AFFINE-DISTANCE SYMMETRY SETS

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ABSTRACT. The affine distance symmetry set (ADSS) of a plane curve is an affinely invariant analogue of the euclidean symmetry set (SS) [7, 6]. We list all transitions on the ADSS for generic 1-parameter families of plane curves. We show that for generic convex curves the possible transitions coincide with those for the SS but for generic non-convex curves, further transitions occur which are generic in 1-parameter families of bifurcation sets, but are impossible in the euclidean case. For a non-convex curve there are also additional local forms and transitions which do not fit into the generic structure of bifurcation sets at all. We give computational and experimental details of these.

Keywords: symmetry set, affine differential geometry, bifurcation set, medial axis, affine skeleton. Subject classification: 58C27, 53A15.

1. INTRODUCTION

Affine-invariant symmetry sets of planar curves were first introduced and studied Giblin and Sapiro (see [12, 14]). The idea was to mimic the numerous different constructions of the euclidean symmetry set to produce analogous affine-invariant symmetry sets for affine plane curves. One of the first, and most striking, observations was that, although the different constructions for the euclidean symmetry set led to *identical* sets, the affine-invariant analogues of these constructions resulted in genuinely *different* sets. Thus there is no single affine-invariant symmetry set, but instead a number of affine-invariant sets which individually capture some aspects of local affine symmetry.

In this article we consider one of the affine-invariant symmetry sets as introduced in [12, 14], namely the *Affine Distance Symmetry Set* (ADSS), defined by replacing euclidean distance with 'affine distance to a curve' in the sense of Izumiya [16]. The local structure of the ADSS was classified in these articles, on the assumption that the curve contained no inflexions. The present article extends this to curves with inflexions and gives a complete list of the transitions on the ADSS of generic 1-parameter families of curves, following the analogous procedure given in [6] for the euclidean symmetry set. We find that ovals (strictly convex smooth closed curves) behave very much as do generic curves relative to the euclidean symmetry set. However, when we allow non-ovals, several transitions which were barred in the euclidean case become possible, and transitions directly involving inflexions are completely new.

The paper is organised as follows. In §2 we introduce the basic notions of affine plane differential geometry needed in the sequel. In §3 we recall the definition of the ADSS, in §4 we describe the theoretically possible transitions on symmetry sets and in §5 we show which of these can actually occur. In §6 we describe the special, and apparently highly degererate (but generic!) transitions which directly involve inflexions. Here we rely on computation and experiment in the absence of a theoretical framework. Finally in §7 we describe further directions for research.

The second author acknowledges the support of an EPSRC research studentship. Several figures were drawn with the Liverpool Surfaces Modelling Package (LSMP); see [17].

2. PLANAR AFFINE DIFFERENTIAL GEOMETRY

Here we briefly present some basic concepts and definitions of planar affine differential geometry. For more information, see for example [12, 18, 19]. Let $\gamma(t)\colon S^1 \to \mathbb{R}^2$ be a simple closed smooth planar curve parametrized by t. A reparametrization using the 'affine arclength' parameter s satisfying

$$[\gamma'(s), \gamma''(s)] = 1, \tag{1}$$

where $'$ denotes derivative with respect to s and $[*, *]$ denotes the determinant of the 2×2 matrix defined by two vectors in $\mathbb{R}^2$, is invariant under affine transformations of determinant 1. (The symmetry set we define in §3 is invariant under arbitrary affine transformations.) The vectors $\gamma'(s)$ and $\gamma''(s)$ are respectively the *affine tangent* and the *affine normal* to γ at $\gamma(s)$.

Geometrically, the straight line in the direction of the affine normal at a point of a curve γ is the locus of centres of conics having (at least) 4-point contact with γ at that point. Since (1) cannot hold at inflexion points of γ, this means that affine differential geometry is not defined at these points: however, since inflexions are affine-invariant, we circumvent this problem in practice by segmenting the curve into convex portions. The limiting affine normal at an inflexion is parallel to the tangent and of infinite length. Note also that for an oval (a closed curve without inflexions) the condition $[\gamma', \gamma''] = 1$ forces an *anticlockwise orientation*.

From expression (1) it follows that for an arbitrary parametrization t,

$$ds = [\dot{\gamma}, \ddot{\gamma}]^{1/3} dt, \tag{2}$$

where $\dot{}$ (dot) denotes derivative w.r.t. t. We also have the following relationship between the affine tangent γ' and the Euclidean tangent T:

$$\gamma' = \kappa^{-1/3} T.$$

Lemma 2.1. *Two curves share the same affine tangent at a point if and only if neither has an inflexion and they have (at least) 3-point contact there. Two curves share the same affine tangent and normal at a point if and only if neither has an inflexion and they have (at least) 4-point contact there.* □

Differentiating (1) w.r.t. s we obtain

$$[\gamma'(s), \gamma'''(s)] = 1,$$

for all s, and therefore

$$\gamma'''(s) + \mu\gamma'(s) = 0, \tag{3}$$

for some real function $\mu(s)$, the the *affine curvature*) of γ: it is the simplest non-trivial affine differential invariant, and defines a curve uniquely up to (equi-) affine transformation (see [2]), just as the euclidean curvature defines a curve up to euclidean transformation. Bracketing both sides of expression (3) with $\gamma''(s)$ gives us

$$\mu(s) = [\gamma''(s), \gamma'''(s)]. \tag{4}$$

Curves of constant affine curvature are *conics*: $\mu < 0$ for a hyperbola, $\mu = 0$ for a parabola and $\mu > 0$ for an ellipse. Two curves having 5-point contact at a point have the same affine tangent, normal and curvature there. In particular the osculating (5-point contact) conic at a non-inflexional point of a curve is a hyperbola, parabola or ellipse according as $\mu <, =, > 0$.

The *centre of affine curvature* at $\gamma(s)$ is the centre of the osculating conic at that point, that is, the point $\gamma(s) + (1/\mu(s))\gamma''(s)$, and the locus of these points is the *affine evolute* of γ, the affine-invariant analogue of the Euclidean evolute: furthermore, with analogy to the Euclidean situation, the affine evolute is the envelope of the affine normal lines to the curve. A point for which $\mu'(s) = 0$ is called an *affine vertex* of a curve, or a *sextactic point*: at such a point there exists a conic having 6-point contact with the curve. The centre of a sextactic conic lies at a cusp of the evolute. There are at least six points on a closed curve for which $\mu'(s) = 0$ (see [2] for a proof of this; see also [9] for a short exposition on the existence of sextactic points).

We now recall the definition of *affine distance*, which is based on area and is invariant under equi-affine transformations.

Definition 2.2. *Let* $\mathbf{x}$ *be a point in the plane, and* $\gamma(s)$ *a planar curve parametrized by affine-arclength* s. *The* **affine distance** *between* $\mathbf{x}$ *and a non-inflexional point* $\gamma(s)$ *on the curve is given by*

$$d(\mathbf{x}, s) \equiv [\mathbf{x} - \gamma(s), \gamma'(s)]. \tag{5}$$

In [16], it is shown that the affine evolute is the bifurcation set of the family of affine-distance functions and this fact is used to study the local structure of the affine evolute.

