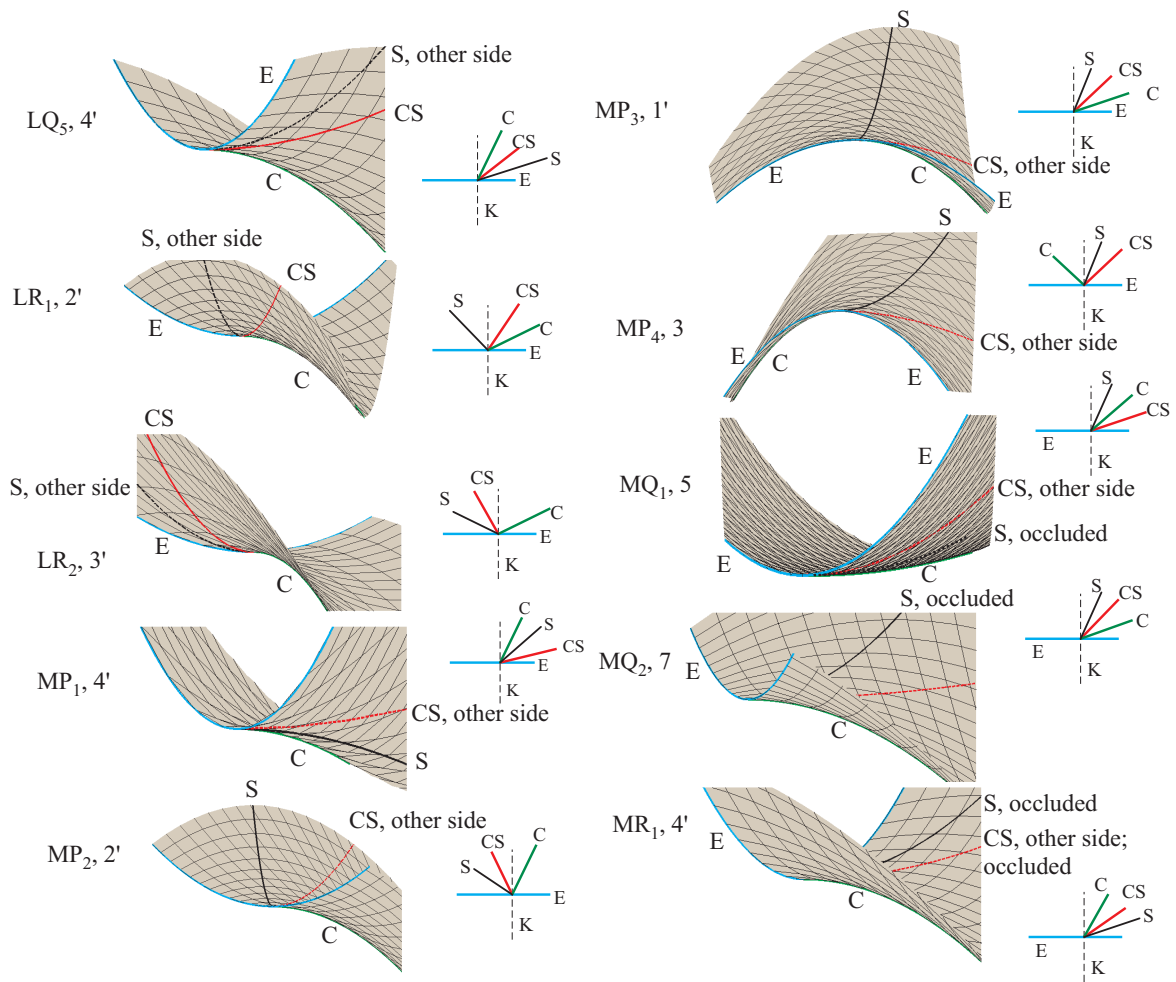
Figure 9: Examples continued from Figure 8. 'Occluded' means that the curve in question is on a part of the surface which is occluded by a nearer part. This can happen when the curve is on the 'side' of the surface facing the viewer or, as noted on the 'other side'. In the single case MQ_1, the occluded curve is indicated by a dashed line since it is not otherwise clear where this curve is located.
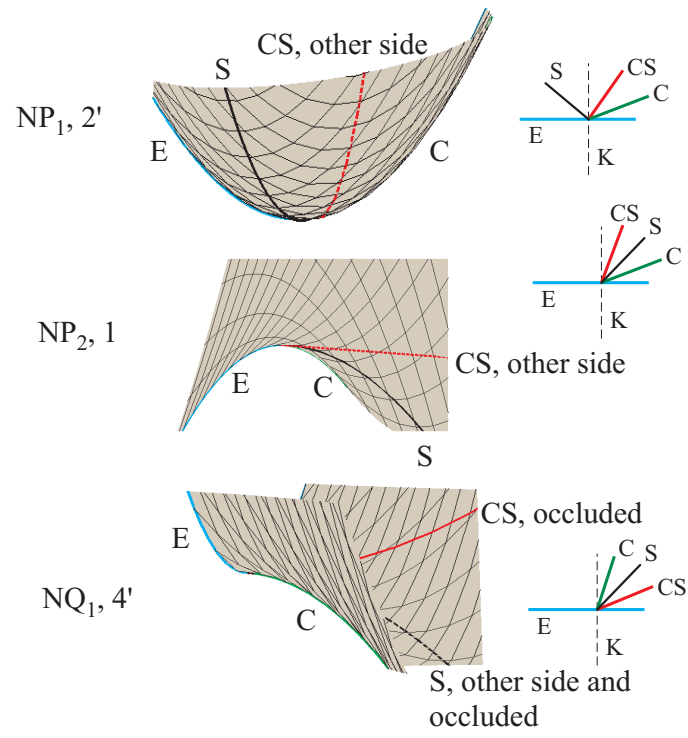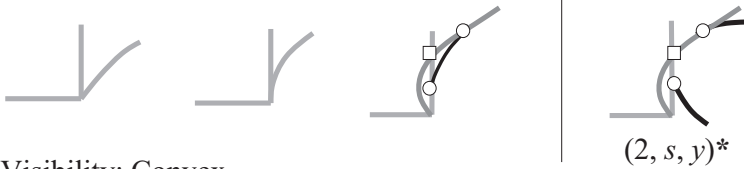
Figure 10: Examples continued from Figures 8 and 9.

