Report on visit by Ricardo Uribe-Vargas, partly funded by RCMM

The purpose of the visit was to continue an investigation begun between Peter Giblin and his former Research Assistant André Diatta, on medial axes and symmetry sets of families of curves which include a singular member. We concentrated on two of the difficult cases where only very limited progress had been made in the past. Our object was to gain insight into these cases by extensive experimentation. This report summarizes the conclusions from this experimentation and the associated theoretical calculations. We hope to make further progress, including theoretical progress, when Dr Uribe-Vargas returns, supported by EPSRC for two months, in the autumn.

1 Umbilic point on a surface

Here we are investigating the curves C_k given by $f(x,y) = k^2$ where k is a small constant (which we can assume is > 0) and $f(x,y) = x^2 + y^2 +$ higher order terms. We assume that the cubic terms are not divisible by $x^2 + y^2$ for genericity. Thus C_k is a plane section of a surface in $\mathbb{R}^3$ parallel and close to the tangent plane at an umbilic (the origin).

Given such a curve C_k we want to know the locus of centres of circles which are tangent to the curve in two points (or more), and the limit points of this locus. This is the *symmetry set* of the curve C_k. If we consider only circles contained inside C_k then we call the locus the *medial axis* of C_k.

It is known that C_k, for k sufficiently small, has six vertices (maxima or minima of curvature) [3]. This gives six endpoints on the symmetry set (and three on the medial axis). Experimental evidence has suggested for some time that the symmetry set of such C_k has

- six cusps—these are at the centres of circles which are osculating at one point and tangent at another point of C_k;
- three triple crossings—these are at the centres of circles tangent to C_k in three distinct points.

The configuration of these cusps and triple crossings appeared from the evidence to be the same for any choice of higher order terms in f, for sufficiently small k; thus the combinatorial structure of the symmetry set appeared to be always the same for any umbilic.

The work done on this during the visit, using the Department's Silicon Graphics computers, consisted of a much more thorough search of examples, seeking for configurations which might be different from the standard one. The examples studied were suggested by Dr Uribe-Vargas, and are described below.

The idea is that the four combinations of $\cos^3\theta,\ \cos^2\theta\sin\theta,\ \cos\theta\sin^2\theta,\ \sin^3\theta$ given by

$$\begin{array}{rcl}\cos\theta & = & \cos^3\theta+\cos\theta\sin^2\theta\\ \sin\theta & = & \cos^2\theta\sin\theta+\sin^3\theta\\ \cos 3\theta & = & \cos^3\theta-3\cos\theta\sin^2\theta\\ \sin 3\theta & = & 3\cos^2\theta\sin\theta-\sin^3\theta\end{array}$$

are independent, and hence combinations of them can represent any cubic terms in f. Writing $x=r\cos\theta, y=r\sin\theta$ we have

$$b_0x^3+b_1x^2y+b_2xy^2+b_3y^3=r^3\left(\frac{3b_0+b_1}{4}\cos\theta+\frac{b_1+3b_3}{4}\sin\theta+\frac{b_1-b_3}{4}\sin 3\theta+\frac{b_0-b_2}{4}\cos 3\theta\right).$$

Moreover, by an initial rotation, we can eliminate one of the four terms, say $\cos 3\theta$. This has the effect of making the coefficients of x^3 and xy^2 always equal: $b_0=b_2$. We sometimes write the degree four terms of f, when present, as $c_0x^4+c_1x^3y+c_2x^2y^2+c_3xy^3+c_4y^4$, and so on.

Example 1.1 Consider $f(x,y)=x^2+y^2-3x^2y+y^3$. This just uses the term $-r^3\sin 3\theta$ above. See Figure 1. The level sets of f will be circles distorted in a way that has 3-fold symmetry. We can then move away from this very symmetrical situation to nearby ones, by adding multiples of $\cos\theta$ and $\sin\theta$ as above. The symmetrical case itself presents some interesting features; in fact, calculations show the following.

(i) The top point of the curve $f(x,y)=k^2$ is of the form $(0,k-\frac{1}{2}k^2+\frac{5}{8}k^3+\ldots$ and the bottom point is $(0,-k-\frac{1}{2}k^2-\frac{5}{8}k^3+\ldots)$.

(ii) The radius of curvature at the top point is $k+4k^2+\frac{77}{8}k^3+\ldots$ and at the bottom point is $k-4k^2+\frac{77}{8}k^3+\ldots$.

(iii) It follows that the centre of curvature at the top point is $(0,-\frac{9}{2}k^2-9k^3+\ldots)$ and at the bottom point is $(0,-\frac{9}{2}k^2+9k^3+\ldots)$. These are the positions of the endpoints of the symmetry set, as in Figure 1, centre, but in this symmetrical case the two branches overlay each other so only one endpoint, namely the first one, is visible.