Using Arnold's standard A_k notation for singularities of functions of one variable, we have:

Proposition 2.3 ([16]). *Away from affine inflexion points of* γ, *the affine distance function* d *defined on* γ *exhibits the following singularities:*

- $A_{\geq 1} \iff \mathbf{x} - \gamma(s)$ *is parallel to* $\gamma''(s)$: $\mathbf{x}$ *is then on the affine normal line to* γ *at* $\gamma(s)$.
- $A_{\geq 2} \iff \mu(s) \neq 0$ *and* $\mathbf{x} = \gamma(s) + \frac{1}{\mu(s)}\gamma''(s)$: $\mathbf{x}$ *is then at the centre of affine curvature of* γ *at* $\gamma(s)$, *that is, on the affine evolute of* γ.
- $A_{\geq 3} \iff \mu(s) \neq 0$, $\mathbf{x} = \gamma(s) + \frac{1}{\mu(s)}\gamma''(s)$ *and* $\mu'(s) = 0$: $\mathbf{x}$ *is then on the affine evolute of* γ *at an affine vertex.*

Proof. See [16]. □

Finally in this section we give some formulae which are useful in converting from arbitrary parametrizations to affine-invariant parametrizations. The proofs are straightforward.

Suppose $\gamma(t)$ is an arbitrary regular parametrization of a plane curve γ. We will use $\dot{}$ (dot) for d/dt, $'$ (prime) for derivative w.r.t. affine-arclength, and write $k(t) = [\dot{\gamma}, \ddot{\gamma}]$. We have

$$\gamma'(t) = k^{-1/3}\dot{\gamma}(t), \quad \gamma'' = k^{-2/3}\ddot{\gamma} - \frac{1}{3}\dot{k}k^{-5/3}\dot{\gamma}. \tag{6}$$

For a graph $\gamma(x) = (x, f(x))$ we have

$$\gamma''(x) = \ddot{f}^{-5/3}\left(-\frac{1}{3}\dddot{f}(x), \ddot{f}(x)^2\right).$$

Thus the affine normal vector is in direction

$$\left(\dddot{f}(x), -3\ddot{f}(x)^2\right).$$

3. The affine distance symmetry set

Recall that the (euclidean) symmetry set of a simple closed plane curve γ is the closure of the locus of centres of circles tangent to γ in two (or more) places. The symmetry set together with the (euclidean) evolute constitute the full bifurcation set of the family of distance-squared functions on γ ([6]).

The analogous symmetry set in the affine case is the *affine distance symmetry set* (ADSS): the closure of the locus of points $\mathbf{x} \in \mathbb{R}^2$ on (at least) two affine normals and affine-equidistant from the corresponding points on the curve. The ADSS of γ is the closure of the set of points $\mathbf{x}$ which are the common centre of (at least) two conics sharing the same affine radius and having (at least) 4-point contact with γ.

The ADSS, together with the affine evolute, form the full bifurcation set of the family of affine distance functions on γ. Using this, we obtain the first four parts of the following theorem, where for example A_1A_2 means an affine-distance function with these two singularity types at the two points of γ, and A_1^3 refers to three type A_1 singularities for the affine-distance function. In parts 5 and 6 of the theorem the affine-distance function is not defined and the result is obtained only by a hands-on calculation [15] with power series expansions. Nevertheless both these situations occur generically as limiting points of the ADSS. We do not know how to fit them into the general theory of bifurcation sets. Some details of the required calculations are given following the statement of the theorem.

Theorem 3.1. *Locally, the affine distance symmetry set of a generic plane curve γ a point $\mathbf{x}$ is as follows.*

1. **Smooth** *when both conics have exactly 4-point contact with γ (A_1^2).*
2. *An* **ordinary cusp** *when one of the conics has 5-point contact with γ ($\mathbf{x}$ is then on the affine evolute of γ too, at a smooth point of it) (A_1A_2).*
3. *An* **endpoint** *when $\mathbf{x}$ is the centre of a 6-point contact conic, that is, a conic tangent to γ at a sextactic point: the endpoint is then in a cusp of the affine evolute (A_3).*
4. *A* **triple crossing** *when there are* three *conics centred at $\mathbf{x}$ having equal affine radius and 4-point contact with γ (A_1^3).*
5. *An* **ordinary cusp** *at the intersection point of two inflexional tangents to γ. This cusp does not lie on the affine evolute, in contrast to case 2 above. In this case we can regard each conic as being a repeated inflexional tangent line. In that case each conic has 6, rather than 4-point contact with γ. See Figure 1.*
6. *A* **(5, 6)-singularity** *(like $\boldsymbol{x^5 = y^6}$) at the point where an inflexional tangent cuts the curve again. In this case we can regard the two conics as being repeated tangent lines, one inflexional tangent and one ordinary tangent. The contacts are therefore 6 and 4, yet this gives a far more degenerate singularity than the preceding case! See Figure 1.*

In order to explain the calculations leading to parts 5 and 6 of the theorem we shall need the following criterion and formula, from [12].

Proposition 3.2 (ADSS Condition). *Suppose $\gamma(s)$ is a smooth, simple closed curve. The necessary and sufficient condition for distinct s_1, s_2, with neither of $\gamma(s_1), \gamma(s_2)$ being an inflexion of the curve, to give a point of the ADSS is*

$$\gamma(s_1) - \gamma(s_2) \text{ parallel to } \gamma''(s_1) - \gamma''(s_2), \tag{7}$$

$'$ being derivative with respect to affine arc-length. In fact

$$\gamma(s_1) - \gamma(s_2) = d_0\left(\gamma''(s_1) - \gamma''(s_2)\right),$$

where d_0 is the common affine distance from the ADSS point to γ at $\gamma(s_1), \gamma(s_2)$.

The corresponding point of the ADSS is

$$\gamma(s_1) + \frac{[\gamma(s_1) - \gamma(s_2),\ \gamma''(s_1)]}{[\gamma''(s_2), \gamma''(s_1)]}\gamma''(s_1). \tag{8}$$

□

We say that the condition (7) defines the *pre-ADSS* : the parameter pairs which are needed to determine the ADSS itself. We do of course include limiting points of (7) which lie on the diagonal $s_1 = s_2$; these give the end-points of the ADSS itself. Some examples of the pre-ADSS are given in the figures in §6.

Remark 3.3. It is interesting to note that smooth points of the pre-ADSS where the curve is tangent (2-point contact) to a line s_1 = constant or s_2 = constant correspond conveniently to cusps as in Theorem 3.1(2), *except* that they also arise for pairs satisfying (7) when the tangent at $\gamma(s_1)$ meets the curve again at $\gamma(s_2)$, or vice versa. This is a generic occurrence and happens, e.g., in Figure 7, left. Cusps of the type in Theorem 3.1(5) do not make themselves evident on the pre-ADSS.

Of course we cannot use (7) or (8) in a neighbourhood of an inflexion, since γ'' is undefined there. In order to obtain results on the limiting behaviour of the ADSS when one or both points of γ are inflexion points we have to resort to 'bare hands', as follows. Take one segment of γ to be the curve γ_1 with an inflexion at the origin, say $\gamma_1(s) = (s, as^3 + bs^4 + \ldots)$. Take another segment of γ to be parametrized by t say; of course s is not affine arclength, and we do not need t to be either. We use (6) to write (7) in terms of s and t and multiply up by $k_1^{5/3}k_2^{5/3}$ to clear