2 Visibility diagrams for corner transitions

The diagrams below illustrate the FC and SFC corner transitions referred to in ******

2.1 Corner transitions by type and visibility, without cast shadows

Basic picture $(2, s, y)$

$(2, s, y)$*

Visibility: Convex

(i)

(ii)

Saddle:(i), (ii), together with (iii) and (iv*) below

(iii)

(iv*)

Notch (iii), (iv*) together with

(v)

(vi*)

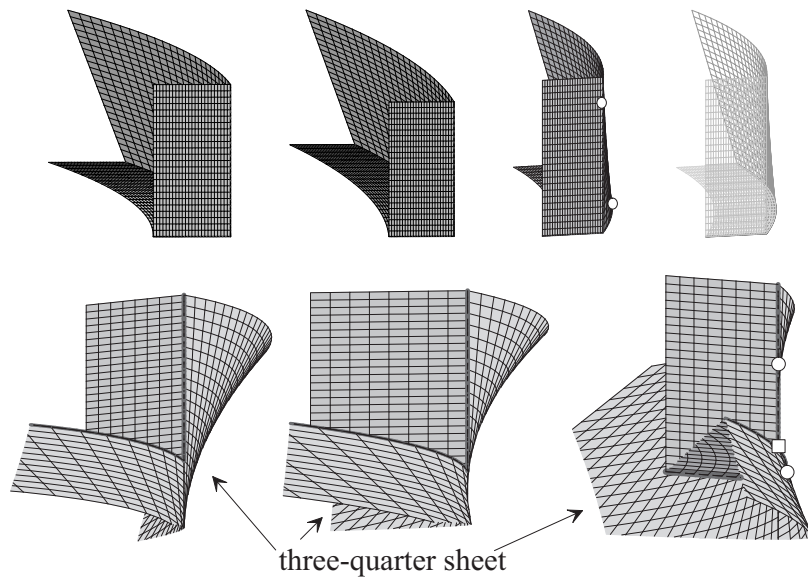Figure 11: Transitions on corners of type $(2, s, y)$