(iv) The centre of the bi-osculating circle lying on the y-axis is $(0,\frac{3}{2}k^2+\frac{27}{8}k^4+\ldots)$. The centre of the triple-tangency circles is the origin: in this case, the 'outer' and 'inner' triple tangency circles are concentric.

(v) Thus the ratio of the lengths $AB:BC$ in Figure 1, right, is 1:3. The point A lies at a crossing of the evolute of the umbilic curve, and the point C lies at a cusp of the evolute. We conjecture that this holds for all umbilics, as $k\to 0$.

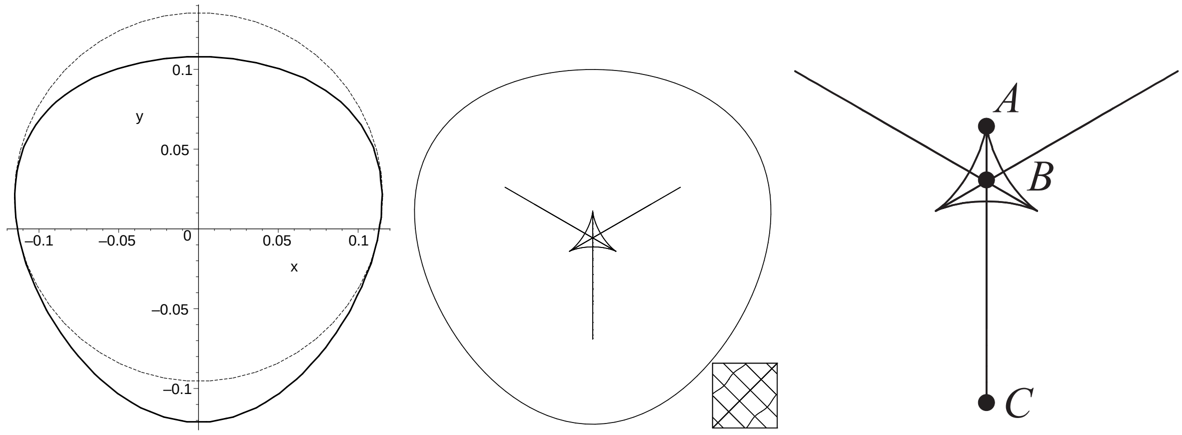

Figure 1: Left: the symmetric umbilic section $x^2+y^2-3x^2y+y^3=k^2$ where $k=0.1$, together with a circle (dashed) which osculates the curve at two points: an A_2^2 circle. Centre: the symmetry set of the same curve; the centre of the circle in the left-hand diagram lies at the double cusp at the top of the curvilinear triangle in the right-hand diagram. The small figure to the right is the *pre-symmetry set*: the set of pairs of parameter values on the curve which admit a bitangent circle. Right: the ratio $AB:BC$ is $1:3$.

Remark 1.2 The method used to draw the symmetry set in Figure 1, using the Liverpool Surfaces Modelling Package (LSMP) is to approximate the curve $f(x,y)=k^2$ as in [2]. That is, we write $x=kr(k,\theta)\cos\theta,\ y=kr(k,\theta)\sin\theta$ and $r=r_0+r_1k+r_2k^2+\ldots$ where the r_i are functions of θ (indeed of $\sin\theta$ and $\cos\theta$) only. For the umbilic, $r(0,\theta)$ is constant, in fact equal to 1 for the form we use here. Conversely, if $x=kr(k,\theta)\cos\theta,\ y=kr(k,\theta)\sin\theta,\ z=r(k,\theta)^2$ with r smooth and $r(0,\theta)$ is constant, them $z=f(x,y)$ has an umbilic at the origin: $f_x=f_y=0$ and $f_{xx}=f_{yy},\ f_{xy}=0$ at $t=0$.

Examples 1.3 We now consider certain deformations of the curve in Example 1.1, of the form

$$f(x,y)=x^2+y^2-3x^2y+y^2+ax(x^2+y^2)+by(x^2+y^2).$$

Note that, as above, this is a general deformation of the cubic terms of f. When $a=0$ we maintain symmetry about one axes, namely the vertical one; moving away from $a=0$ breaks this symmetry. The object of studying these was to discover whether configurations different from what we regarded as the 'standard configuration' were possible. An example of a 'standard configuration' is in Figure 2, where $a=-0.2,\ b=-0.4$.

In fact, performing a circuit of the origin in the (a,b)-plane, the standard configuration is maintained: Figure 3.