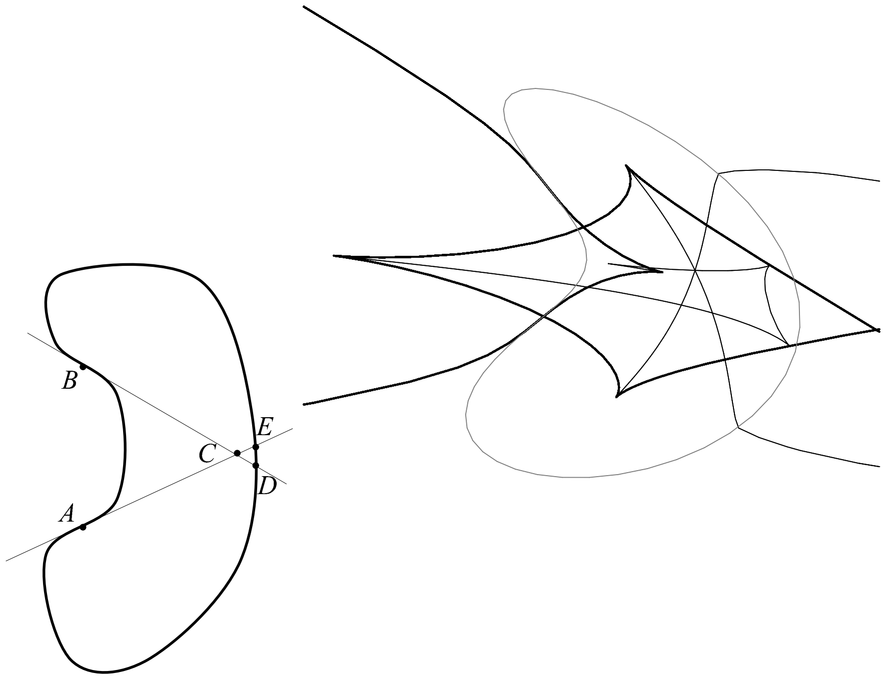


FIGURE 1. Left: Inflexional tangents at A and B intersect at C, where the ADSS will have an ordinary cusp; at D and E the ADSS will have a singularity of type (5,6). See Theorem 3.1, parts 5 and 6. Right: an actual example of a curve γ (in grey) exhibiting these features on the ADSS (thinner black curve). The affine evolute is also drawn (thicker black curve); it has inflexions at the inflexions of γ and four cusps in the figure—at the right there is a crossing, not a cusp, where the figure is clipped. The ADSS has endpoints in the four cusps of the affine evolute (Theorem 3.1, part 3), two cusps on the affine evolute (part 2), a cusp at the intersection of inflexional tangents of γ (part 5), and two (5,6) singularities where inflexional tangents of γ meet the curve again (part 6).

denominators, where $k_i = [\dot{\gamma}_i, \ddot{\gamma}_i]$, the dots referring to differentiation with respect to s or t. It is then convenient to express k_1 as a power series in s, and hence to obtain

$$k_1^{5/3} = (6a)^{5/3}\overline{s}\left(1 + \frac{10b}{a}s + \dots\right),$$

where $\overline{s} = s^{5/3}$. Writing $\gamma_i = (X_i, Y_i)$ we arrive at the pre-ADSS condition replacing (7) of the form $c_1 = c_2\overline{s}$, where

$$\begin{aligned} c_1 &= (X_1 - X_2)k_2^{5/3}\left(k_1\ddot{Y}_1 - \frac{1}{3}\dot{k}_1\dot{Y}_1\right) - (Y_1 - Y_2)k_2^{5/3}\left(k_1\ddot{X}_1 - \frac{1}{3}\dot{k}_1\dot{X}_1\right), \\ c_2 &= (6a)^{5/3}\left(1 + \frac{10b}{a}s + \dots\right)\left((X_1 - X_2)\left(k_2\ddot{Y}_2 - \frac{1}{3}\dot{k}_2\dot{Y}_2\right)\right) - (Y_1 - Y_2)\left(k_2\ddot{X}_2 - \frac{1}{3}\dot{k}_2\dot{X}_2\right). \end{aligned}$$

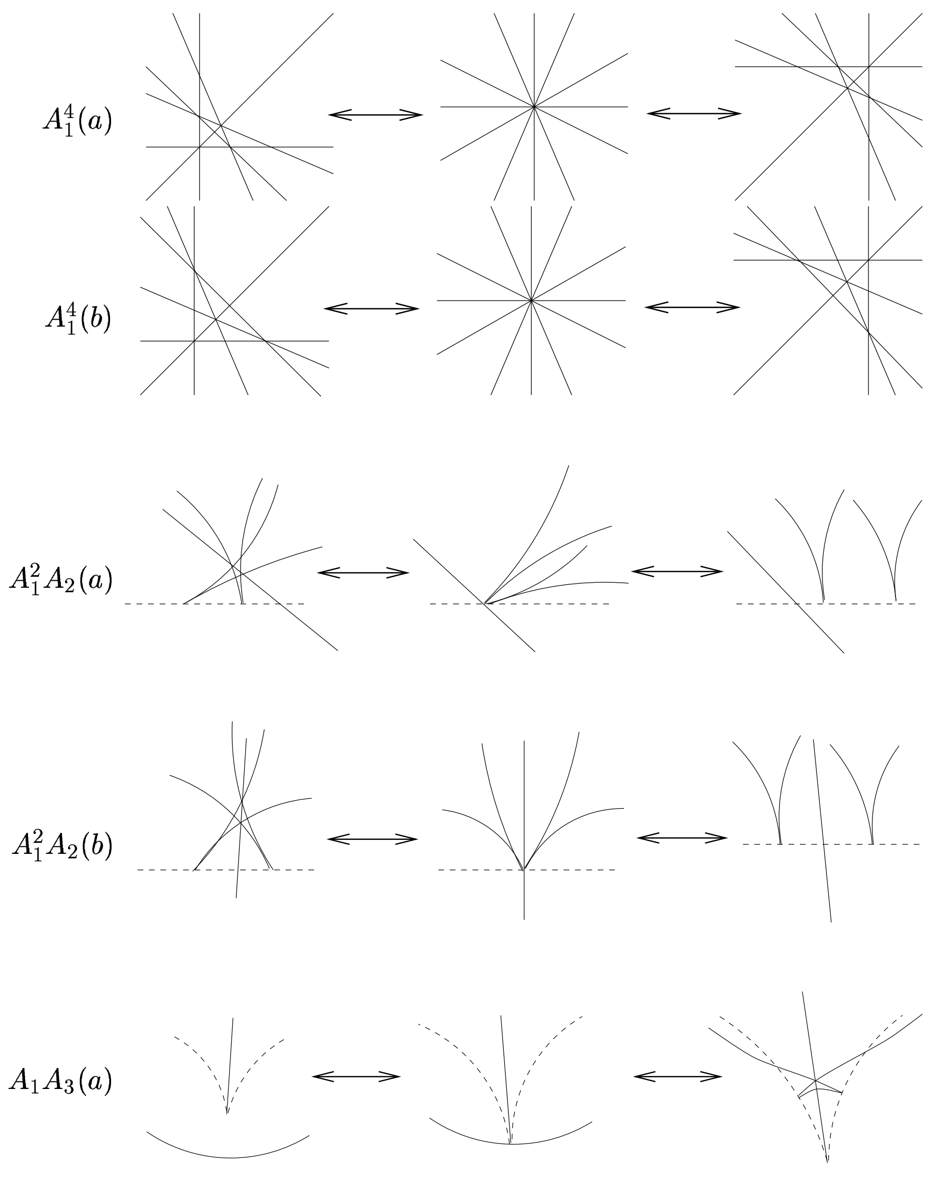
Finally, to make the functions smooth everywhere we actually use for the pre-ADSS condition

$$c_1^3 = c_2^3\overline{s}^3, \quad \text{that is } c_1^3 = c_2^3 s^5. \tag{9}$$

This can be expanded as a power series in s and t for computational purposes. The result can be substituted in (8) to obtain a local power series expansion of the ADSS. In this way we find the results 5 and 6 of Theorem 3.1. (The full calculations are in [15].) □

4. Transitions on bifurcation sets

In the study of 1-parameter families of Euclidean Symmetry Sets in [6], a full list of all the possible transitions that may occur on the full bifurcation set of a generic 2-parameter family of functions of one variable is obtained. We shall reproduce here in Figure 2 only the list which is relevant to the current situation; for the other cases ('Morse' transitions and those involving D singularities) see [6].