three-quarter sheet

Figure 12: Top row: a transition on a convex corner of type $(2, s, y)$(ii). Note that it is the arrangement of crease edges and contour which is important, not their shapes, when comparing actual examples with the schematic diagrams. The right-hand figure is a wireframe view of the figure to its left, showing the occluded self-intersection of creases in the image. Bottom row: transition on a notch corner of type $(2, s, y)$(iv*).
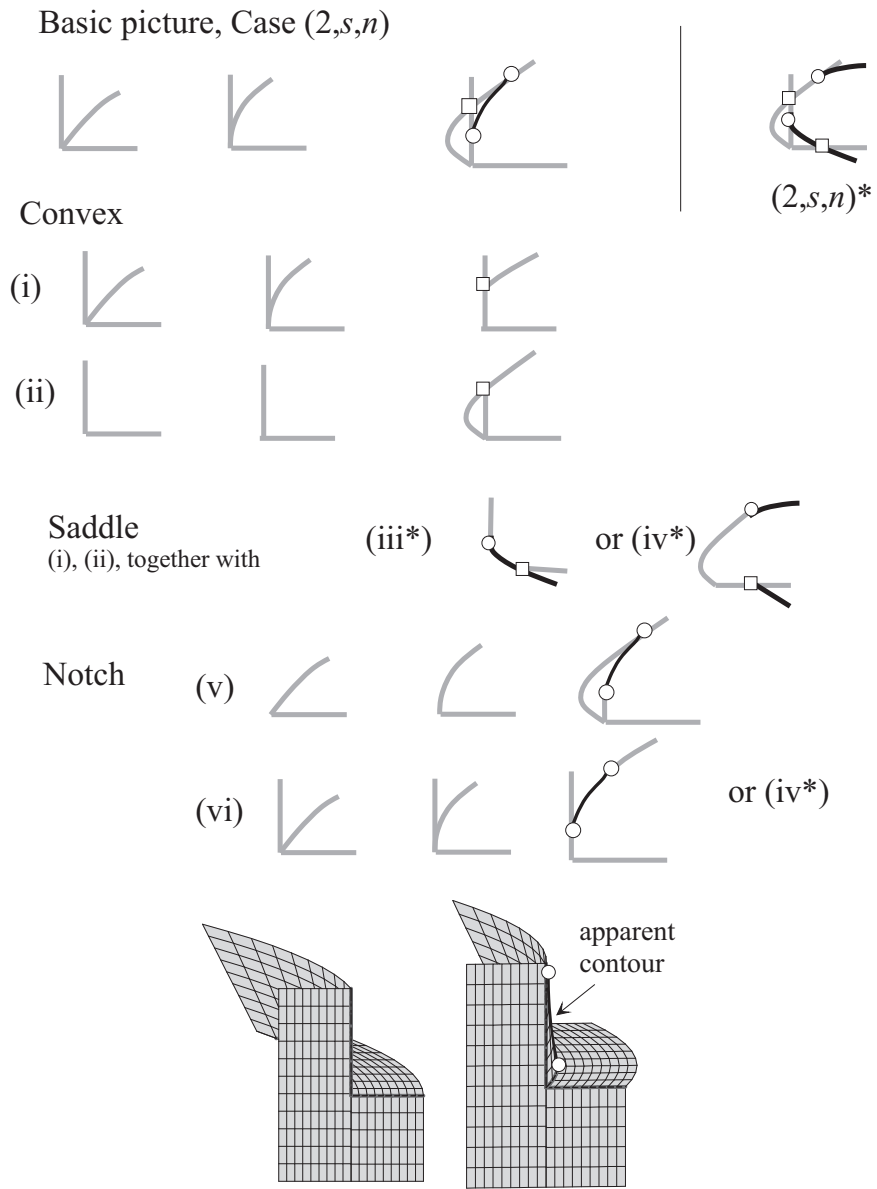
Basic picture, Case (2,*s*,*n*)

Convex

(i)

(ii)

(2,*s*,*n*)*

Saddle
(i), (ii), together with

(iii*)   or (iv*)

Notch   (v)

(vi)

or (iv*)

apparent
contour

Figure 13: Transitions on corners of type $(2, s, n)$; bottom row: type $(2, s, n)(\mathrm{v})$

Basic picture, (2,*o,n*)

(2,*o,n*)*

Visibility:

Convex (i)

Concave (ii)

Saddle (iii*)

Notch: (i*), (ii)
or (iv)