When we start to add degree 3 terms generated by $\cos\theta$ and $\sin\theta$ (that is $x(x^2+y^2)$ and $y(x^2+y^2)$) and also degree 4 terms the situation changes, in that long and short branches can now cross. An example is shown in Figure 4. Examples have been found

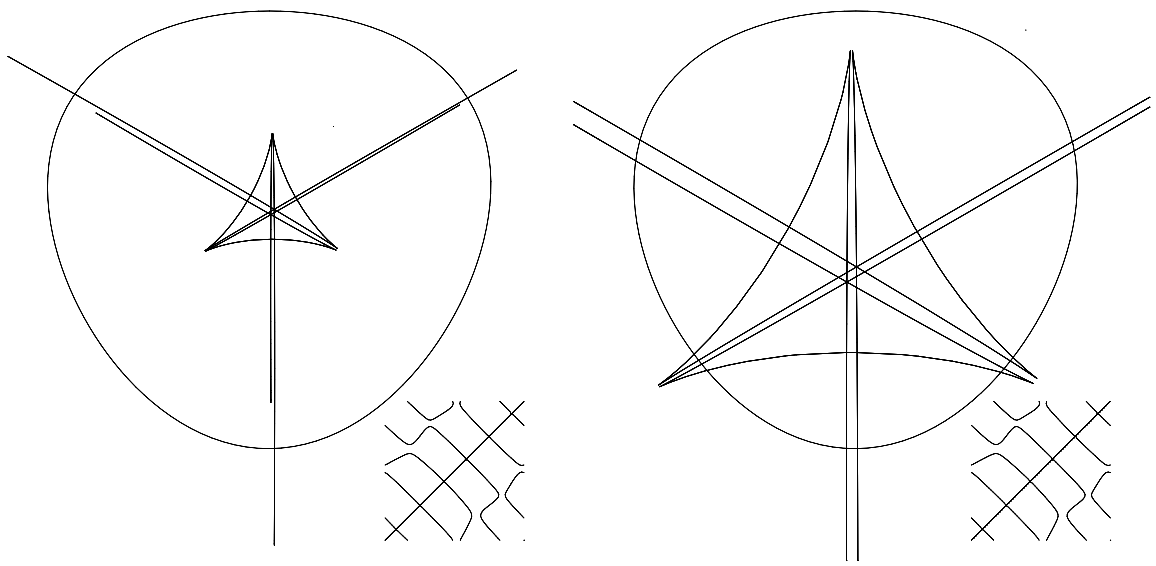

Figure 2: A 'standard' configuration for the symmetry set of an umbilic curve. The symmetry set is slightly enlarged compared with the curve in the left-hand figure and greatly enlarged in the right-hand figure. The branches extending to the endpoints do not cross. There is almost symmetry about a vertical line but not about the lines at $\pm 60°$ to the vertical, on account of the 'short' and 'long' branches. The 'long' (resp. 'short') branches end in the centre of curvature at a *minimum* (resp. *maximum*) of curvature. The three branches of the symmetry set join maximum-maximum, minimum-minimum and maximum-minimum. The last is in some sense special. This example has $a = -0.2,\ b = -0.4$.

where one, two or three pairs cross, and where the crossings are at any place relative to the triple points. We have not as yet been able to prove that these crossings persist as $k \to 0$. It may be that for small enough k the generic picture is always that of Figure 3.

Problems 1.4 The following problems remain, and will be the subject of further investigation.

1. We lack rigorous proofs that there are always 6 cusps and 2 triple crossings on the symmetry set in the umbilic case, for sufficiently small k, with the same combinatorial structure as displayed in Figures 3. Writing T for triple point, C for cusp, L for long endpoint branch and S for short, all branches of the symmetry set in these examples and other studied are of the form $LTCCTS$, $STCCTS$ or $LTCCTL$.

2. Do the crossings of the branches out to the endpoints of the symmetry set persist as $k \to 0$ or is the generic picture that given by cubic terms alone?

3. What is the flow of the radius function on the symmetry set? It will necessarily have a turning point at a cusp, and will reach a maximum at an endpoint given