In [6] it is shown that not all of these transitions may actually occur for the euclidean symmetry set: the transitions $A_1^4(b), A_1^2A_2(b), A_1A_3(b)$ are ruled out by geometrical considerations, whereas the respective (a) transitions do occur.

We now carry out a similar analysis of the transitions on 1-parameter families of affine distance symmetry sets, in order to classify the transitions which may actually occur on the ADSS of a

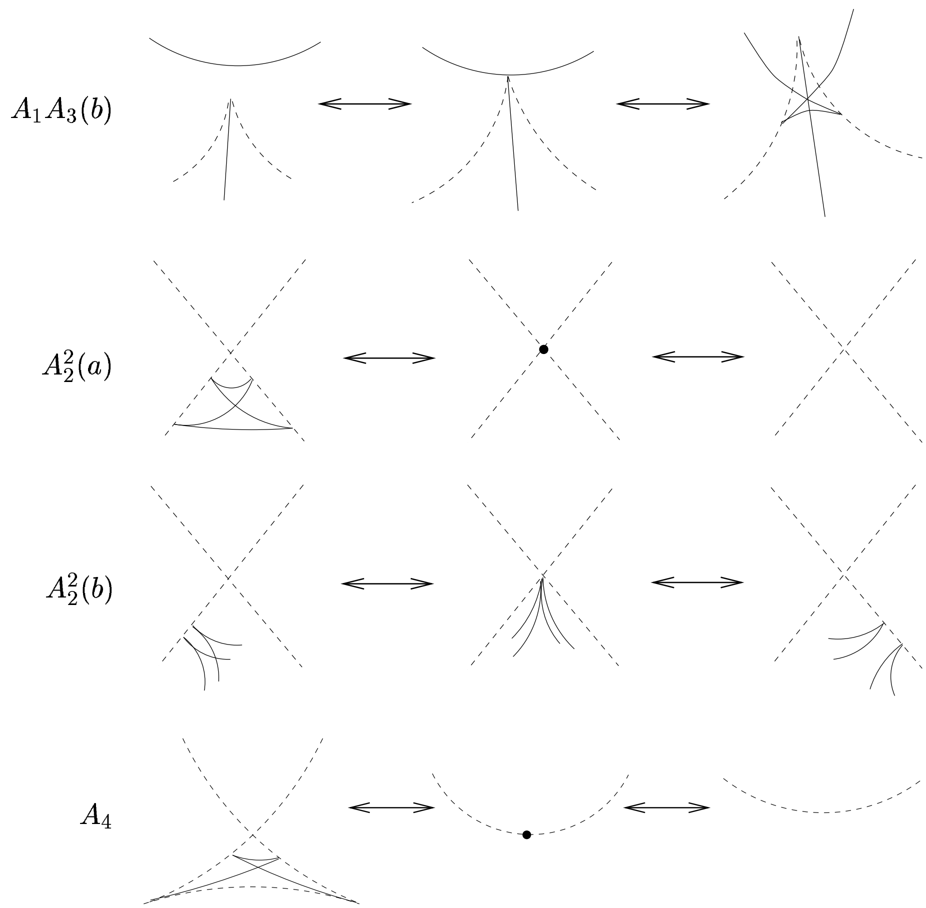


FIGURE 2. Local transitions on symmetry sets in generic 1-parameter families of plane curves, omitting the 'Morse' transitions and those related to D singularities.

smooth plane curve as this curve is deformed through a 1-parameter family. In the next section we avoid inflexion points of the underlying curve γ. Nevertheless it will turn out that there is a striking difference between the cases of oval and non-oval curves γ.

In §5 we illustrate the method with an example, that of the A_1A_3 transitions. Similar procedures apply to the other transitions; the details are in [15].

In §6 we give some details of the strange transitions which occur when we have inflexions on the curve γ, as in Theorem 3.1(5) and (6). At present we are not able to predict the details of these transitions theoretically: as in the theorem, we are forced to do bare-hands calculations and experiments since the affine-distance function to which we wish to apply the techniques of singularity theory is undefined at the relevant points. The transitions appear to be, from the usual standpoint, highly degenerate, though in the present context they are generic.

5. Transitions on the ADSS

We will now sketch proofs of the following two theorems, the main results of this article. For this section we avoid inflexion points of the underlying curve γ. (Compare §6.) The proofs proceed on a case-by-case basis and we illustrate with a typical case below, that of A_1A_3.

Theorem 5.1. *The transitions $A_1^4(a)$, $A_1^2A_2(a)$, $A_1A_3(a)$, $A_2^2(a)$, $A_2^2(b)$ and A_4 (as illustrated in Figure 2) may occur on the Affine Distance Symmetry Set of a generic family of ovals, but the transitions $A_1^4(b)$, $A_1^2A_2(b)$, $A_1A_3(b)$ may not.*

The crucial point to note about the above is that the proof is restricted to ovals only: the proof depends fundamentally on the fact that we are restricting the family of curves to ovals, and if we lose this restriction, then there is no reason to rule out the $A_1^4(b)$, $A_1^2A_2(b)$, $A_1A_3(b)$ transitions from occurring on the ADSS. In fact, our arguments show, by finding explicit conditions on curve segments (e.g. (10) below), that the other transitions *do* occur on families of non-oval plane curves, and in fact by means of examples it is possible to observe these 'extra' transitions occurring on the ADSS of a non-oval. (This task is non-trivial due to the extremely complicated nature of the ADSS.) We are able to conclude:

Theorem 5.2. *The transitions $A_1^4(a)$, $A_1^4(b)$, $A_1^2A_2(a)$, $A_1A_3(a)$, $A_1A_3(b)$, $A_1^2A_2(b)$, $A_2^2(a)$, $A_2^2(b)$ and A_4 (as illustrated in Figure 2) may occur on the ADSS of a generic family of plane curves.*

Example: the A_1A_3 transitions

We follow the procedure as outlined in [6] in the A_1A_3 singularity case in order to illustrate the methods by which we hope to classify the transitions that may occur on 1-parameter families of Affine Distance Symmetry Sets.

Consider the standard multi-versal unfolding of an A_1A_3 singularity, given by

$$G\colon \mathbb{R}^{(2)} \times \mathbb{R}^3 \to \mathbb{R},$$

where $\mathbb{R}^{(2)}$ denotes parameters t_1, t_2 (near zero), $\mathbb{R}^3$ denotes the space of unfolding parameters $\mathbf{y} = (y_1, y_2, y_3)$, and multi-versal unfolding G is given by the two unfoldings

$$\begin{aligned} G_1(t_1, \mathbf{y}) &= t_1^2, \\ G_2(t_2, \mathbf{y}) &= \pm t_2^4 + t_2^2 + t_2 y_2 + y_3. \end{aligned}$$

Note that there is a choice of sign in G_2: this ambiguity will not effect our calculations, and without loss of generality we will from now on take the positive sign.

Step One: Finding the 'Big Bifurcation Set'

The first task is to find the 'Big Bifurcation Set' (BBS) of standard unfolding G, which sits in $\mathbf{y}$-space: this object contains all the possible bifurcation sets in a neighbourhood of the A_1A_3 singularity of which G is a multi-versal unfolding. The A_1A_3-point itself sits at the origin in this space. The individual bifurcation sets can be recovered as the level sets of a generic function on the BBS. The BBS will comprise an A_1^2-set (the 'big symmetry set') and a A_2-set (the 'big evolute'), situated in $\mathbb{R}_{\mathbf{y}}$-space. The A_1^2-set itself is in two parts: the first is the 'swallowtail' surface defined by

$$\begin{cases} y_2 = -4t_2^3 - 2t_2 y_1, \\ y_3 = 3t_2^4 + t_2^2 y_1, \end{cases}$$

and the second is the half-plane $\{y_1 \leq 0, y_2 = 0\}$. The A_2-set is the cuspidal edge in the y_3-direction, with $y_1 \leq 0$, given by

$$\left.\begin{aligned} y_1 &= -6t_2^2 \\ y_2 &= 8t_2^3 \\ y_3 &\text{ arbitrary} \end{aligned}\right\}$$

Figure $3(a)$ shows the BBS.