apparent contour   hidden corner

(2,*o,n*)(ii)   (2,*o,n*)(i*)
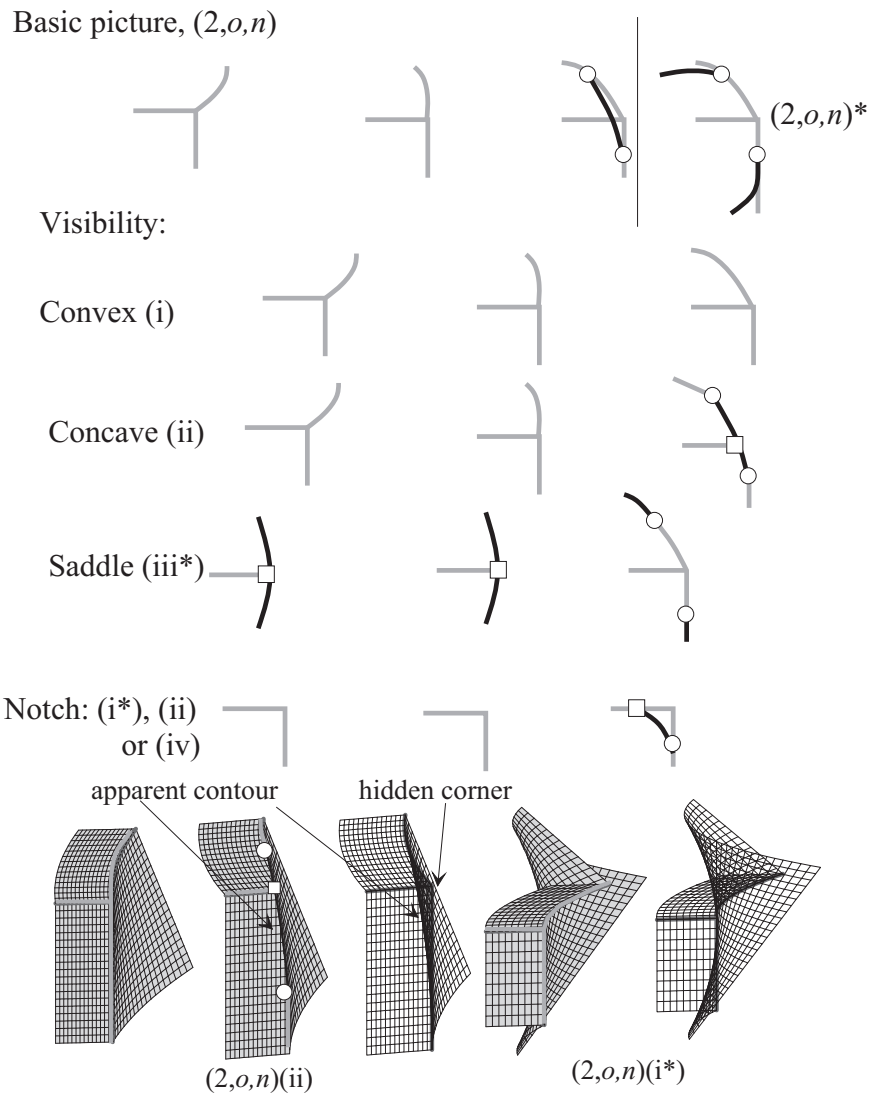
Figure 14: Transitions on corners of type $(2, o, n)$. Bottom row: the first three figures are concave type $(2, o, n)$(ii) with the third a wireframe view of the figure to its left. The fourth figure is a notch of type $(2, o, n)$(i*) where the contour is completely hidden, but shown in the wireframe figure to its right.
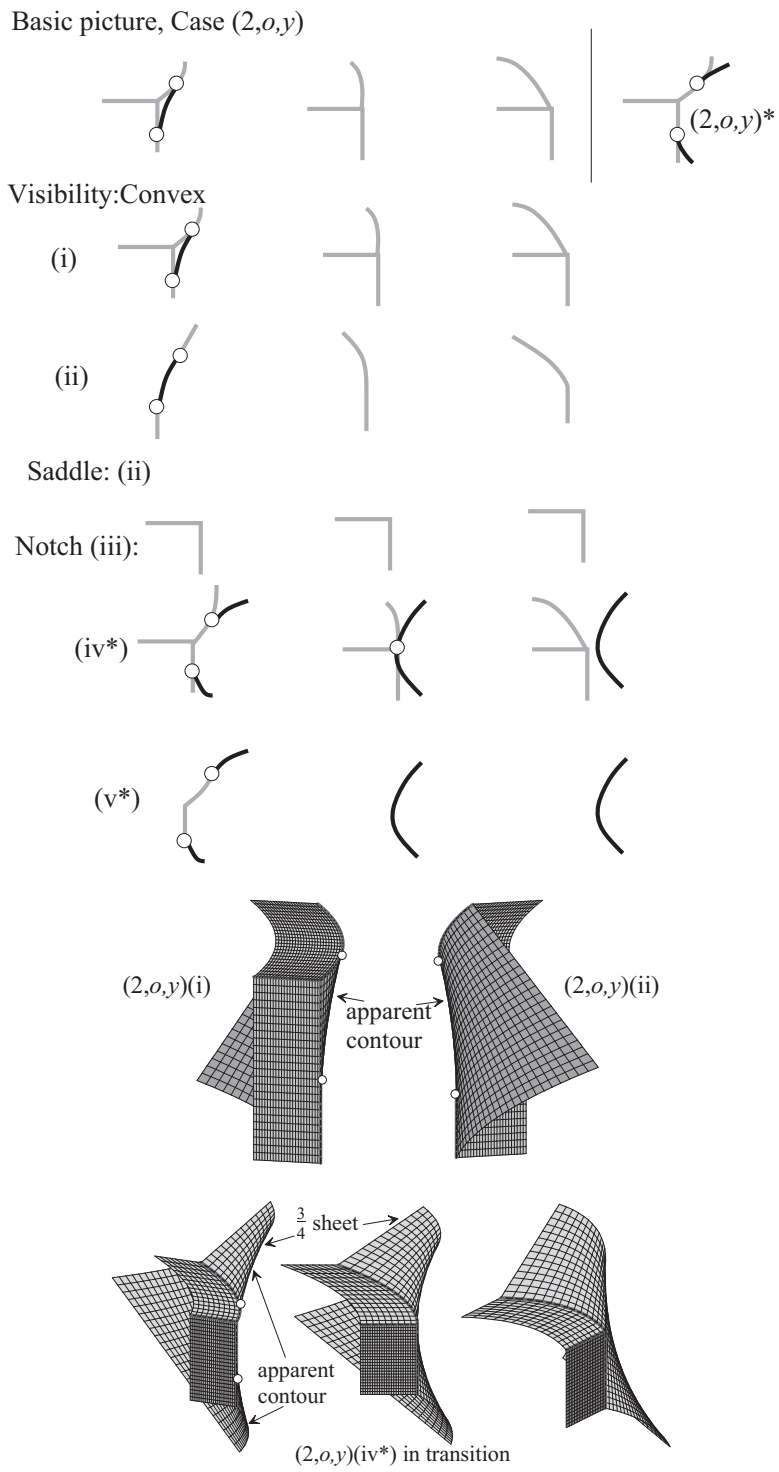
Basic picture, Case (2,*o*,*y*)