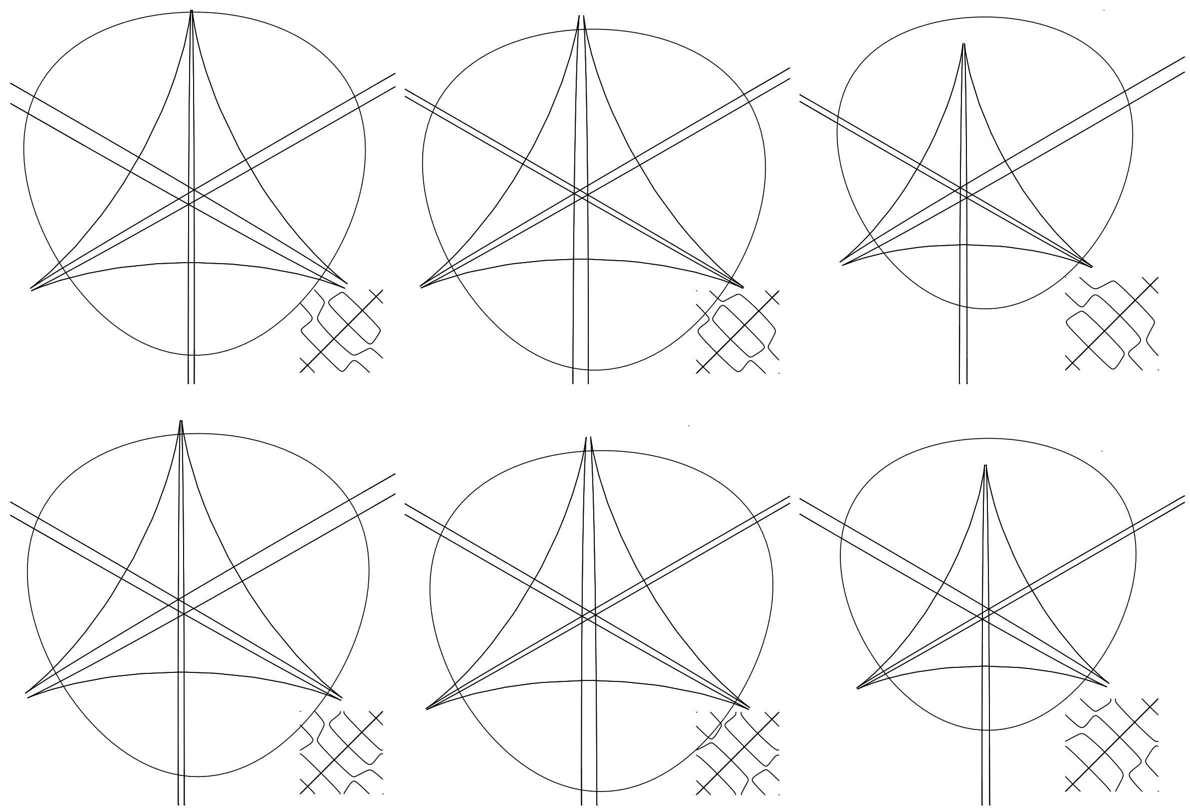

Figure 3: Clockwise from top left: $(a, b) = (0.2, 0.4)$, $(0.4, 0)$, $(0.2, -0.4)$, $(-0.2, -0.4)$, $(-0.4, 0)$, $(-0.2, 0.4)$. The configuration is essentially the same in each case but short and long branches (not distinguished here) can switch over when passing through a symmetric configuration with $a = 0$. Note the realignments of the triple points in the centres of the figures and also in the 'pre-symmetry sets' shown as small boxed figures for each example. These sets show Morse transitions where branches cross and re-connect in the opposite way; this is an 'exchange of cusps' or 'nib transition' on the symmetry set. The pairs of close branches on the symmetry sets from the endpoints do not cross on their way to the first cusps.

by a minimum of curvature, and vice versa. But other turning points, and their geometrical meaning, are unclear.

2 Elliptic cusp of Gauss

In this section we consider the case of a surface in $\mathbb{R}^3$ with an elliptic cusp of Gauss (also called a *godron*), and plane sections of the surface close to the tangent plane at this point. Again the sections will be closed curves, or empty, but now the closed curves have two inflexions and four vertices [3]. The general form of a cusp of Gauss, up to euclidean

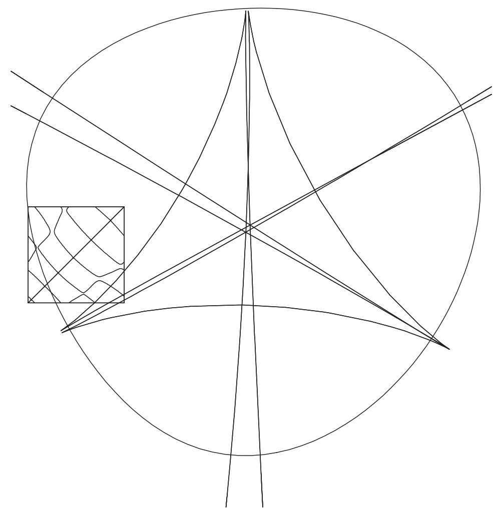

Figure 4: A perturbation of a symmetric umbilic $f(x,y) = x^2 + y^2 - 3xy^2 + y^3$ by the addition of degree 3 terms of the '$\cos\theta$' and '$\sin\theta$' kind, in fact $2x(x^2+y^2) + \frac{1}{2}y(x^2+y^2)$ and also degree 4 terms $x^4 - 8x^3y - 8xy^3 + y^4$. This allows all three pairs of branches going out to endpoints to cross. This picture has $f(x,y) = k^2$ where $k = 0.1$. We do not know whether this behaviour can persist as $k \to 0$.