Step Two: Finding the 'bad planes'

We call a plane through the origin in $\mathbb{R}^3_{\mathbf{y}}$ a *bad plane* if it contains the limit of tangent spaces to a stratum of the BBS at smooth points tending to the origin. Our task is to find all of these bad planes: it is precisely these planes which we wish to avoid as kernel planes to generic linear functions on the BBS. Let such a linear function be

$$h = a_1 y_1 + a_2 y_2 + a_3 y_3,$$

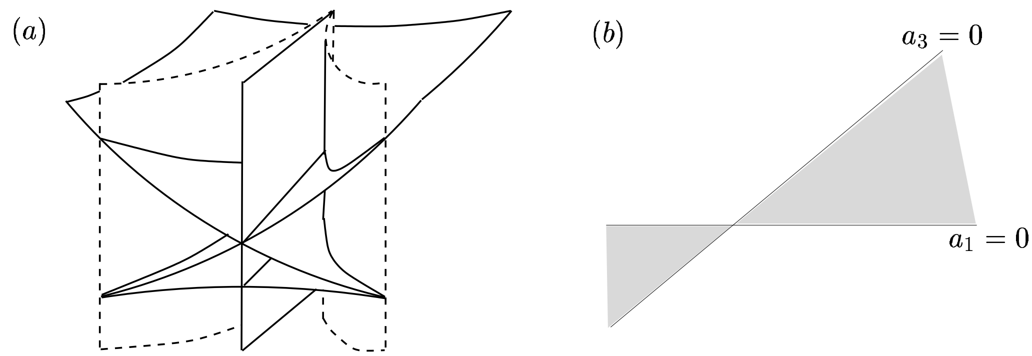


FIGURE 3. *(a) The Big Bifurcation Set for the standard unfolding of an A_1A_3 singularity. It consists of a swallowtail surface, a half-plane, and the 'big evolute' which is the cuspidal edge, shown dashed. (b) Each point in this $\mathbb{R}P^2$ represents a plane through the origin in $\mathbf{y}$-space: the lines $a_1 = 0$ and $a_3 = 0$ represent the set Δ corresponding to the 'bad directions'; these are the kernel directions of non-generic linear functions on the BBS.*

Consideration of the limiting tangent planes shows that the only 'bad' planes are those orthogonal to $(1, 0, 0)$ and $(0, 0, 1)$. We denote this set of bad planes in $\mathbb{R}P^2$ by Δ, shown in Figure 3(b). The components of $\mathbb{R}P^2 - \Delta$ represent collections of normals to planes which, as kernels of $dh(0)$, give stratified C^0-equivalent functions h: that is, each component in the region swept out by normals to planes giving stratified C^0-equivalent families of sections.

Remark 5.3. For relevant remarks on stratified C^0 equivalence, and in particular a discussion of why this is the correct equivalence to use here, see [6, p.199].

Step Three: Families of sections (level sets of generic functions) We can distinguish between the regions of Figure 3(b) by considering the sign of a_1a_3. We find:

Proposition 5.4 (A_1A_3 condition)**.** *A point $(a_1\colon a_2\colon a_3)$ is in a shaded/unshaded region of Figure 3(b) depending on whether*

$$a_1 a_3$$

is positive/negative respectively, and the corresponding full bifurcation set exhibits a transition of type $A_1A_3(a)/A_1A_3(b)$ (see Figure 4). □

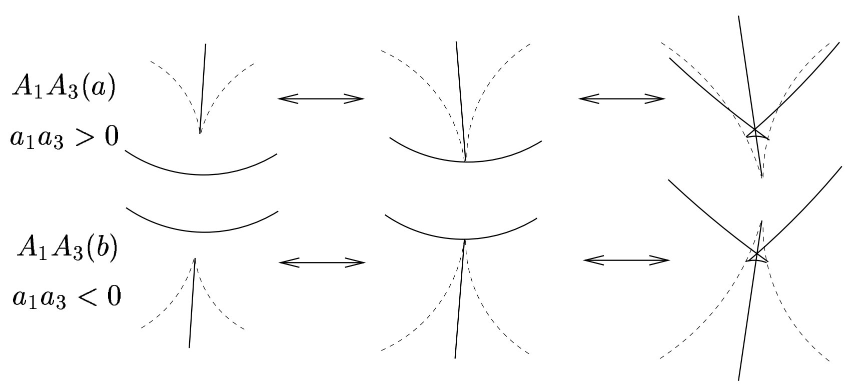


FIGURE 4. *The A_1A_3 transitions for $a_1a_3 > 0$ and $a_1a_3 < 0$. The evolute (A_2-set) is shown as a dashed line.*

Step Four: Relating standard model to the ADSS

It remains to relate the A_1A_3 condition of Proposition 5.4, which distinguishes between the occurrence of the two different A_1A_3 transitions on a generic full bifurcation set, to the particular family of functions at hand, namely those given by affine distance. The calculations are from now on specific to this case.

Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, and denote by $\mathbf{x}_0$ the A_1A_3-point on the ADSS. Then the family of affine distance functions on the family of curve segments will be

$$F\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2, (0, 0, \mathbf{x}_0) \to \mathbb{R},$$

given by

$$F_i(t_i, u, \mathbf{x}) = [\mathbf{x} - \gamma_{u,i}(t_i), \gamma'_{u,i}(t_i)] = \begin{vmatrix} x_1 - X_{u,i}(t_i) & X'_{u,i}(t_i) \\ x_2 - Y_{u,i}(t_i) & Y'_{u,i}(t_i) \end{vmatrix}$$

for $i = 1, 2$, where $'$ (prime) will always denote $\partial/\partial t_i$, and t_i is assumed to be the affine-arclength parameter along the corresponding curve segment γ_i. We are able to show that

$$a_1 \equiv \left.\frac{\partial B_1}{\partial y_1}\right|_{\mathbf{y}=\mathbf{0}} \text{ and } a_3 \equiv \left.\frac{\partial B_1}{\partial y_3}\right|_{\mathbf{y}=\mathbf{0}}$$

where B_1 is equivalent to the map h on the standard A_1A_3-set. We then deduce that

$$I_3 = \left.\begin{pmatrix} \frac{\partial^2}{\partial t^2}\left(\frac{\partial F_2}{\partial u}\right) & \frac{\partial^2}{\partial t^2}\left(\frac{\partial F_2}{\partial x_1}\right) & \frac{\partial^2}{\partial t^2}\left(\frac{\partial F_2}{\partial x_2}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial F_2}{\partial u}\right) & \frac{\partial}{\partial t}\left(\frac{\partial F_2}{\partial x_1}\right) & \frac{\partial}{\partial t}\left(\frac{\partial F_2}{\partial x_2}\right) \\ \frac{\partial F_2}{\partial u} - \frac{\partial F_1}{\partial u} & \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_1} & \frac{\partial F_2}{\partial x_2} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}\right|_{(A(t,\mathbf{0}),\mathbf{x}_0)} \times \left.\begin{pmatrix} \frac{\partial B_1}{\partial y_1} & \frac{\partial B_1}{\partial y_2} & \frac{\partial B_1}{\partial y_3} \\ \frac{\partial B_2}{\partial y_1} & \frac{\partial B_2}{\partial y_2} & \frac{\partial B_2}{\partial y_3} \\ \frac{\partial B_3}{\partial y_1} & \frac{\partial B_3}{\partial y_2} & \frac{\partial B_3}{\partial y_3} \end{pmatrix}\right|_{\mathbf{y}=\mathbf{0}}$$