(2,*o*,*y*)*

Visibility:Convex

(i)

(ii)

Saddle: (ii)

Notch (iii):

(iv*)

(v*)

(2,*o*,*y*)(i)

apparent
contour

(2,*o*,*y*)(ii)

$\frac{3}{4}$ sheet

apparent
contour

(2,*o*,*y*)(iv*) in transition

Figure 15: Transitions on corners of type $(2, o, y)$

Basic picture, (1,*s,y*)

Visibility: Convex

(i)

(ii)

Saddle: (i), (ii)

Notch (iii)

(iv)

(v*)

(vi*)

apparent
contour

apparent
contour
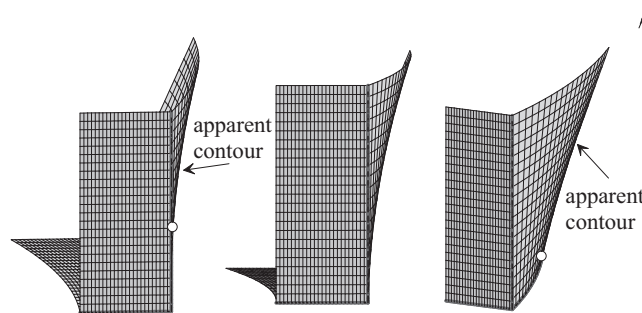
(1,*s,y*)(i) in transition

Figure 16: Transitions on corners of type $(1, s, y)$

Basic picture, (1,*o*,*y*)

Visibility: Convex

(i)

(ii)

Saddle: (ii)

Notch: (iii)

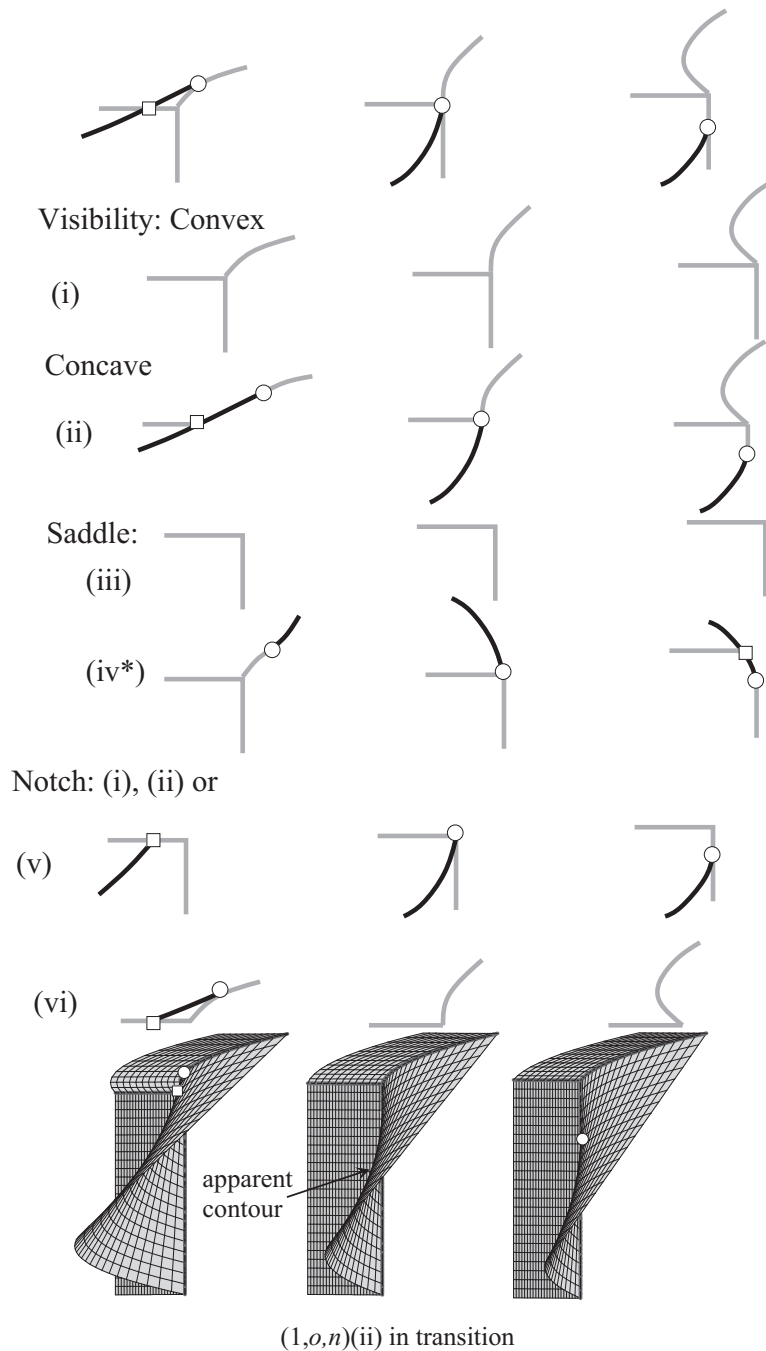(iv*)