motions and scaling, is

$$z = f(x,y) = x^2 + b_0x^3 + b_1x^2y + b_2xy^2 + c_0x^4 + c_1x^3y + c_2x^2y^2 + c_3xy^3 + c_4y^4 + \ldots, \quad (1)$$

where $b_2 \neq 0$ (to guarantee a nonsingular parabolic curve at the origin) and $b_2^2 \neq 4c_4$ for a non-degenerate cusp of Gauss. This means that the contact with the tangent plane has type A_3 and not higher. When $b_2^2 - 4c_4 < 0$ the cusp of Gauss is *elliptic* and close to the origin the set $f(x,y) = k^4$ is a closed curve.

Example 2.1 The basic example $f_0(x,y) = k^4$ $(k > 0)$ where $f_0(x,y) = x^2 - 2xy^2 + \rho y^4$. The parameter ρ here is from the Platonova normal form; there is an extensive investigation of the different ranges of ρ in [5]. For an elliptic cusp of Gauss, $\rho > 1$. Although special, this example gives a lot of geometrical insight into the way in which the general section $f(x,y) = k^4$ behaves as $k \to 0$. Many of the examples below have ρ just greater than the critical value 1.

Remark 2.2 The coefficient -2 in f_0 is not significant; by scaling x and y by say λ we can write $x^2 - axy^2 + \rho y^4 = k^4$ $(a \neq 0,\ a^2 < 4\rho)$ as $x^2 - a\lambda xy^2 + \rho\lambda^2y^4 = k^4/\lambda^2$. Then choosing $\lambda = 2/a$ we get $x^2 - 2xy^2 + (4\rho/a^2)y^4 = \frac{1}{4}k^4a^2$, which is the 'basic example' with revised values of ρ and k.

The approximate paramterization of the curve $f_0(x,y) = k^4$ from [2] (compare Remark 1.2) becomes in this case an exact parametrization. We write $x = k^2r^2\cos\theta,\ y =$

$kr\sin\theta$; then r becomes a function of θ only in this case:

$$r^{-4} = \cos^2\theta - 2\cos\theta\sin^2\theta + \rho\sin^4\theta = (\cos\theta - \sin^2\theta)^2 + (\rho - 1)\sin^4\theta.$$

Note that for $\rho > 1$ this always gives a smooth function r.

Since the curve $C_{0k}:\ f_0(x,y) = k^4$ has two inflexions we must allow both for bitangent circles which are *coherent* and *non-coherent*. Here we take an orientation of the curve C_{0k} and call the circle coherent provided the two tangencies with C_{0k} are oriented the same way round the circle. There are as $k \to 0$ four vertices on C_{0k} and special conditions attach to curves, whether convex or not, with four vertices. (See [4, §6].) In particular, provided the circles of curvature at the vertices where C_{0k} meets the x-axis intersect, C_{0k} can be inverted into a *convex* curve ([4, Theorem 6.3]). If this is the case, then there cannot be any circle which is osculating for C_{0k} at one point and ordinarily tangent at another point (an A_2A_1 circle), since these would carry over to the inverted curve and such a circle would have to be coherent (for a convex curve) and this would imply that C_{0k} had at least 6 vertices, using [4, Theorem 7.1]. The same applies to the existence of A_1^3 circles, that is circles tritangent to C_{0k}: for a convex curve these would have to be coherent circles and this implies at least 6 vertices using the same theorem in [4].

The curve C_{0k} meets the x-axis where $x = \pm k^2$; calculation shows that the radii of curvature of C_k at these points are equal to $\frac{1}{2}$ for all k; hence the circles of curvature will certainly intersect for small k. Consequently:

Proposition 2.3 *For small k there can be no A_1A_2 circles and hence no cusps on the symmetry set of C_{0k}. With four vertices there must be two branches of the symmetry set joining the centres of curvature at these vertices, one branch going to infinity.*

We give more details of this example below. Extensive experimentation suggested that the same holds for a general function $f(x,y)$ corresponding to an elliptic cusp of Gauss. A typical example is shown in Figure 5. Note that the symmetry set of f_0 is special: the branch associated with the minima of curvature overlays itself, because of the global symmetry about $y = 0$. Making a small perturbation, as in Figure 5, separates the branches but does not appear to introduce cusps on the symmetry set, for small enough k.