where I_3 is the (3×3) identity matrix. We will denote by JB the matrix of partial derivatives of B_1, B_2 and B_3, evaluated at $\mathbf{y} = \mathbf{0}$. We will not need the $\partial F_2/\partial u$ components, since we only require terms from the top row of JB, which are given as cofactors in the matrix of partial derivatives of F_1, F_2. If we write

$$A(t, \mathbf{0}) \equiv \alpha_1 t + \alpha_2 t^2 + \dots$$

for coefficients $\alpha_i \in \mathbb{R}$ $(\alpha_1 \neq 0)$, then the system becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} * & \alpha_2 Y_2'' + \alpha_1^2 Y_2''' & -\alpha_2 X_2'' - \alpha_1^2 X_2''' \\ * & \alpha_1 Y_2'' & -\alpha_1 X_2'' \\ * & Y_2' - Y_1' & -X_2' + X_1' \end{pmatrix} \times \left.\begin{pmatrix} \frac{\partial B_1}{\partial y_1} & \frac{\partial B_1}{\partial y_2} & \frac{\partial B_1}{\partial y_3} \\ \frac{\partial B_2}{\partial y_1} & \frac{\partial B_2}{\partial y_2} & \frac{\partial B_2}{\partial y_3} \\ \frac{\partial B_3}{\partial y_1} & \frac{\partial B_3}{\partial y_2} & \frac{\partial B_3}{\partial y_3} \end{pmatrix}\right|_{\mathbf{y}=\mathbf{0}}$$

This tells us that

$$\begin{aligned} \left.\frac{\partial B_1}{\partial y_1}\right|_{\mathbf{y}=\mathbf{0}} &= \begin{vmatrix} \alpha_1 Y_2'' & -\alpha_1 X_2'' \\ Y_2' - Y_1' & -X_2' + X_1' \end{vmatrix}, \\ &= -\alpha_1[\gamma_2' - \gamma_1', \gamma_2''], \\ \left.\frac{\partial B_1}{\partial y_3}\right|_{\mathbf{y}=\mathbf{0}} &= -\alpha_1^3[\gamma_2'', \gamma_2''']. \end{aligned}$$

Thus

$$a_1 a_3 = \alpha_1^4[\gamma_2' - \gamma_1', \gamma_2''] \cdot [\gamma_2'', \gamma_2'''],$$

is the expression that we wish to interpret. Now as usual we denote the affine curvature of γ_2 at $t_2 = 0$ by $\mu_2 \equiv [\gamma_2'', \gamma_2''']$, and thus we have:

Theorem 5.5 (A_1A_3 condition for the ADSS). *The ADSS at an A_1A_3-point exhibits a transition of type $A_1A_3(a)/A_1A_3(b)$ depending upon whether*

$$-\mu_2[\gamma_1' - \gamma_2', \gamma_2''] \tag{10}$$

is positive/negative respectively. □

In what follows we interpret this condition for ovals, showing that the expression (10) can take only one sign for ovals. Then we disregard the condition that the curves are ovals and show that this expression can take *both* negative and positive signs for generic plane curves.

Interpretation of A_1A_3 condition for ovals We will assume that our curve points γ_1 and γ_2 lie on the same oval, with corresponding affine tangents γ_1', γ_2'. We use the following result, which follows from the fact that affine arclength forces an anticlockwise orientation on an oval.

Lemma 5.6 (Oval Condition). *If γ_i, γ_j are two distinct points on an oval parametrized by affine-arclength, then*

$$[\gamma_i - \gamma_j, \gamma_i'] > 0,$$

where as usual $'$ (prime) denotes derivative w.r.t. affine-arclength. □

We also use the ADSS Condition of Proposition 3.2. Now since we have an A_3 singularity of the affine distance function at γ_2, we know that the A_1A_3 ADSS point $\mathbf{x}_0$ can be expressed as

$$\mathbf{x}_0 \equiv \gamma_2 + \frac{1}{\mu_2}\gamma_2'',$$

(see Proposition 2.3), and the fact that γ_1 and γ_2 must be the same affine distance d_0 from $\mathbf{x}_0$ implies that $d_0 = -1/\mu_2$, and therefore

$$\mathbf{x}_0 \equiv \gamma_1 + \frac{1}{\mu_2}\gamma_1''.$$

We substitute this into the Oval Condition $[\gamma_1 - \gamma_2, \gamma_1'] > 0$ to get

$$\begin{aligned}
& \left[\frac{1}{\mu_2}(\gamma_2'' - \gamma_1''), \gamma_1'\right] > 0, \\
\iff \quad & \frac{1}{\mu_2}([\gamma_2'', \gamma_1'] + 1) > 0, \\
\iff \quad & \frac{1}{\mu_2}(1 - [\gamma_1', \gamma_2'']) > 0, \\
\iff \quad & \frac{1}{\mu_2}([\gamma_2' - \gamma_1', \gamma_2'']) > 0, \text{ since } [\gamma_2', \gamma_2''] = 1,
\end{aligned}$$

which proves that the expression (10) takes only positive values for ovals. Thus the transition $A_1A_3(b)$ *will not occur* on the ADSS of a family of ovals. The transition $A_1A_3(a)$ may occur, and indeed explicit examples can be constructed ([15]).

Proposition 5.7. *The transition $A_1A_3(a)$ may occur generically on the ADSS of a family of ovals, but the transition $A_1A_3(b)$ does not.* □

Interpretation of A_1A_3 condition for non-ovals

We will now show that, if we disregard the assumption that the points γ_1 and γ_2 lie on the same oval, then the expression (10) may take both negative and positive values. We will construct two situations in turn, one with $\mu_2 > 0$ and the other with $\mu_2 < 0$, and show that (10) is positive and negative respectively.

Case (i): $\mu_2 > 0$

Consider Figure 5(a), where without loss of generality we have fixed γ_2, γ_2' and $\mathbf{x}_0$, and also the γ_1 point and the tangent line at this point. We can deduce from $[\gamma_2', \gamma_2''] = 1$ that γ_2'' has the

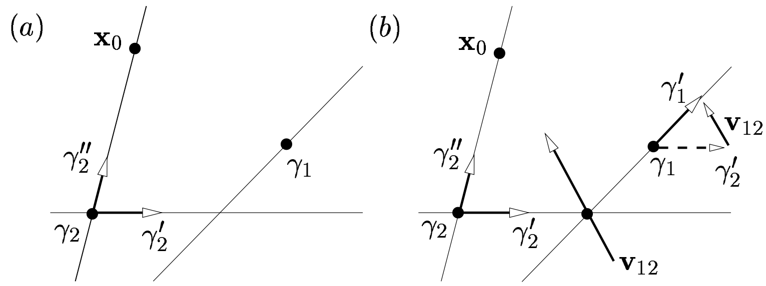


FIGURE 5. *(a) Fix γ_2, γ_2', $\mathbf{x}_0$, γ_1 and the tangent direction at γ_1. Then we can deduce γ_2'', and we see that $\mu_2 > 0$. (b) It then follows that γ_1' is as shown, and hence we can deduce $\mathbf{v}_{12}$. It is then clear that $[\mathbf{v}_{12}, \gamma_2''] < 0$.*

direction and length as illustrated. Then, since the γ_2 point corresponds to an A_3 singularity of the distance function, we know that

$$\mathbf{x}_0 \equiv \gamma_2 + \frac{1}{\mu_2}\gamma_2'',$$

and thus $\mu_2 > 0$. Also, since $\mathbf{x}_0$ must be the same affine distance from γ_1 as it is from γ_2, we can deduce that γ_1' has length and direction as shown in Figure 5(b), and from this it follows that the vector $\mathbf{v}_{12} \equiv \gamma_1' - \gamma_2'$ has orientation as shown.