(v*)

apparent
contour

(1,*o*,*y*)(i) in transition
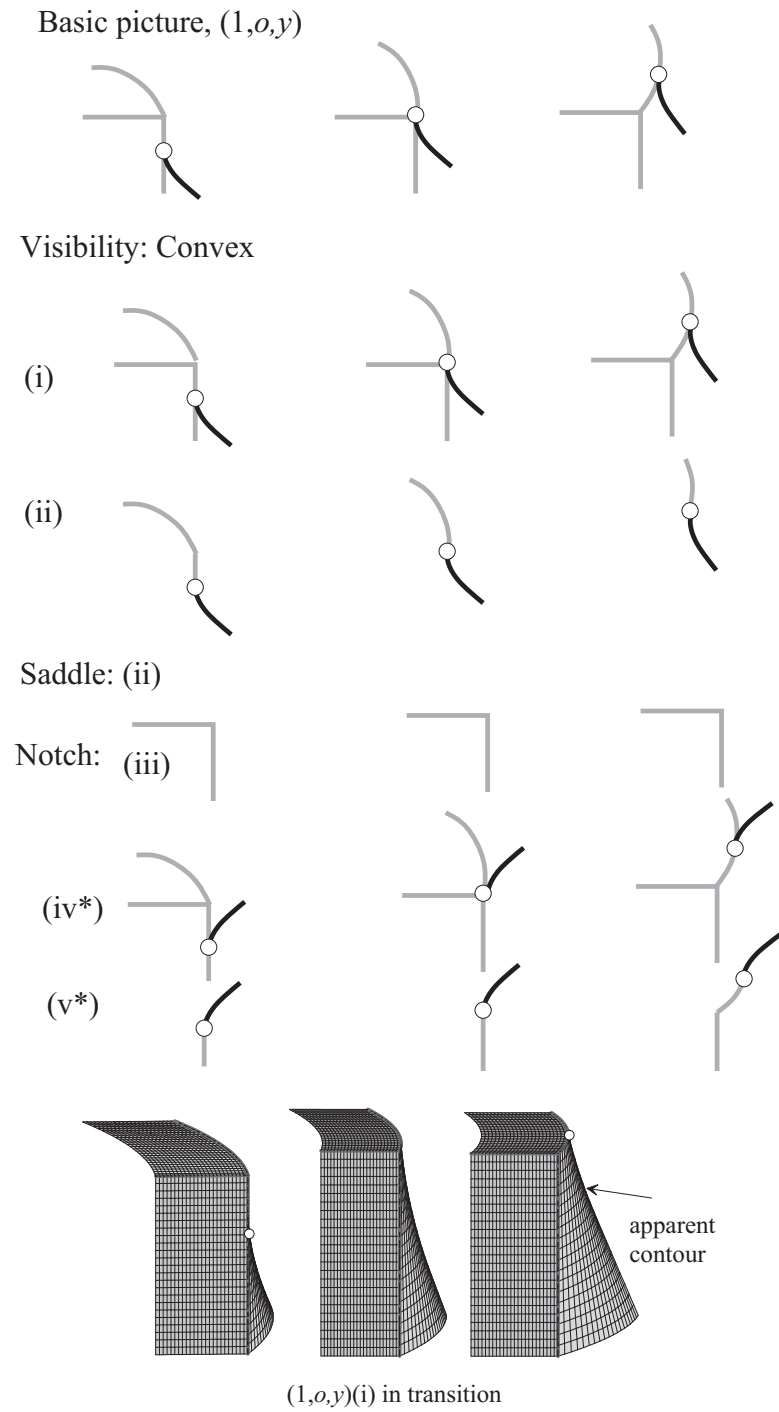
Figure 17: Transitions on corners of type $(1, o, y)$. Note that in this case there is in fact no qualitative distinction between the starred and unstarred cases.