In fact using methods and results from [3] we do find the same result as Proposition 2.3 in general. See Proposition 2.6 below.

Next, a note on the shape of the curve C_{0k} as a function of ρ. See Figure 6.

We now give some more details of Example 2.1 and related matters.

Example 2.4 We have noted that, for any elliptic cusp of Gauss the associated curve $f(x,y) = k^4$ for small k has exactly four vertices and two inflexions. This configuration of vertices and inflexions does not in itself prevent the appearance of cusps on the symmetry set, as the following example shows. Start with a curve $x = p\cos t + q\sin t,\ y = \sin^3 t + \mu\sin t$ where initially $\mu = 0$. This curve has two inflexions on the x-axis, corresponding to $t = 0, \pi$. It is possible to choose p and q so that the curve has exactly four vertices;

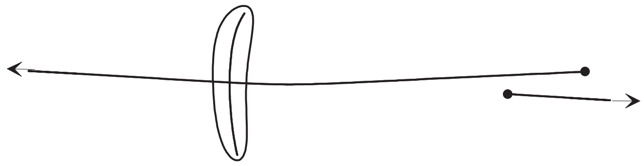

Figure 5: Curve C_k given by an elliptic cusp of Gauss, in fact given by $k^4 = x^2 + x^3 + 1.8x^2y + 1.3xy^2 + 0.8x^3y - 0.6x^2y^2 + 0.7xy^3 + 1.8y^4$ where $k = 0.2$. This behaviour, which is believed to be generic for small k, shows two nonsingular branches of the symmetry set, one going to infinity (indicated by the arrows) and having endpoints (marked by dots) at the centres of curvature at the minima of curvature. The other branch connects the maxima of curvature within the curve C_k.

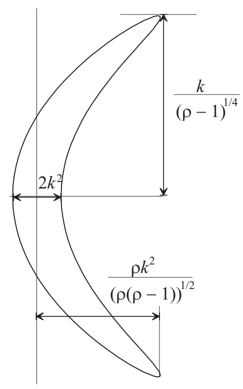

Figure 6: The shape of the curve $x^2 - 2xy^2 + \rho y^4 = k^4$. Scaling by $1/k^2$, the width, both at the 'waist' and overall, stays constant and, as $k \to 0$, the height tends to infinity. We conjecture that the same holds for any cusp of Gauss.

an example is in Figure 7, left. If we invert this curve with respect to a point not on the x-axis, then the resulting curve will have a bi-osculating circle (A_2A_2 contact), which will be non-coherent, since this is true of the bi-inflexional tangent. However we change μ slightly and then invert, to produce a curve with A_2A_1 circles, close to a 'moth transition' [1]. See Figure 7, right, which shows the symmetry set having a characteristic 'moth' form with four cusps. The circles giving rise to the cusps on the symmetry set will necessarily be non-coherent.

It is also possible to have (non-coherent) biosculating circles for the basic Example 2.1, as illustrated in Figure 8, but as k decreases these disappear since, as noted above in Proposition 2.3, no cusps are possible for small enough k.

What happens as k is steadily decreased, fixing $\rho > 1$? The moth configurations shrink to points and disappear, but it is very hard to determine the exact moment, in terms of ρ, when this happens. This is because the conditions for bi-osculating circles are hard to determine explicitly.

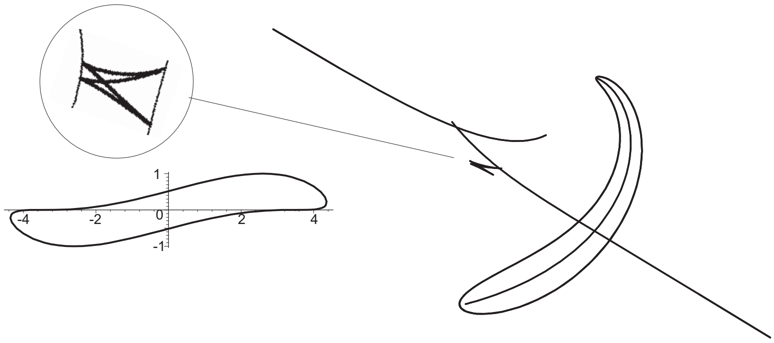

Figure 7: Left: starting with $p = 3.5,\ q = 2.6,\ \mu = 0$ in Example 2.4, this produces the curve on the left, with a (non-coherent) bi-inflexional line and exactly four vertices. Then changing μ to -0.2 and inverting with respect to $(-4, 4)$ we get the four-vertex curve on the right, with symmetry set having the 'expected' two branches. one going to infinity, but also a 'moth' configuration of four cusps. The cameo above the left-hand figure is an enlargement of the 'moth', showing also two branches of the evolute which the moth straddles.