We recall here the following proposition from [12].

Proposition 5.8 (Concurrent Tangents Condition). *Suppose two points $\gamma(s_1)$, $\gamma(s_2)$ contribute point $\mathbf{x}$ to the ADSS of a curve γ, parametrized by affine-arclength s. (As usual, we use $'$ (prime) to denote derivative w.r.t. s.) Then the tangent line to the ADSS at $\mathbf{x}$ is*

- *in the direction $\gamma'(s_1) - \gamma'(s_2)$, and*
- *concurrent with the corresponding tangent lines at $\gamma(s_1), \gamma(s_2)$.*

□

In the current context this tells us that $\mathbf{v}_{12}$ is in the direction of the line joining the intersection of the tangents lines at γ_1 and γ_2.

Thus

$$[\mathbf{v}_{12}, \gamma_2''] < 0,$$

and therefore

$$-\mu_2[\mathbf{v}_{12}, \gamma_2''] > 0.$$

Remark 5.9. In this case, γ_1 and γ_2 may lie on the same oval with corresponding affine tangent vectors γ_1' and γ_2', although they need not.

Case (ii): $\mu_2 < 0$

Consider Figure 6(a), where without loss of generality we have fixed γ_2, γ_2' and $\mathbf{x}_0$, and also the point γ_1 and the corresponding tangent line through this point. Since $[\gamma_2', \gamma_2''] = 1$, we can deduce the direction and length of γ_2'' as shown. Then, since the γ_2 point corresponds with the A_3 singularity of the affine distance function, we know that

$$\mathbf{x}_0 \equiv \gamma_2 + \frac{1}{\mu_2}\gamma_2'',$$

and hence $\mu_2 < 0$. Also, since $\mathbf{x}_0$ must be the same affine distance from γ_1 as it is from γ_2, we can deduce that γ_1' has direction and length as shown in Figure 6(b), and from this it follows that $\mathbf{v}_{12} \equiv \gamma_1' - \gamma_2'$ has orientation as shown. (As before, the Concurrent Tangent Condition tells us the direction of $\mathbf{v}_{12}$.) Thus

$$[\mathbf{v}_{12}, \gamma_2''] < 0,$$

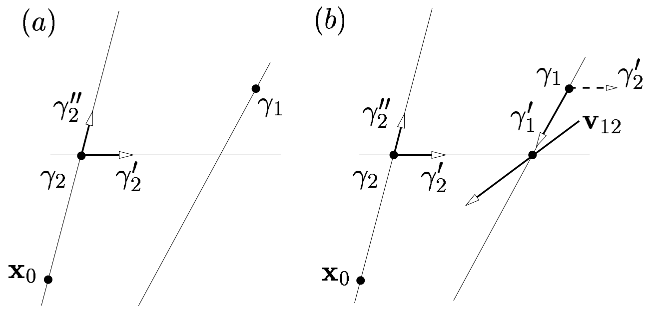


FIGURE 6. *(a) Fix γ_2, γ_2', $\mathbf{x}_0$, γ_1 and the tangent direction at γ_1. Then we can deduce γ_2'', and we see that $\mu_2 < 0$. (b) It then follows that γ_1' is as shown, and hence we can deduce $\mathbf{v}_{12}$. It is then clear that $[\mathbf{v}_{12}, \gamma_2''] < 0$.*

and therefore

$$-\mu_2[\mathbf{v}_{12}, \gamma_2''] < 0.$$

Remark 5.10. In this case, γ_1 and γ_2 cannot lie on the same oval with corresponding affine tangent vectors γ_1' and γ_2'.

Hence we have shown that the expression (10) can take *both* signs when we disregard the oval assumption, and thus we have:

Proposition 5.11. *The ADSS of a generic family of plane curves may exhibit transitions of type $A_1A_3(a)$ and $A_1A_3(b)$.* □

It is now possible to take two polynomial branches of a smooth curve γ and calculate the condition on the coefficients which separates the two cases. Explicit families can now be constructed which exhibit the transitions. This is done in [15].

For the other cases, here are the conditions which determine which of the two alternative transitions occur. In all of these, dropping the oval condition permits both signs of the expression to be realised. The details of calculations are in [15]. As with the A_1A_3 transition, the crucial point is that, for non-ovals, both signs can occur so that both transitions are possible. The notation is that of Figure 2.

Proposition 5.12. (1) A_1^4 *(a) or (b) according as*

$$[\gamma_1' - \gamma_2', \gamma_2' - \gamma_3'][\gamma_2' - \gamma_3', \gamma_3' - \gamma_4'][\gamma_3' - \gamma_4', \gamma_4' - \gamma_1'][\gamma_4' - \gamma_1', \gamma_1' - \gamma_2']$$

is positive or negative.

(2) $A_1^2A_2$ *(a) or (b) according as $[\gamma_1' - \gamma_2', \gamma_1''][\gamma_1' - \gamma_3', \gamma_1'']$ is positive or negative. Here γ_1 is the branch contributing the A_2 singularity.*
(3) A_2A_2 *(a) or (b) according as $\mu_1'\mu_2'$ is positive or negative. Here μ is the affine curvature. In the situation of the Euclidean symmetry set both cases occur and are distinguished by the signs of the derivative of Euclidean curvature.*
(4) *The single A_4 transition also occurs generically on the ADSS.* □

6. Transitions involving inflexions

In this section we present some experimental results which show how the ADSS transforms when inflexions on the curve γ are involved. We do not as yet know how to fit these transitions into the framework of singularity theory. We shall briefly consider four cases:

(1) Two inflexions merging locally in a higher inflexion cause an ordinary cusp at the intersection of the inflexional tangents, as in Theorem 3.1(5), to disappear.
(2) Two inflexions merging in a higher inflexion cause two ordinary cusps, at the intersection with another fixed inflexional tangent, as in Theorem 3.1(5), to interact.

(3) The inflexional tangent at $\gamma(s_1)$ meets the curve γ again in two points which come into coincidence; as in Theorem 3.1(6) two (5,6) singularities on the ADSS then merge.
(4) Two inflexions merging cause two (5,6) singularities to merge since the inflexional tangents meet γ in two further points which come into coincidence.

1. The cusp at the intersection of two inflexional tangents is very close to an endpoint of the ADSS (in a cusp of the affine evolute). As the inflexions merge another branch of the ADSS approaches and at the moment of transition this branch is tangent to the curve γ. The rest of the ADSS has locally become identified with γ itself. After the transition, when γ no longer has inflexions locally, the ADSS has three branches all ending in cusps of the affine evolute. See Figure 7.

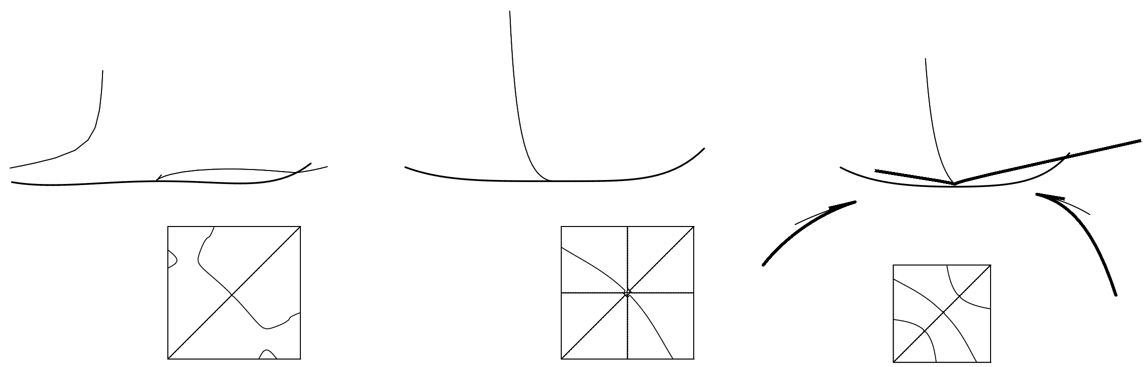


FIGURE 7. Left: a curve with two nearby inflexions. The ADSS has a single cusp at the intersection of the two inflexional tangents (Theorem 3.1(5)). Below is drawn the pre-ADSS, plus the diagonal. The cusp on the ADSS is not evident on the pre-ADSS and the horizontal/vertical tangents to the pre-ADSS do not indicate cusps on the ADSS. See Remark 3.3. Centre: the moment where the two inflexions merge. The pre-ADSS (below) has become highly singular: even ignoring the diagonal part there are three branches through the singular point. The right-hand diagram shows the curve, now having no inflexions locally, together with the ADSS—three branches with endpoints—and also for good measure the affine evolute (drawn heavily), which can be seen to have cusps at the endpoints of the ADSS.