Basic picture, (1,*o,n*)

Visibility: Convex

(i)

Concave

(ii)

Saddle:

(iii)

(iv*)

Notch: (i), (ii) or

(v)

(vi)

apparent
contour

(1,*o,n*)(ii) in transition

Figure 18: Transitions on corners of type $(1, o, n)$. Note that in this case there is in fact no qualitative distinction between the starred and unstarred cases.
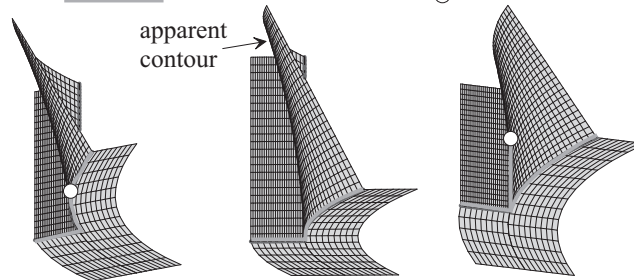
Basic picture, (1,*s*,*n*)

Visibility: Convex:

(i)

(ii)

Saddle: (i), (ii) or

(iii*)

Notch:

(iv)

apparent
contour
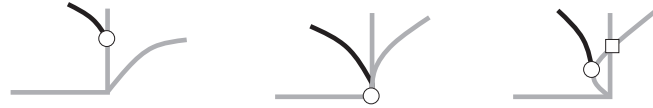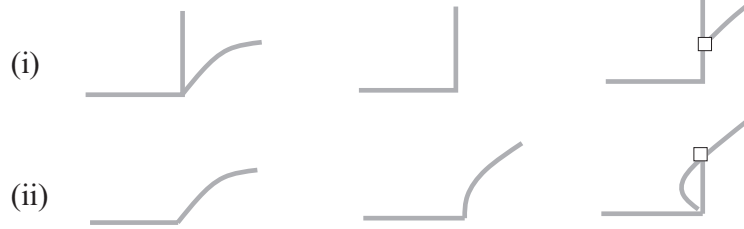
(1,*s*,*n*)(iv) in transition

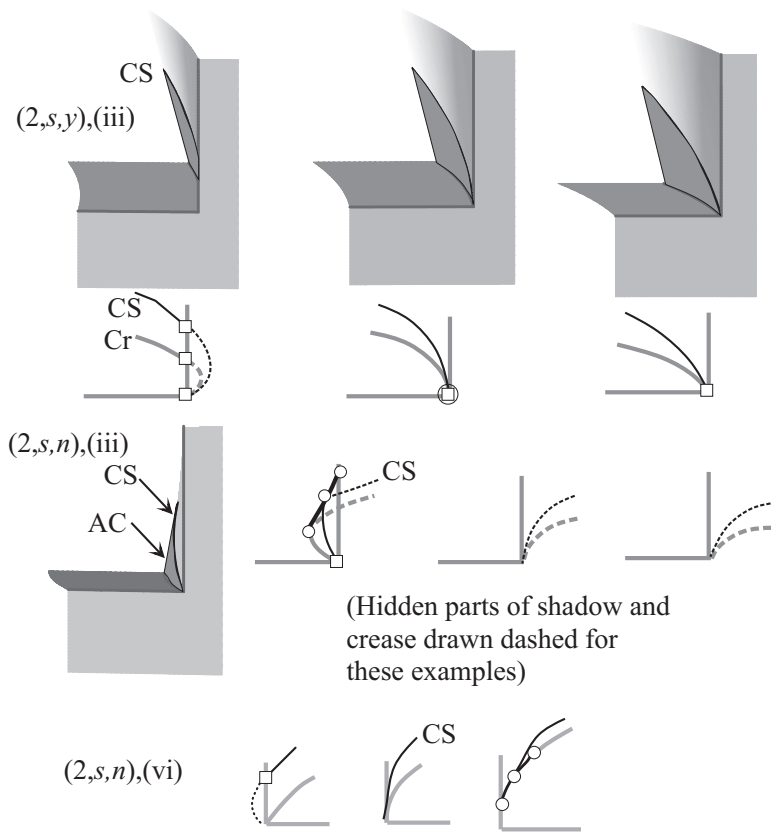Figure 19: Transitions on corners of type $(1, s, n)$