However we can look at the way in which vertices of the curve C_{0k} evolve as k decreases. In fact this does not happen in a simple way. Instead of six vertices changing to four, the first event is that six vertices become eight, when the vertex at $(-k^2, 0)$ becomes degenerate. Expanding the curve C_{0k}, given by $x^2 - 2xy^2 + \rho y^4 = k^4$ about the points $(-k^2, 0)$ and $(k^2, 0)$ we get, respectively,

$$x = -k^2 + y^2 + \frac{\rho - 1}{2k^2}y^4 + \frac{(\rho-1)^2}{8k^6}y^8 + \dots, \quad x = k^2 + y^2 - \frac{\rho - 1}{2k^2}y^4 - \frac{(\rho-1)^2}{8k^6}y^8 + \dots.$$

For a curve with local equation $x = y^2 + ay^4 + by^5 + cy^6 + \dots$, which therefore has a vertex at $(0, 0)$, with centre of curvature $(\frac{1}{2}, 0)$, the conditions for the circle of curvature to have various contacts with the curve at $(0, 0)$ are as follows:

$$\text{4-point (ordinary vertex)}: \ a \neq 1; \ \text{5-point: } a = 1,\ b \neq 0; \ \text{6-point: } a = 1,\ b = 0,\ c \neq 2,$$

etc. Bearing in mind that $\rho > 1$ it follows that, for C_{0k}, a higher vertex at $(k^2, 0)$ is impossible while at $(-k^2, 0)$ the condition is $k^2 = \frac{1}{2}(\rho - 1)$ for a higher vertex, and the circle of curvature has exactly 6-point contact. It follows that this vertex actually gives birth to *three* vertices as k decreases, thus increasing the number from six to eight. What subsequently happens is that four of these collapse in pairs to leave four vertices altogether, which then persist for all smaller k. We illustrate this with some further calculations and figures.

There is a moment at which of the eight vertices give eight cusps on the evolute, two lying at the same place on the x-axis. Calculation shows that this happens when $k^2 = (\rho - 1)/(2\rho^{3/2})$, which is of course always less than the above value of k^2 at which six vertices become eight. Shortly after this, two pairs of vertices disappear to give the final configuration of four vertices, and four cusps on the evolute. See Figure 9.

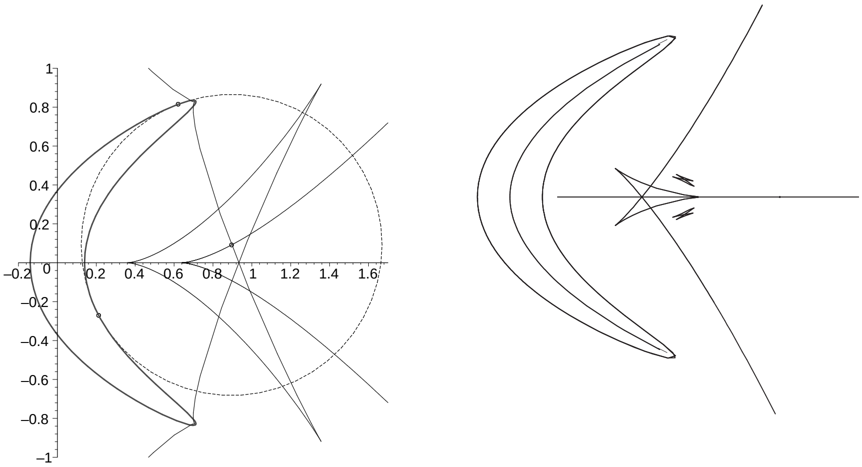

Figure 8: Left: example with $\rho = 1.04,\ k = 0.3738$ in $C_{0k}:\ x^2 - 2xy^2 + \rho y^4 = k^4$. The thin curve is the evolute of C_{0k}, going to infinity in two directions, and the six cusps indicate that C_{0k} has six vertices. Also shown dashed is a bi-osculating circle which is non-coherent. The centre of this circle is necessarily at a self-crossing of the evolute, marked in the figure, but the numbers are chosen here so that the two osculating circles with this centre actually coincide. The symmetry set will acquire an isolated point at the centre of the bi-osculating circle and increasing k slightly will result in two four-cusped 'moth' figures, symmetric about the x-axis because of the global symmetry of C_{0k}. Right: this is illustrated by taking $k = 0.45$; here the symmetry set is shown, not the evolute.