2. Here, there are two cusps as in Theorem 3.1(5)—see Figure 8, left. One branch of γ has two inflexions very close together and the other branch has one inflexion. The two cusps approach one another, touch, and separate, as shown in Figure 8. After separation, there are no inflexions to cause cusps on the ADSS; these cusps of of the type in Theorem 3.1(2), and can be predicted from the pre-ADSS which has two horizontal tangent lines, as shown in the figure.

3. The ADSS 'pulls away' from one branch of the curve γ in the manner of Figure 9.

4. Two inflexional tangents at nearby inflexions on one branch of γ meet another branch of γ and at each of these intersections the ADSS has a (5,6) singularity. These merge as the inflexions come into coincidence, and separate into two ordinary cusps. See Figure 10. Note the vertical tangents to the pre-ADSS in the right-hand figure.

7. Conclusion and further research

We have considered the affine distance symmetry set (ADSS) of a plane curve, which is defined in a way closely analogous to the euclidean symmetry set. For the case of oval curves

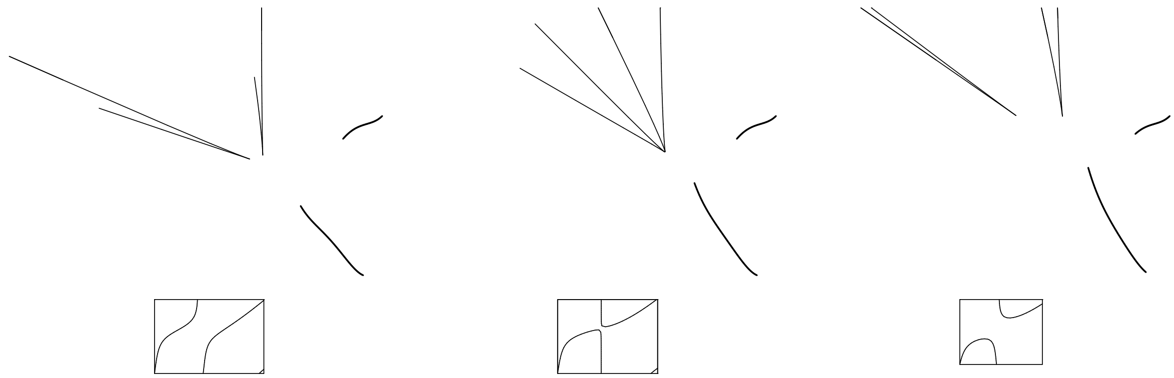


FIGURE 8. Left: One branch of γ has two inflexions which are very close together. The tangents to γ there meet the tangent to the other inflexional branch of γ, creating two cusps on the ADSS. The pre-ADSS is shown below. As the two inflexions on the first branch of γ merge (centre) the pre-ADSS undergoes a transition reminiscent of a Morse transition. After the two inflexions on the first branch have disappeared (right) there are still two cusps on the ADSS, caused now by the two horizontal tangents of the pre-ADSS.

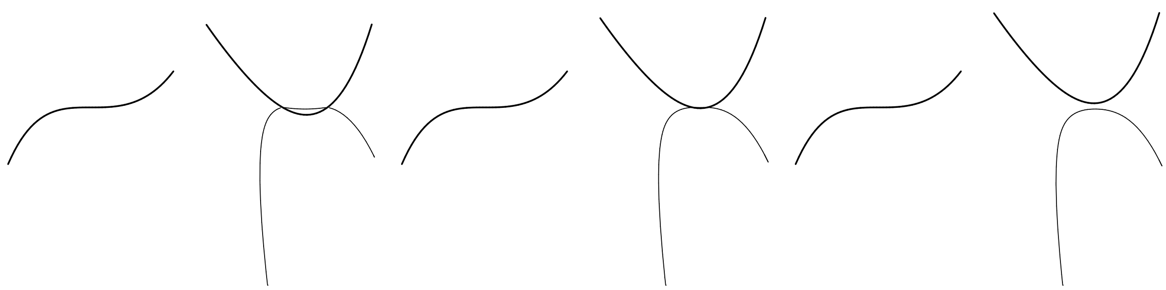


FIGURE 9. An inflexional tangent meets the curve again in two points which come into coincidence. The two $(5, 6)$ singularities predicted by Theorem 3.1(6) (left) merge (centre) into a nonsingular branch of the ADSS (right).

the transitions occurring on the ADSS in a generic 1-parameter family of curves are in fact identical with those occurring on the euclidean symmetry set of a generic family of curves. When we come to consider curves with inflexions, two things happen. Firstly other transitions, barred in the case of euclidean symmetry sets and ADSS for ovals, now occur. Secondly, there are transitions which involve inflexions directly, and these do not resemble those of the euclidean symmetry set at all. It would clearly be desirable to embrace these, and the anomalous structures of the ADSS, in the same framework of bifurcation sets which allows us to analyse the more regular cases.

In the euclidean case, there is a subset of the symmetry set called the 'medial axis', which is obtained by restricting the bitangent circles to ones whose radius equals the minimum distance from their centre to the curve γ ('maximal circles'). A similar restriction is possible to turn the ADSS into the affine distance medial axis, and some preliminary work has been done on this in [13].

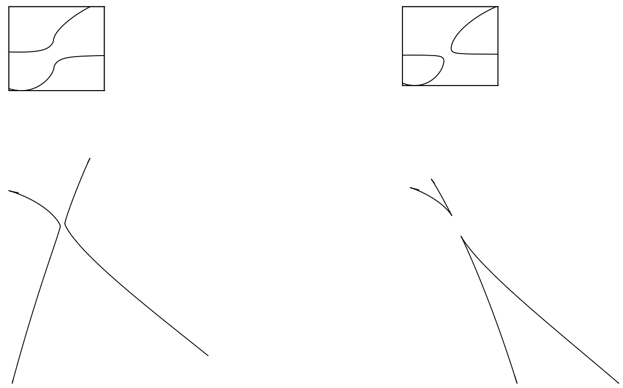


FIGURE 10. Here, γ consists of a curve segment γ_1 with two inflexions very close together, and another segment γ_2 without inflexions which intersects the two inflexional tangents transversely. At these two intersection points the ADSS will have $(5, 6)$ singularities, as in Theorem 3.1(6). The segment γ_1 will be off the picture, and γ_2 is not shown, but it goes roughly horizontally through the two fairly obvious kinks in the ADSS in the left hand figure. The pre-ADSS is shown above. After the inflexions have merged and disappeared the ADSS is left with two ordinary cusps, as in the right-hand figure.

There are several other promising candidates for the role of an affinely invariant symmetry set. Some of these are explored in [14, 1] but the transitions which occur in 1-parameter families have not been investigated.

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