2.2 Corners and cast shadows: notch cases

Figure 20: Two examples of notch corners with cast shadow transitions and a schematic picture. CS = cast shadow of a crease
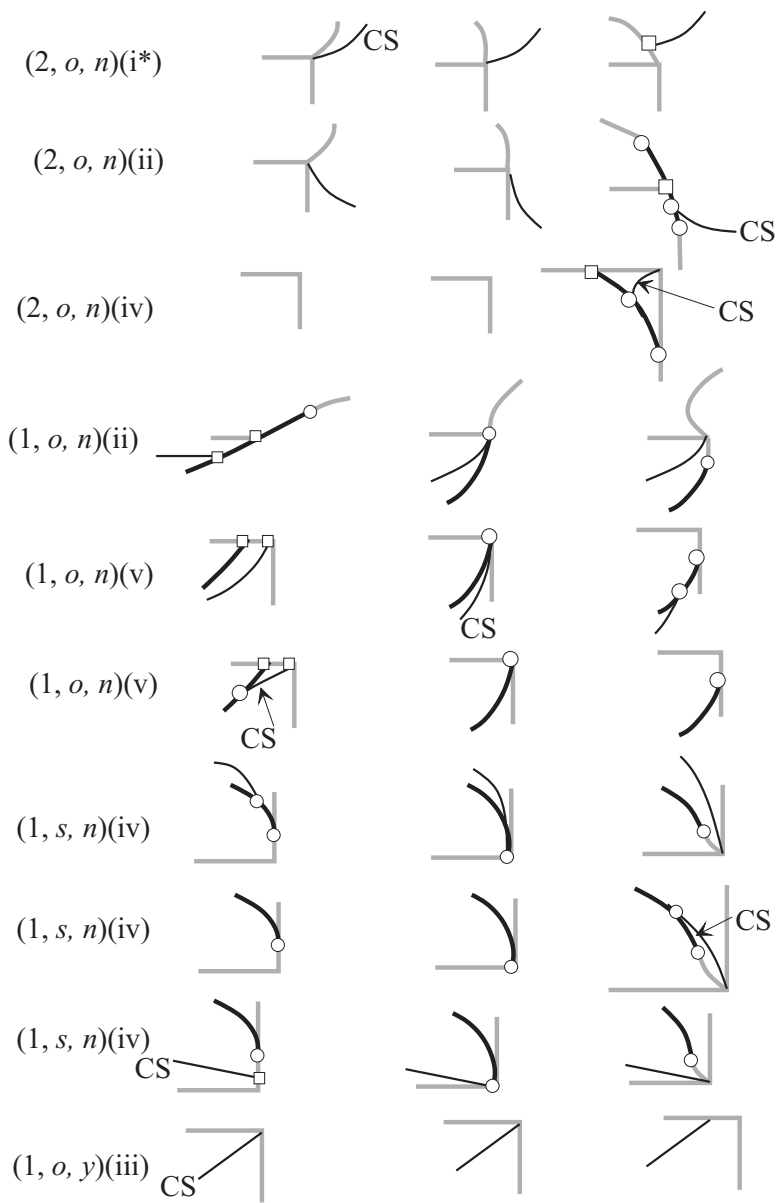
(2, *o*, *n*)(i*)

(2, *o*, *n*)(ii)

(2, *o*, *n*)(iv)

(1, *o*, *n*)(ii)

(1, *o*, *n*)(v)

(1, *o*, *n*)(v)

(1, *s*, *n*)(iv)

(1, *s*, *n*)(iv)

(1, *s*, *n*)(iv)

(1, *o*, *y*)(iii)

Figure 21: Schematic pictures of the remaining notch corners with cast shadow transitions. CS = cast shadow of a crease

2.3 Corners and cast shadows: saddle cases. CS = cast shadow of a crease

(2,*o*,*n*) (iii*)

(1,*o*,*y*) (ii)
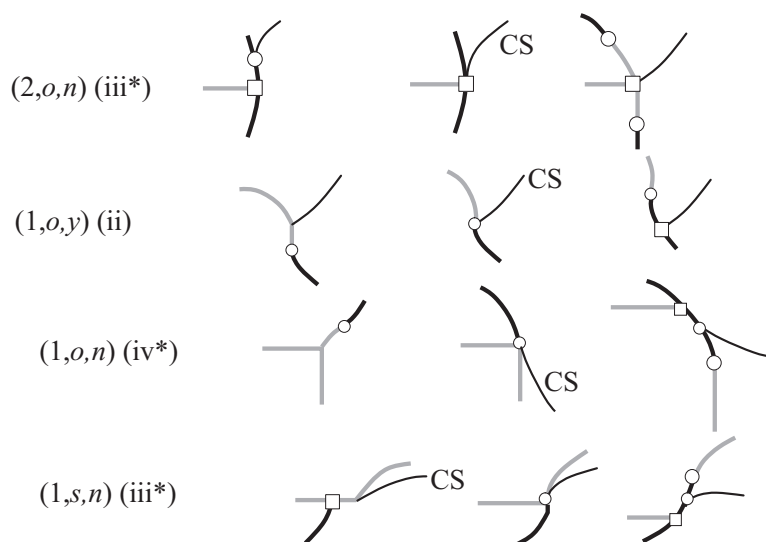
(1,*o*,*n*) (iv*)

(1,*s*,*n*) (iii*)

Figure 22: Schematic pictures of saddle corners with cast shadow transitions