General case 2.5 Finally let us look at the general case of an elliptic cusp of Gauss, given by (1). We use the methods and results of [3].

The 'vertex curve', that is the locus of vertices of all curves $f(x, y) = c$ for constants c consists of two smooth branches, one tangent to the y-axis and one—the one which concerns us here—tangent to the line $x(c_3 - b_1b_2) = y(b_2^2 - 4c_4)$ ([3, Prop.6.1]). This is the branch along which the vertices corresponding to a minimum of curvature move.

This smooth branch can easily be parametrized, and then we can use a standard formula for the square of the curvature of an implicit curve $f(x, y) = c$, namely,

$$\kappa^2 = \frac{(f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2)^2}{(f_x^2 + f_y^2)^3}.$$

When we do this and then consider the limit as $c \to 0$ (or $k \to 0$ in our previous notation) we find that the limit of the curvatures at the vertices of minimum curvature is precisely (up to sign) equal to b_2. (Note that in Example 2.1 $b_2 = -2$ and the curvature was 2 up to sign.) Thus as in the particular example we have two circles of curvature of finite radius $|b_2|$ at two points which in the limit lie along the direction $x(c_3 - b_1b_2) = y(b_2^2 - 4c_4)$, not parallel to the y-axis, whereas the tangent to $f(x, y) = c$ tends to the y-axis. These circles are then bound to intersect, which means that as above we can invert the curve into a convex curve and this rules out the possibility of cusps or triple crossings on the symmetry set. We therefore have:

Proposition 2.6 *For small k there can be no A_1A_2 circles and hence no cusps on the symmetry set of $C_k: \ f(x,y) = k^4$. With four vertices there must be exactly two branches of the symmetry set joining the centres of curvature at these vertices, one branch going to infinity.*

References

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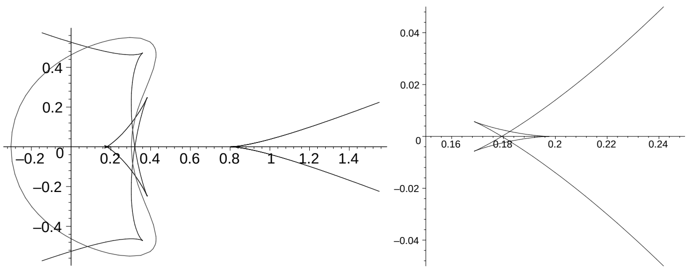
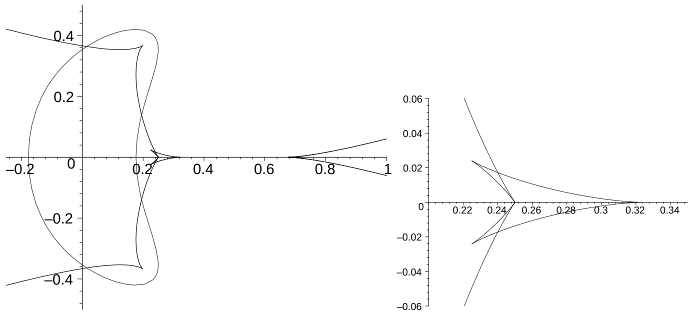
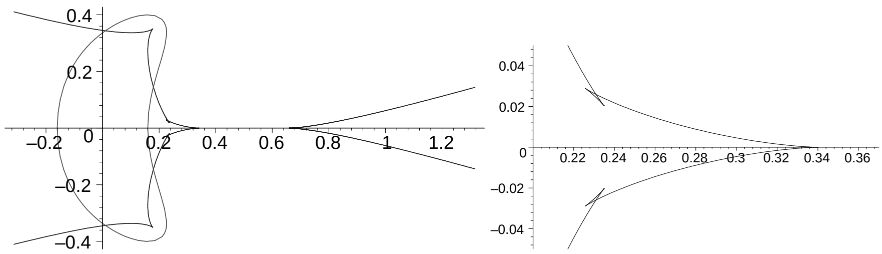


$k = 0.55$: six vertices have just turned into eight.



$k = 0.4204$: the configuration has eight cusps, two at the same place on the x-axis.



$k = 0.4$: two pairs of cusps are about to disappear.

Figure 9: Here $\rho = 2$; the curve C_{0k} and its evolute are shown, with details as appropriate. The value of k at which six vertices turn into eight (hence six cusps on the evolute turn into eight) is then $k = \frac{1}{2}\sqrt{2} = 0.707...$ and the value at which two cusps are at the same point on the x-axis is $\frac{1}{2}2^{-1/4} = 0.420....$ Shortly after this four vertices disappear in pairs to leave the stable situation of four vertices which then persists as $k \to 0$.